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On a Problem Concerning Two Integer Sequences

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§1. Introduction

Let $\mathscr{A}$ and $\mathscr{B}$ be sets of natural numbers. Then $\mathscr{B}$ is said to be a k-set with respect to $\mathscr{A}$ if $\mathscr{B}$ contains k integers say $b_1, \ldots, b_k$ such that $b_i + b_j$ $(1 \leqslant j < l \leqslant k)$ all lie outside $\mathscr{A}$. In this paper we shall prove the following

THEOREM. *Let $\varepsilon > 0$ and k be a natural number $\geqslant 2$. Then there exists a positive integer $s^*(\varepsilon, k)$ with the following property.*

Suppose $\mathscr{A} = \{a_1, \ldots, a_t\}$ and $\mathscr{B} = \{b_1, \ldots, b_s\}$ are sets of positive integers where $s \geqslant s^*(\varepsilon, k)$ and $t \leqslant (1-\varepsilon)\, s$. Then $\mathscr{B}$ is a k-set with respect to $\mathscr{A}$.

We note that the above theorem is certainly best possible in the sense that it fails to hold if $\varepsilon > 0$ is replaced by the number 0; for by taking $\mathscr{B}$ to be the integers $1, 2, \ldots, s$ and $\mathscr{A}$ to be the even integers $2, 4, \ldots, 2s$, it is clear that we cannot select from $\mathscr{B}$ three integers with all their pairwise sums not appearing in $\mathscr{A}$.

§2. Some Lemmas

In this section we prove a number of lemmas necessary for the proof of the theorem.

LEMMA 1. *Suppose $\alpha > 0$ and $\beta > 0$. Then there exists $n^*(h, \alpha, \beta)$ such that if $n \geqslant n^*$ and $\mathscr{X}$ is a set of N natural numbers, where $N \geqslant \alpha n$, contained in a progression of length n, and $\mathscr{Y}$ is a set of natural numbers with*[1]

$$|(2\mathscr{X}) \cap \mathscr{Y}| \leqslant (1-\beta)\, N, \tag{1}$$

then $\mathscr{X}$ is an h-set with respect to $\mathscr{Y}$.

Proof. In view of (1) there exist in $\mathscr{X}$ integers $x_1, \ldots, x_{N^*}$, $N^* \geqslant \beta N$ so that $2x_1, \ldots, 2x_{N^*}$ are not in $\mathscr{Y}$. We then proceed to extract h integers from among $x_1, \ldots, x_{N^*}$ so that their pairwise sums all appear in $2x_1, \ldots, 2x_{N^*}$. Since $x_1, \ldots, x_{N^*}$ are N^* integers contained in a progression of length n, $N^* \geqslant \alpha\beta n$, and $n \geqslant n^*(h, \alpha, \beta)$, we have, by a theorem of Varnavides [1], that there exist a number $c = c(\alpha, \beta)$ and cN^2 triples of integers $x_{i_j}, x_{i_l}, x_{i_m}$ so that $x_{i_j} + x_{i_l} = 2x_{i_m}$. Thus there exists one x_{i_j}, which we shall denote by x_{i_1}, so that there are $\beta_1 N$, where $\beta_1 = \beta_1(\alpha, \beta)$, integers x_{i_l} such that $x_{i_1} + x_{i_l} = 2x_{i_m}$. We now repeat the argument with these $\beta_1 N$ integers x_{i_l} and so on.

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[1]) Here $2\mathscr{X}$ simply denotes the set of integers $2x$, where $x \in \mathscr{X}$

By choosing n^* sufficiently large in terms of h, α, β, the argument may be repeated $h-1$ times to yield the assertion of the lemma.

LEMMA 2. *Let $\lambda \geqslant 2$ and $w \geqslant 2$ be a natural number. Then, provided $v \geqslant 2(8)^{2w-3} \lambda^{2w-2}$, $n \geqslant (8\lambda)^{2w-1}$ and $n \geqslant v$, the following is true:*

If $\mathscr{X}$ is a set of n natural numbers and $\mathscr{X}'$ is a subset consisting of v numbers such that $|\mathscr{X}+\mathscr{X}'| \leqslant \lambda |\mathscr{X}|$, then there exists a subset $\mathscr{S}$ in $\mathscr{X}$ of the form[2] *$\mathscr{S} = \{x_0\} + \{0, x_1\} + \cdots + \{0, x_w\}$ consisting of 2^w distinct integers.*

It will be shown presently that Lemma 2 is a simple consequence of the following

LEMMA 2′. *Let $\eta \geqslant 1$ and $\mathscr{T}$, $\mathscr{U}$ be sets of natural numbers so that*

$$|\mathscr{U}| \geqslant 2\eta$$

and

$$|\mathscr{T}+\mathscr{U}| \leqslant \eta |\mathscr{T}|.$$

Then there exists a number $d \neq 0$ so that

$$t_i - t_j = d \tag{2}$$

has at least $(4\eta)^{-1} |\mathscr{T}|$ solutions in $t_i, t_j \in \mathscr{T}$.

Proof of Lemma 2′. We denote $|\mathscr{T}|$ by T, and $|\mathscr{U}|$ by U. Let $s_1, \ldots, s_p$, $p \leqslant \eta T$ be the elements of $\mathscr{T}+\mathscr{U}$ and suppose

$$t_1^{(l)} + u_1^{(l)} = t_2^{(l)} + u_2^{(l)} = \cdots = t_{q_l}^{(l)} + u_{q_l}^{(l)} = s_l\,; \qquad 1 \leqslant l \leqslant p,$$

where $t_i^{(l)} \in \mathscr{T}$ and $t_i^{(l)} \in \mathscr{U}$. From the above set of equations we see that there are, for a fixed l, $\frac{1}{2} q_l (q_l - 1)$ pairs of equations of type $t_i^{(l)} + u_i^{(l)} = t_j^{(l)} + u_j^{(l)}$. We may rewrite each of these equations as $t_i^{(l)} - t_j^{(l)} = u_j^{(l)} - u_i^{(l)}$ or $t_j^{(l)} - t_i^{(l)} = u_i^{(l)} - u_j^{(l)}$. Thus altogether there are $\sum_{l=1}^{p} q_l (q_l - 1)$ differences $t_i - t_j$ $(i \neq j)$ obtained in this way. As there are at most U^2 differences $u_i - u_j$ $(i \neq j)$, there exists one such difference say d $(d \neq 0)$ so that there are N distinct solutions for (2) in t_i, $t_j \geqslant$ where

$$N \geqslant U^{-2} \sum_{l=1}^{p} q_l (q_l - 1).$$

Clearly the above yields

$$N \geqslant U^{-2} \left(\frac{1}{2} \sum_{l=1}^{p} q_l^2 - p \right).$$

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[2]) For sets $\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_r$ of integers, $\mathscr{X}_1 + \cdots + \mathscr{X}_r$ denotes the set of integers of the form $x_1 + \cdots + x_r$ where $x_1 \in \mathscr{X}_1, \ldots, x_r \in \mathscr{X}_r$.

Now $\sum_{l=1}^{p} q_l = TU$ so that an applications of Schwarz's inequality yields

$$(TU)^2 = \left(\sum_{l=1}^{p} q_l\right)^2 \leqslant \sum_{l=1}^{p} 1 \sum_{l=1}^{p} q_l^2 \leqslant 2p(U^2N+p).$$

Since $p \leqslant \eta T$ and $U \geqslant 2\eta$, the above implies $2pU^2N \geqslant (TU)^2/2$ from which $N \geqslant T^2/4p \geqslant$ $\geqslant T/4\eta$ follows. This completes the proof of Lemma 2′.

Proof of Lemma 2. We apply Lemma 2′ with $\mathcal{T} = \mathcal{X}$, $\mathcal{U} = \mathcal{X}'$ and $\eta = \lambda$ and assert the existence of a subset $\mathcal{X}_1$ of $\mathcal{X}$ consisting of $\geqslant (8\lambda)^{-1}\,|\mathcal{X}|$ integers and a number d_0 so that $(d_0 + \mathcal{X}_1) \cap \mathcal{X}_1 = \emptyset$ and $d_0 + \mathcal{X}_1 \subset \mathcal{X}$. We now apply Lemma 2′ again with $\mathcal{T} = \mathcal{X}_1$ and $\mathcal{U} = \mathcal{X}'$ and $\eta = 8\lambda^2$ and assert the existence of a subset $\mathcal{X}_2$ of $\mathcal{X}_1$ consisting of $\geqslant (8^2\lambda^2)^{-1}\,|\mathcal{X}_1| \geqslant (8\lambda)^{-3}\,|\mathcal{X}|$ integers and a number d_1 so that $(d_1 + \mathcal{X}_2) \cap \mathcal{X}_1 = \emptyset$ and $d_1 + \mathcal{X}_2 \subset \mathcal{X}_1$. Since $n \geqslant (8\lambda)^{2w-1}$ and $v \geqslant 2(8)^{2w-3}\,\lambda^{2w-2}$ we may repeat the argument w times to obtain a sequence of numbers $d_0, d_1, \ldots, d_w$ so that $\{d_w\} + \{0, d_{w-1}\} +$ $+ \cdots + \{0, d_0\}$ consists of 2^w numbers all belonging to $\mathcal{X}$. This completes the proof of Lemma 2.

Before stating the next lemma we introduce the concept of a type c set. Henceforth we shall regard ε and k as being fixed.

Let $c \geqslant 2$ be any positive number. We define the numbers $n^*(c)$ and $u(c)$ by

$$n^*(c) = n^*\left(k, \frac{\varepsilon}{8c}, \frac{\varepsilon}{4}\right), \tag{3}$$

where $n^*(h, \alpha, \beta)$ is the number appearing in Lemma 1; and

$$u(c) = [20\varepsilon^{-1} n^*(c)\, c \log c]. \tag{4}$$

DEFINITION 1. A set $\mathcal{X}$ consisting of at least 2^u natural numbers, where $u = u(c)$ is defined by (4), is said to be a set of type c, or simply a type c set, if it contains a subset $\mathcal{S}$ of the form $\{x_0\} + \cdots + \{0, x_u\}$, consisting of 2^u distinct integers such that $|\mathcal{X} + \mathcal{S}| \leqslant c|\mathcal{X}|$.

LEMMA 3. *Let $c \geqslant 2$ and $n \geqslant 2^u$, where $u = u(c)$ is defined by* (4). *Suppose $\mathcal{X}$ is a type c set consisting of n positive integers. Then, with at most* $4^{-1}\,\varepsilon|\mathcal{X}|$ *exceptions, the integers in $\mathcal{X}$ is contained in the union of disjoint arithmetic progressions each of length $\geqslant n^*(c)$, where $n^*(c)$ is defined by* (3), *and each such progression has at least a proportion $\varepsilon(8c)^{-1}$ of its integers appearing in $\mathcal{X}$.*

Proof. Since $\mathcal{X}$ is of type c, it has, by definition, a subset $\mathcal{S} = \{x_0\} + \cdots + \{0, x_u\}$ so that

$$|\mathcal{X} + \mathcal{S}| \leqslant c|\mathcal{X}|.$$

We define $\mathscr{X}^{(i)}$ $(i=0, 1, \ldots, u)$ by

$$\mathscr{X}^{(i)}=\mathscr{X}+\mathscr{S}^{(i)}$$

where

$$\mathscr{S}^{(i)}=\{x_0\}+\cdots+\{0, x_i\}.$$

Clearly $\mathscr{X}^{(0)}\subset\mathscr{X}^{(1)}\subset\cdots\subset\mathscr{X}^{(u)}$. Since $|\mathscr{X}^{(u)}|=|\mathscr{X}+\mathscr{S}|\leqslant c|\mathscr{X}|$, there exists some j $(1\leqslant j<u)$ so that

$$|\mathscr{X}^{(j+1)}|\leqslant(1+u^{-1}\,2\log c)\,|\mathscr{X}^{(j)}|. \tag{5}$$

We partition the integers in $\mathscr{X}^{(j)}$ into maximal progressions $\bmod x_{i+1}$. In view of (5) the number of progressions is at most $u^{-1}(2\log c)\,|\mathscr{X}^{(j)}|\leqslant u^{-1}(2\log c)\,(cn)$. Thus the number of elements of $\mathscr{X}^{(0)}$ contained in all progressions each with $<n^*(c)$ elements is $\leqslant u^{-1}$ $(2\log c)\,(cn)\,n^*(c)\leqslant\varepsilon n/8$, in view of (4). Next we consider those progressions having each at most a proportion $(8c)^{-1}$ ε of its elements in $\mathscr{X}^{(0)}$. The total number of elements of $\mathscr{X}^{(0)}$ contained in progressions of this type is $\leqslant cn(8c)^{-1}$ $\varepsilon=\varepsilon n/8$. Thus each of the remaining progressions is of length $\geqslant n^*(c)$ and has at least a proportion $(8c)^{-1}$ ε of its elements contained in $\mathscr{X}^{(0)}$. Therefore the total number of elements of $\mathscr{X}^{(0)}$ falling into these remaining progressions is $\geqslant n(1-\varepsilon/8-\varepsilon/8)=$ $=(1-\varepsilon/4)\,n$. Since $\mathscr{X}^{(0)}=\mathscr{X}+\{x_0\}$, we have the assertion in the lemma.

Before stating Lemma 4 we introduce one further definition, that of an r-good set with respect to a given set.

DEFINITION 2. Let r be a natural number, and $\mathscr{Z}$ be a set of natural numbers. Then a set $\mathscr{X}$ of natural numbers is said to be an r-good set with respect to $\mathscr{Z}$ if it contains r integers $x_1, \ldots, x_r$ so that on putting[3])

$$\mathscr{X}_j=\mathscr{X}-\{x_1, \ldots, x_j\}, \quad j=1, \ldots, r,$$

we have

$$(x_j+\mathscr{X}_j)\cap\mathscr{Z}=\emptyset, \quad j=1, \ldots, r.$$

LEMMA 4. *Let* $c\geqslant 2$ *and* $u=u(c)$ *be defined by* (4). *Let* $v\geqslant 2(8)^{2u-3}\,c^{2u-2}$, $n\geqslant(8c)^{2u-1}$ *and* $n\geqslant v$. *Suppose* $\mathscr{X}$ *consists of* n *natural numbers and is not of type* c. *Then, for any given set* $\mathscr{Z}$ *of natural numbers such that* $|\mathscr{Z}|\leqslant cn-n$, *there exists a subset* $\mathscr{X}^*$ *of* $\mathscr{X}$ *which is* 1-*good with respect to* $\mathscr{Z}$ *and* $|\mathscr{X}^*|\geqslant v^{-1}|\mathscr{X}|$.

Proof. We take a subset $\mathscr{X}'$ of $\mathscr{X}$ with $|\mathscr{X}'|=v$. First suppose $|\mathscr{X}+\mathscr{X}'|\leqslant c|\mathscr{X}|$. Then,

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3) Here $\mathscr{X}_j$ is the complement of $\{x_1, \ldots, x_j\}$ in $\mathscr{X}$.

by Lemma 2, there exists a subset $\mathscr{S}$ of $\mathscr{X}$ of the form $\mathscr{S}=\{x_0\}+\cdots+\{0, x_u\}$. Since $\mathscr{X}$ is not of type c, $|\mathscr{X}+\mathscr{S}|>c|\mathscr{X}|$. Thus, on recalling that $|\mathscr{Z}|\leqslant c|\mathscr{X}|-|\mathscr{X}|$, we can assert that there is an element $x\in\mathscr{S}$ and a subset $\mathscr{X}''$ of $\mathscr{X}$ so that

$$|\mathscr{X}''|\geqslant(2^u)^{-1}\,|\mathscr{X}|$$

and

$$(\mathscr{X}''+x)\cap\mathscr{Z}=\emptyset.$$

We may take $\mathscr{X}^*=\mathscr{X}''+\{x\}$. Since $2^u\leqslant v$, $\mathscr{X}^*$ is a 1-good set as desired.

On the other hand, if $|\mathscr{X}+\mathscr{X}'|>c|\mathscr{X}|$, then again there exists $y\in\mathscr{X}'$ and a subset $\mathscr{X}'''$ of $\mathscr{X}$ so that

$$|\mathscr{X}'''|\geqslant v^{-1}|\mathscr{X}|$$

and

$$(\mathscr{X}'''+y)\cap\mathscr{Z}=\emptyset.$$

We may now take $\mathscr{X}^*=\mathscr{X}'''+\{y\}$. This completes the proof of the lemma.

§3. Proof of the Theorem

As the proof of the theorem is rather involved we shall first deduce Lemma A as a consequence of the lemmas in §2, and then deduce from it Lemma B. In the actual proof of the theorem, the only reference will be to Lemma B.

First we introduce the following definition.

DEFINITION 3. We define the numbers s_j, c_j, u_j, v_j, $j=1,\ldots,k-1$, as follows.

$$s_j=\begin{cases}[(4k)^{-1}\,\varepsilon s] & j=1\\ [(4k)^{-1}\,\varepsilon\lceil v_{j-1}^{-1}s_{j-1}\rceil] & j=2,\ldots,k-1\end{cases} \tag{6}$$[4]

$$c_j=\begin{cases}2(4k\varepsilon^{-1})+1 & j=1\\ 2(4k\varepsilon^{-1})^j\,v_{j-1}\ldots v_1+1, & j=2,\ldots,k-1\end{cases} \tag{7}$$

$$u_j=[20\varepsilon^{-1}n_j^*c_j\log c_j], \tag{8}$$

where $n_j^*=n^*(c_j)$ is the function defined by (3),

$$v_j=2(8)^{2u_j-3}\,c_j^{2u_j-2}. \tag{9}$$

[4]) $\lceil\alpha\rceil$ denotes the smallest integer $\geqslant\alpha$.

We suppose $\mathscr{A}$ and $\mathscr{B}$ are sets satisfying the hypothesis of the theorem and we suppose s so large that the number s_{k-1}, defined in the definition 3, satisfies

$$s_{k-1} \geqslant (8c_{k-1})^{2u_{k-1}-1}. \tag{10}$$

We note a number of consequences of definition 3 and (10) for future references. We have

$$s_j \geqslant (8c_j)^{2u_j-1}, \quad j=1,\dots,k-1 \tag{11}$$

$$2\{(4k\varepsilon^{-1})\,v_1+(4k\varepsilon^{-1})^2\,v_1v_2+\cdots+(4k\varepsilon^{-1})^{k-2}\,v_1\dots v_{k-2}\} \leqslant \frac{\varepsilon s}{2}, \tag{12}$$

and

$$s_jc_j - s_j \geqslant s \geqslant |\mathscr{A}|. \tag{13}$$

We obtain (11) by observing that (6)–(9) clearly imply $c_1<c_2<\cdots<c_{k-1}$, $u_1<\cdots<u_{k-1}$ and $s>s_1>\cdots>s_{k-1}$, and then using (10).

To obtain (12) we observe that (6) implies $s \geqslant (\frac{1}{2})\,(4k\varepsilon^{-1})^{k-1}\,v_1\dots v_{k-1}s_{k-1}$ which in view of (10) gives $4(4k\varepsilon^{-1})^{k-1}\,v_1\dots v_{k-1} \leqslant \varepsilon s/2$; it is now clear that (12) follows from this inequality.

Finally (13) is a consequence of (6) and (7) and the fact that $|\mathscr{A}| \leqslant (1-\varepsilon)\,s$.

Henceforth, an r-good set (see Definition 2) will always be taken to mean one with respect to $\mathscr{A}$, and an r-set will also be taken to mean one with respect to $\mathscr{A}$.

LEMMA A. *Let* $1 \leqslant j \leqslant k-1$ *and* $\mathscr{D}_j$ *be a subset of* $\mathscr{B}$ *consisting of* s_j *integers, where* s_j *is given by* (6). *Suppose* $\mathscr{D}_j$ *is not a k-set. Then either* $\mathscr{D}_j$ *is a set of type* c_j *and*

$$|2\mathscr{D}_j \cap \mathscr{A}| \geqslant \left(1-\frac{\varepsilon}{2}\right)|\mathscr{D}_j| \tag{14}$$

or $\mathscr{D}_j$ *contains a subset* $\mathscr{D}_j^*$ *which is* 1-*good and*

$$|\mathscr{D}_j^*| = \lceil v_j^{-1}\,|\mathscr{D}_j|\rceil = \lceil v_j^{-1}s_j\rceil, \tag{15}$$

where v_j *is given by* (9).

Proof. Suppose $\mathscr{D}_j$ is of type c_j. Since $s_j \geqslant 2^{u_j}$ in view of (11), we may apply Lemma 3 with $c=c_j$, $n=s_j$ and $\mathscr{X}=\mathscr{D}_j$ to assert that, with at most $4^{-1}\varepsilon|\mathscr{D}_j|$ exceptions, the integers in $\mathscr{D}_j$ are contained in the union of disjoint arithmetic progressions each of length $\geqslant n_j^*$ and each having at least a proportion $\varepsilon(8c_j)^{-1}$ of its integers in $\mathscr{D}_j$. We shall suppose (14) does not hold and deduce a contradiction.

Since $(1-\varepsilon/2) \leqslant (1-\varepsilon/4)\,(1-\varepsilon/4)$, the assumption that (14) does not hold implies that there exists an arithmetic progression $\mathscr{P}$ so that

$$|\mathscr{P}| \geqslant n_j^* = n^*(k, (8c_j)^{-1}\,\varepsilon, \varepsilon/4)$$
$$|\mathscr{P} \cap \mathscr{D}_j| \geqslant \varepsilon (8c_j)^{-1}\,|\mathscr{P}|$$

and

$$|(2(\mathscr{P} \cap \mathscr{D}_j)) \cap \mathscr{A}| \leqslant (1-\varepsilon/4)\,|\mathscr{D}_j \cap \mathscr{P}|.$$

Thus by Lemma 1, $\mathscr{P} \cap \mathscr{D}_j$ is a k-set, and this contradicts the hypothesis in the lemma concerning $\mathscr{D}_j$. Therefore in the case when $\mathscr{D}_j$ is of type c_j we have (14).

Now suppose $\mathscr{D}_j$ is *not* of type c_j. Since $|\mathscr{A}| \leqslant c_j s_j - s_j$ by (13), and $s_j \geqslant (8c_j)^{2u_j-1}$ by (11), we may apply Lemma 4 with $c = c_j$, $\mathscr{X} = \mathscr{D}_j$, $n = s_j$ and $\mathscr{Z} = \mathscr{A}$ to yield a 1-good subset $\mathscr{D}_j^*$ of $\mathscr{D}_j$ satisfying (15).

LEMMA B. *Let* $0 \leqslant j \leqslant k-2$ *and* $\mathscr{G}^{(j)}$ *be a subset of* $\mathscr{B}$ *so that*

$$|\mathscr{G}^{(j)}| = \begin{cases} s & \text{if } s=0 \\ \lceil s_j v_j^{-1} \rceil - 1 & \text{if } j>0 \end{cases} \tag{16}$$

where s_j, v_j *are given by* (6) *and* (9). *Suppose* $\mathscr{G}^{(j)}$ *is not a k-set. Then we may partition the integers in* $\mathscr{G}^{(j)}$, *with at most* s_{j+1} *exceptions, into subsets which are either of type* c_{j+1} *each consisting of* s_{j+1} *elements or* 1-*good sets each consisting of* $\lceil v_{j+1}^{-1} s_{j+1} \rceil$ *elements. Furthermore, if* $\mathscr{Y}$ *is a type* c_{j+1} *set in the partition then* $\mathscr{Y}$ *satisfies*

$$|2\mathscr{Y} \cap \mathscr{A}| \geqslant \left(1 - \frac{\varepsilon}{2}\right)|\mathscr{Y}|. \tag{17}$$

Proof. We take a subset $\mathscr{D}_{j+1}$ of $\mathscr{G}^{(j)}$ consisting of s_{j+1} elements and apply Lemma A to assert that either $\mathscr{D}_{j+1}$ is a set of type c_{j+1} so that (17) is satisfied with $\mathscr{Y} = \mathscr{D}_{j+1}$ or else $\mathscr{D}_{j+1}$ contains a subset $\mathscr{D}_{j+1}^*$ which is 1-good. Put

$$\mathscr{F} = \begin{cases} \mathscr{D}_{j+1} & \text{if } \mathscr{D}_{j+1} \text{ is of type } c_{j+1} \\ \mathscr{D}_{j+1}^* & \text{if } \mathscr{D}_{j+1} \text{ is not of type } c_{j+1}. \end{cases}$$

We apply the same argument to $\mathscr{G}^{(j)} - \mathscr{F}$ and so on. In this way we may partition $\mathscr{G}^{(j)}$, with at most s_{j+1} exceptions of its integers, into subsets having the properties asserted in the lemma.

We are now in a position to prove our theorem. For $j=0$, Lemma B yields a partition of $\mathscr{B}$ into subsets

$$\mathscr{C}_1^{(1)}, \ldots, \mathscr{C}_{l_1}^{(1)} \tag{18}$$

$$\mathscr{E}_1^{(1)}, \ldots, \mathscr{E}_{h_1}^{(1)} \tag{19}$$

and

$$\mathscr{M}^{(1)}, \tag{20}$$

where

$$|2\mathscr{C}_i^{(1)} \cap \mathscr{A}| \geqslant \left(1-\frac{\varepsilon}{2}\right) |\mathscr{C}_i^{(1)}|, \qquad i=1, \ldots, l_1 \tag{21}$$

$\mathscr{E}_i^{(1)}$ $(i=1, \ldots, h_1)$ are 1-good sets satisfying

$$|\mathscr{E}_i^{(1)}| = \lceil v_1^{-1} s_1 \rceil, \qquad i=1, \ldots, h_1, \tag{22}$$

and

$$|\mathscr{M}^{(1)}| \leqslant s_1 \leqslant (4k)^{-1} \varepsilon s. \tag{23}$$

We note that since $|\mathscr{A}| \leqslant (1-\varepsilon) s$, inequalities (21) and (23) imply that there are indeed 1-good sets $\mathscr{E}_i^{(1)}$ in the partition.

For each of the 1-good sets $\mathscr{E}_i^{(1)}$, we let $\tilde{\mathscr{E}}_i^{(1)}$ be the subset of $\mathscr{E}_i^{(1)}$ so that there exists an element b_i in $\mathscr{E}_i^{(1)}$ with

$$\tilde{\mathscr{E}}_i^{(1)} = \mathscr{E}_i^{(1)} - \{b\}$$

and

$$\tilde{\mathscr{E}}_i^{(1)} \cap \mathscr{A} = \emptyset.$$

We apply Lemma B with $j=1$ and $\mathscr{G}^{(1)} = \tilde{\mathscr{E}}_i^{(1)}$ and obtain for each $\tilde{\mathscr{E}}_i^{(1)}$ a partition of all its integers with at most $s_2 = [(4k)^{-1} \varepsilon |\mathscr{E}_i^{(1)}|]$ exceptions, into sets of type c_2 and 1-good sets. We further note that each such 1-good set may with the adjunction of an element from $\mathscr{B}$ become a 2-good set. Thus, at this second stage, we have partitioned $\mathscr{B}$ into subsets

$$\mathscr{C}_1^{(2)}, \ldots, \mathscr{C}_{l_2}^{(2)} \tag{24}$$

$$\mathscr{E}_1^{(2)}, \ldots, \mathscr{E}_{h_2}^{(2)} \tag{25}$$

and

$$\mathscr{M}^{(2)} \tag{26}$$

where

$$|2\mathscr{C}_i^{(2)} \cap \mathscr{A}| \geqslant \left(1-\frac{\varepsilon}{2}\right) |\mathscr{C}_i^{(2)}|, \qquad i=1, \ldots, l_2, \tag{27}$$

$\mathscr{E}_i^{(2)}$ are 1-good sets satisfying

$$|\mathscr{E}_i^{(2)}|=\lceil v_2^{-1}s_2\rceil, \qquad i=1,\ldots,h_2 \tag{28}$$

and

$$|\mathscr{M}^{(2)}|\leqslant 2(4k)^{-1}\,\varepsilon s+2(4k\varepsilon^{-1}v_1)\leqslant\frac{\varepsilon}{2}s \tag{29}$$

[For (29) see (12)]. Again the existence of sets $\mathscr{E}_i^{(2)}$ is a consequence of (27), (29) and $|\mathscr{A}|\leqslant(1-\varepsilon)\,s$.

It is clear that the argument employed above may be repeated. At the $(k-1)$-th stage we may assert the partitioning of $\mathscr{B}$ into subsets

$$\mathscr{C}_1^{(k-1)},\ldots,C_{l_{k-1}}^{(k-1)} \tag{30}$$

$$\mathscr{E}_1^{(k-1)},\ldots,\mathscr{E}_{h_{k-1}}^{(k-1)} \tag{31}$$

and

$$\mathscr{M}^{(k-1)},$$

where

$$|2\mathscr{C}_i^{(k-1)}\cap\mathscr{A}|\geqslant\left(1-\frac{\varepsilon}{2}\right)|\mathscr{C}_i^{(k-1)}|, \qquad i=1,2,\ldots, \tag{32}$$

$\mathscr{E}_i^{(k-1)}$ are 1-good sets satisfying

$$|\mathscr{E}_i^{(k-1)}|=\lceil v_{k-1}^{-1}s_{k-1}\rceil, \qquad i=1,2,\ldots \tag{33}$$

and

$$|\mathscr{M}^{(k-1)}|\leqslant(k-1)\,(4k)^{-1}\,\varepsilon s+2\{(4k\varepsilon^{-1})\,v_1+\cdots+(4k\varepsilon^{-1})^{k-2}\,v_1\ldots v_{k-2}\}\leqslant\frac{\varepsilon}{2}s \tag{34}$$

[for (34) see (12)]. Again the existence of sets $\mathscr{E}_i^{(k-1)}$ is a consequence of (32), (34) and $|\mathscr{A}|\leqslant(1-\varepsilon)\,s$. These 1-good sets may with the adjunction with $k-2$ elements from $\mathscr{B}$, become $(k-1)$-good sets. Since a $(k-1)$-good set is already a k-set we have completed the proof of the theorem.

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