A Geometrical Isoperimetric Inequality and Applications to Problems of Mathematical Physics

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0. Introduction

The classical isoperimetric inequality states that among all closed curves of given circumference the circle encloses the largest area. This inequality has been considerably generalized by A. D. Alexandrow. He derived [1] inequalities for the case where the curve lies on an abstract surface, and obtained lower bounds for the length of the curve in terms of the area of the domain and an expression involving the curvature of the surface. In this paper we consider a curve Γ_0 on an abstract surface whose endpoints lie on a curve Γ_1. With the help of Alexandrow's inequality we construct lower bounds for the length of Γ_0. These bounds depend on the area of the domain between Γ_0 and Γ_1, the curvature of the surface and the geodesic curvature of Γ_1. By use of the geometrical inequalities we derive a monotony property of the Green's function. The geometrical inequalities lead also to an estimate for the fundamental frequency of an inhomogeneous membrane with partially free boundary. The result extends the Rayleigh-Faber-Krahn inequality [12] and its generalizations obtained by Nehari [11] and the author [2, 3, 6]. At the end we indicate how to generalize the concept of Schwarz symmetrization [12] for functions which do not vanish at the whole boundary. This symmetrization combines in a certain way the ones defined in [2] and [3]. The principal results of this paper have already been announced in [4].

1. Geometrical Inequality

1.1. Let D be a simply connected domain in the complex z-plane $(z=x+iy)$ with a piecewise analytic boundary ∂D. ∂D is divided into two connected arcs $\Gamma_0=\bar{\Gamma}_0$ and Γ_1 such that $\Gamma_1\cap\Gamma_0=\emptyset$ and $\partial D=\Gamma_0\cup\Gamma_1$. The boundary ∂D is given by the parametric representation $z(s)$, where s is the arc-length. The function $z(s)$ is analytic except at the corners. Let $z_1=z(s_1)$, $z_2=z(s_2),\ldots, z_n=z(s_n)$ be the corners belonging to Γ_1. We suppose that the boundary is orientated such that $-i(dz/ds)\equiv -i\,\dot{z}(s)$ gives the outer normal of D.

Furthermore we assume that $\dot{z}(s_i+0)$ and $\dot{z}(s_i-0)$ are well-defined for all $i==1, 2,\ldots, n$. We denote by $\beta_i\in[-\pi,\pi]$ the angle $\arg\{\dot{z}(s_i+0)\}-\arg\{\dot{z}(s_i-0)\}$, and by $\kappa(s)$ the curvature of ∂D. $\kappa(s)$ is analytic on Γ_1 except at the corners where it has to be interpreted as a Dirac measure. We shall write $\int_{s_i-0}^{s_i+0}\kappa(s)\,ds=\beta_i$.

Let $u(x, y)$ be a real function of class $C^2(\bar{D})$. Let $\{z_j\}_{j=n+1}^{n+m}$ be a set of points in $\bar{D}-\Gamma_0$, none of which should coincide with a corner $z(s_i)$. We introduce the function

$$U(x, y)=-\frac{1}{\pi}\sum_{j=n+1}^{n+m}\omega_j \log|z-z_j|,$$

where $\omega_j \in \mathbf{R}$ for $j=n+1, \ldots, n+m$. For each Borel set $\gamma \subseteq \Gamma_1$ we define

$$\mu(\gamma)=\int_\gamma \left[\kappa(s)+\frac{1}{2}\frac{\partial u}{\partial n}\right] ds+\int_{\gamma-\{z_j\}_{j\geqslant n+1}} \frac{1}{2}\frac{\partial U}{\partial n} ds.$$

Let us write

$$\mu^+=\sup_{\gamma\subseteq\Gamma_1}\{\mu(\gamma)\}$$

and

$$\nu^+=\sum_{\substack{z_j\in D\\ j\geqslant n+1}}\max(\omega_j, 0)+\sum_{\substack{z_j\in \Gamma_1\\ j\geqslant n+1}}\max\left(\frac{\omega_j}{2}, 0\right).$$

Let $\varrho(x, y)=e^{u(x, y)+U(x, y)}$. *Throughout this paper we will assume that* $\varrho(x, y)$ *satisfies the differential inequality*

$$\Delta \log\varrho(x, y)+2C\varrho(x, y)\geqslant 0 \quad \text{in} \quad \bar{D}-\{z_j\}_{j=1}^{n+m}, \tag{1.1}$$

C being an arbitrary real number. We shall use the following notations $A(B)=\int_B \varrho(x, y)\,dx\,dy$ where $B\subseteq\bar{D}$ is an arbitrary subdomain; and $L(\Gamma)=\int_\Gamma \sqrt{\varrho}\,ds$, where Γ is an arc in $\bar{D}$. Consider a domain D' in the complex z'-plane ($z'=x'+iy'$) and a positive function $\varrho'(x', y')$ in D'. Γ_0' is a connected arc lying on $\partial D'$.

DEFINITION: The triple (D, Γ_0, ϱ) is *conformally equivalent* to $(D', \Gamma_0', \varrho')$, if there exists a conformal mapping $f: D'\to D$ such that $D=f(D')$, $\Gamma_0=f(\Gamma_0')$ and

$$\varrho'(x', y')=\left|\frac{df}{dz'}\right|^2 \varrho(x, y)\Big|_{\substack{x=x(x', y')\\ y=y(x', y')}}. \tag{1.2}$$

We shall write $S(\alpha, R)$ for the circular sector $\{0\leqslant\theta\leqslant\alpha, r\leqslant R\}$ (r, θ polar coordinates), $\hat{\Gamma}_0$ for its boundary arc $\{r=R\}$, and $\hat{\varrho}(r)=(1+Cr^2/4)^{-2}$. In this case we have $m=0$ and therefore $U(x, y)=0$. It is easily checked that $\hat{\varrho}$ satisfies (1.1) with the equality sign.

The purpose of this section is to establish the following result.

THEOREM 1.1. *Let D and ϱ be defined as before. Suppose that* $0\leqslant\mu^+ +\nu^+\equiv\pi-\alpha$

where $\alpha > 0$. *Then the inequality*

$$L^2(\Gamma_0) \geqslant (2\alpha - CA(D))\, A(D) \tag{1.3}$$

holds. Equality is achieved if and only if (D, Γ_0, ϱ) *is conformally equivalent to* $(S(\alpha, R), \hat{\Gamma}_0, \hat{\varrho})$.

Before we give the proof we indicate a geometrical interpretation of the result. Following [1, 9, 10] we introduce an abstract surface $\mathfrak{M}$ in the isothermic representation; i.e. in the domain D of the z-plane a Riemann metric is given by the line element $d\sigma^2 = \varrho(x, y)\, ds^2$. With respect to this metric $A(D)$ represents the area of D and $L(\Gamma_0)$ the length of Γ_0. The function $[\kappa(s) + (1/2)\, \partial/\partial n \log\varrho]/\sqrt{\varrho}$ is the geodesic curvature of ∂D and $K = -(\Delta \log\varrho/2\varrho)$ is the Gaussian curvature of $\mathfrak{M}$. The surface $(S(\alpha, R), \hat{\varrho})$ can be interpreted as a sector on a surface of constant Gaussian curvature C. If we identify the segments $\theta = \alpha$ and $\theta = 0$ then $(S(\alpha, R), \hat{\varrho})$ is isometric to a regular cone in a space of constant curvature C [1; p. 17, 450, 513]. (1.3) yields an isoperimetric inequality for abstract surfaces.

Proof of Theorem 1.1. The proof uses an idea developed by Nehari [11]. Let $f(w)$ $(w = \xi + i\eta)$ be the analytic function which maps the semicircle $Se = \{w;\, |w| < 1, \operatorname{Im}\{w\} > 0\}$ conformally onto the region D and transforms the segment $-1 < w < 1$ into the boundary arc Γ_1.

In this proof we shall often write for short $h(x, y) = h(z)$ for a real function in the z-plane, and $h(\xi, \eta) = h(w)$ for a real function in the w-plane. For a given function $\varrho(x, y) = \varrho(z)$ we define in $\bar{S}e$ the function $p(\xi, \eta) = p(w) = \varrho(f(w))\, |f'(w)|^2$. Let $w_i = f^{-1}(z(s_i))$, $i = 1, 2, \ldots, n$, and let $w_j = f^{-1}(z_j)$, $j = n+1, \ldots, n+m$.

Since Γ_1 is piecewise analytic, $|f'(w)|^2$ exists and is continuous on $f^{-1}(\Gamma_1) - \{w_i\}_{i=1}^n$. Furthermore, $|f'(w)|^2$ has at w_i for $i = 1, 2, \ldots, n$ the development

$$|f'(w)|^2 = |w - w_i|^{-2\beta_i/\pi}\, H_i(|w - w_i|^{2\beta_i/\pi})$$

where H_i is a regular function with $H_i(0) \neq 0$ [7, p. 364]. In $Se - \{w_j\}_{j=1}^n$ the function $\log|f'(w)|^2$ is harmonic. There, we have $\Delta_w \log p(w) = \Delta_w \log\varrho(f(w))$ where Δ_w denotes the Laplace operator in the w-plane. In view of (1.1) we get in $Se - \{w_j\}_{j \geqslant 1}$ the inequality

$$\Delta_w \log p(w) + 2Cp(w) \geqslant 0. \tag{1.4}$$

Let

$$P(w) = \begin{cases} p(w) & \text{in } Se \\ p(\bar{w}) & \text{in } \{w\colon |w| < 1, \operatorname{Im}\{w\} \leqslant 0\}. \end{cases}$$

Let $S = \{|w| < 1\}$ be the unit circle in the w-plane and let $S^- = S - \{-1 < w < 1\} - \{w_i\}_{i=1}^{m+n} - \{\bar{w}_i\}_{i=1}^{m+n}$. In order to simplify the notations we set $\gamma^- = \{-1 < w < 1\} -$

$-\{w_i\}_{i=1}^{m+n}$. By the previous observations $P(w)$ satisfies inequality (1.4) in S^-. Let $g(w, w^*)$ be the Green's function of the unit circle which vanishes on the boundary. With the help of this function we can write

$$\begin{aligned}\log P(w) = &-\int_{S^-} g(w, w^*)\, \Delta \log P(\xi^*, \eta^*)\, d\xi^* d\eta^* \\ &-\oint_{|w^*|=1} \log P(w^*) \frac{\partial g(w, w^*)}{\partial n_{w^*}} |dw^*| \\ &-2\int_{\gamma^-} g(w, w^*) \frac{\partial \log p(\xi^*, \eta^*)}{\partial \eta^*} |dw^*| \\ &+4\sum_{i=1}^{n} g(w, w_i)\, \beta_i + 2\sum_{\substack{w_i \neq \bar{w}_i \\ i \geqq n+1}} \{g(w, w_i) + g(w, \bar{w}_i)\}\, \omega_i \\ &+2\sum_{\substack{w_i = \bar{w}_i \\ i \geqq n+1}} g(w, w_i) \cdot \omega_i .\end{aligned} \tag{1.5}$$

n_w stands for the outer normal of S.

We observe that

$$h(w) = \oint_{|w^*|=1} \log P(w^*) \frac{\partial g(w, w^*)}{\partial n_{w^*}} |dw^*|$$

is harmonic in S. Because of (1.5), $\log P$ admits a representation of the form

$$\log P(w) = 2\int_S g(w, w^*)\, d\omega(e_{w^*}) - h(w). \tag{1.6}$$

$\omega(e)$ is the mass distribution associated with $P(w)$, and e is a Borel set. The integral (1.6) has to be interpreted in the sense of Lebesgue-Radon. Consider on S the Riemann metric $d\tilde{\sigma}^2 = P(w)\, |dw|^2$. By a result of Reshetnjak [1, 9] $\omega(e)$ corresponds to Alexandrow's curvature for the surface $\mathfrak{M} = (S, P(w))$.

$$A(\mathfrak{M}) = \int_{|w|<1} P(w)\, d\xi\, d\eta$$

denotes the area of $\mathfrak{M}$, and

$$L(\partial\mathfrak{M}) = \oint_{|w|=1} \sqrt{P(w)}\, |dw|$$

is the length of $\partial\mathfrak{M}$. We have

$$A(\mathfrak{M})=2A(D) \tag{1.7}$$

and

$$L(\partial\mathfrak{M})=2L(\Gamma_0). \tag{1.8}$$

As in [1, p. 513] we define $\omega_C(e)=\omega(e)-C\int_e P(w)\,d\xi\,d\eta$. By means of Riesz's decomposition theorem it follows that

$$\omega_C(S)=\omega_C^+(S)-\omega_C^-(S), \quad \text{where} \quad \omega_C^\pm(S)=\sup_{e\subseteq S}\{\pm\omega_C(e)\}.$$

Alexandrow [1; p. 514] proved, under our assumptions regarding the function $P(w)$, the

LEMMA 1.1. *If $\delta\equiv\omega_C^+(S)<2\pi$, then $L(\partial\mathfrak{M})$ and $A(\mathfrak{M})$ satisfy the inequality*

$$L^2(\partial\mathfrak{M})\geqslant(4\pi-2\delta-CA(\mathfrak{M}))\,A(\mathfrak{M}). \tag{1.9}$$

Equality holds if and only if $(S,P(w))$ is conformally equivalent to

$$(S,\hat{P}(w)) \quad \text{with} \quad \hat{P}(w)=\frac{(2\pi-\delta)^2}{4\pi^2}\,b\,|w|^{-\delta/\pi}\left(1+\frac{Cb}{4}\,|w|^{(2\pi-\delta)/\pi}\right)^{-2}.$$

b stands for an arbitrary positive number.

For the proof we refer to [1]. It should be noticed that the surface $(S,\hat{P}(w))$ is isometric to a regular cone in a space of constant curvature C.

The next step is to evaluate δ. Because of (1.4) and (1.5) we have $\omega_C^+(e)=0$ for each $e\subseteq S^-$. If $\beta\subseteq\gamma^-$ we obtain

$$\begin{aligned}\omega_C(\beta)&=-\int_\beta \frac{\partial}{\partial\eta}(\log p(\xi,\eta))\,|dw|=\\ &-\int_\beta \frac{\partial}{\partial\eta}(\log\varrho(f(w)))\,|dw|-2\int_\beta \frac{\partial}{\partial\eta}(\log|f'|)\,|dw|.\end{aligned} \tag{1.10}$$

Since $f(w)$ is a conformal mapping, it follows that

$$\int_\beta \frac{\partial}{\partial\eta}(\log\varrho(f(w)))\,|dw|=-\int_{f(\beta)} \frac{\partial}{\partial n}(\log\varrho(z))\,|dz|. \tag{1.11}$$

Furthermore, we have

$$\int_{\beta} \frac{\partial}{\partial \eta}(\log|f'|)\,|dw| = \mathrm{Re}\left\{\int_{\beta} \frac{\partial}{\partial \eta}(\log f')\,|dw|\right\} =$$

$$-\mathrm{Im}\left\{\int_{\beta} \frac{f''}{f'}\,|dw|\right\} = -\int_{\beta} d(\arg f') = -\int_{f(\beta)} \kappa\, ds. \tag{1.12}$$

On the set $\{w_i\}_{i\geqslant 1} \cup \{\bar{w}_i\}_{i\geqslant 1}$ we have

$$\omega_C(w_i) = \omega_C(\bar{w}_i) = \begin{cases} 2\beta_i & \text{if } \; i=1,2,\dots,n \\ \omega_i & \text{if } \; i=1+n,\dots,m+n. \end{cases} \tag{1.13}$$

The only contribution to δ comes from the set $\gamma^- \cup \{w_i\}_{i=1}^{n+m} \cup \{\bar{w}_i\}_{i=1}^{n+m}$. (1.10), (1.11), (1.12) and (1.13) lead to

$$\omega_C^+(S) = \sup_{e \subseteq f(\gamma^-)} \left\{\int_e \frac{\partial}{\partial n}(\log \varrho(z))\, ds + 2\int_e \kappa(s)\, ds\right\}$$

$$+2\sum_{i=1}^{n} \max(\beta_i, 0) + 2\sum_{\substack{z_i \in D \\ i \geqslant n+1}} \max(\omega_i, 0) + \sum_{\substack{z_i \in \Gamma_1 \\ i \geqslant n+1}} \max(\omega_i, 0). \tag{1.14}$$

Hence, it follows that $\omega_C^+(S) = 2\mu^+ + 2\nu^+$. Inserting this expression into (1.9) and observing (1.7) and (1.8) we obtain

$$L^2(\Gamma_0) \geqslant \left(\pi - \mu^+ - \nu^+ - \frac{CA(D)}{2}\right) 2A(D)$$

which yields (1.3). For the triple $(S(\alpha, R), \hat{\Gamma}_0, \hat{\varrho})$ with $0 < \alpha \leqslant \pi$ we have $u(x, y) = \log(1 + (Cr^2/4))^{-2}$, $U(x, y) = 0$, $\mu^+ = \pi - \alpha$, $\nu^+ = 0$. A straightforward calculation gives

$$A(S(\alpha, R)) = \frac{2\alpha R^2}{4 + CR^2} \quad \text{and} \quad L^2(\hat{\Gamma}_0) = \frac{\alpha^2 R^2}{\left(1 + \frac{CR^2}{4}\right)^2}.$$

In this case (1.3) holds with the equality sign. This completes the proof of Theorem 1.1.

1.2. The following results are consequences of Theorem 1.1.

COROLLARY 1.1. *Let D satisfy the assumptions of Sec.* 1.1. *Suppose that*

$\pi - \sup_{\beta \subseteq \Gamma_1} \{\int_\beta \kappa \, ds\} \equiv \alpha > 0$. *Then*

$$\left\{ \int_{\Gamma_0} ds \right\}^2 \geqslant 2\alpha \int_D dx \, dy. \tag{1.15}$$

Equality holds if and only if $D = S(\alpha, R)$ *and* $\Gamma_0 = \hat{\Gamma}_0$.

This result follows immediately from Theorem 1.1 by setting $\varrho \equiv 1$, $u = U = 0$, $C = 0$. The same theorem was obtained in [3] by more elementary methods, and applied to estimate the logarithmic capacity and the fundamental frequency of a membrane.

COROLLARY 1.2. *Let* D *satisfy the assumptions of Sec.* 1.1. *Suppose that* $\Delta \log \varrho \geqslant 0$ *in* D. *If*

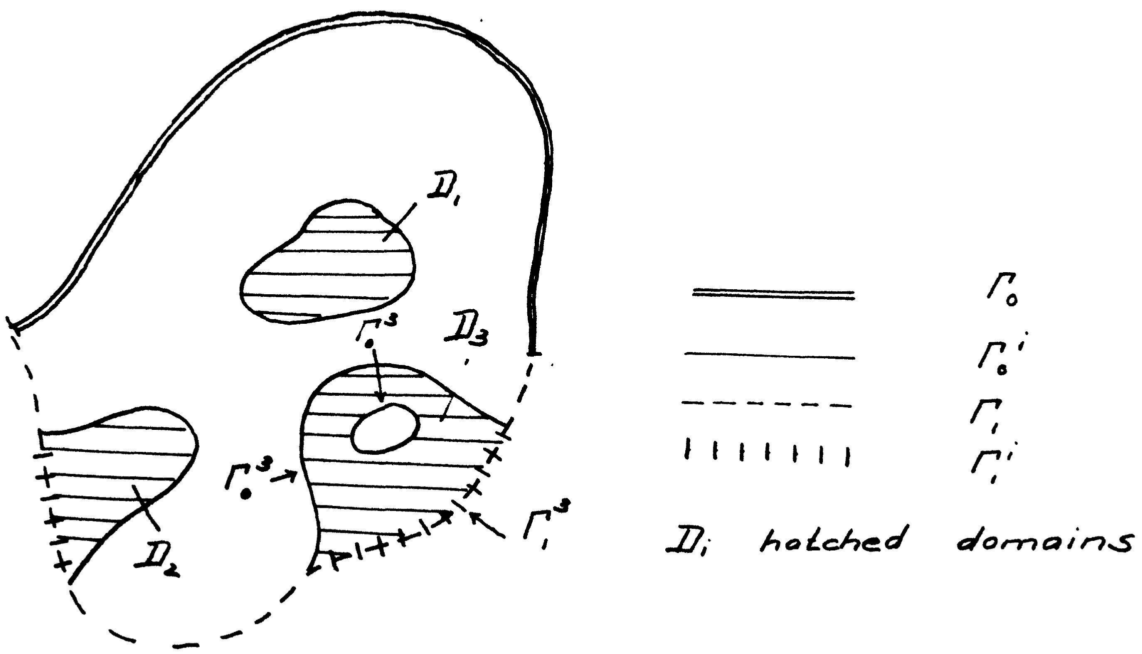
$$\pi - \sup_{\beta \subseteq \Gamma_1} \left\{ \int_\beta \left[\kappa(s) + \frac{\partial \log \varrho}{2 \partial n} \right] ds \right\} \equiv \alpha > 0,$$

then

$$\left\{ \int_{\Gamma_0} \sqrt{\varrho} \, ds \right\}^2 \geqslant 2\alpha \int_D \varrho \, dx \, dy. \tag{1.16}$$

Equality holds if and only if (D, Γ_0, ϱ) *is conformally equivalent to* $(S(\alpha, R), \hat{\Gamma}_0, 1)$.

1.3. Here we extend the results of Sec. 1.1. to a slightly more general situation. Let D and ∂D satisfy the same assumptions as in Sec. 1.1. Consider in D a collection $\{D_i\}_{i=1}^k$ of domains, and let the boundary ∂D_i, $i = 1, \ldots, k$, be divided into two not necessarily connected sets Γ_0^i and Γ_1^i where $\Gamma_1^i \subseteq \Gamma_1$. We assume $\varrho(x, y)$ and α to be defined as in 1.1. We write $L_i = L(\Gamma_0^i)$, $A_i = A(D_i)$, $L = \sum_{i=1}^k L_i$ and $A = \sum_{i=1}^k A_i$.



COROLLARY 1.3. *Let D, Γ_1, ϱ satisfy the same assumptions as in Theorem* 1.1. *Then the following inequality holds provided* $2\alpha > CA$

$$L^2 \geqslant (2\alpha - CA)\, A. \tag{1.17}$$

Equality holds if and only if $D_i = D$ and (D, Γ_0, ϱ) is conformally equivalent to $(S(\alpha, R), \hat{\Gamma}_0, \hat{\varrho})$.

Proof. By the same type of reasoning as in the proof of Theorem 1.1. we show that for $i = 1, 2, \ldots, k$

$$L_i^2 \geqslant (2\alpha - CA_i)\, A_i. \tag{1.18}$$

From this inequality (1.18) it follows that

$$\left(\sum_{i=1}^{2} L_i\right)^2 \geqslant \left(\sum_{i=1}^{2} \sqrt{(2\alpha - CA_i)\, A_i}\right)^2 = 2\alpha (A_1 + A_2) - C(A_1^2 + A_2^2)$$
$$+ 2[(2\alpha - CA_1)\, A_1]^{1/2}\, [(2\alpha - CA_2)\, A_2]^{1/2}. \tag{1.19}$$

An elementary argument shows that the right-hand side of (1.19) is, whatever the sign of C is, larger than $2\alpha(A_1 + A_2) - C(A_1 + A_2)^2$. By induction we conclude that

$$\left(\sum_{i=1}^{k} L_i\right)^2 \geqslant \left(2\alpha - C \sum_{i=1}^{k} A_i\right) \sum_{i=1}^{k} A_i$$

which completes the proof of Corollary 1.3.

2. Applications

2.1. Let D, Γ_0, Γ_1 and ϱ satisfy the conditions of Section 1.1. Consider in D the following Green's function

$$\left.\begin{aligned} &\Delta_z G(z, z^*) = -\delta_{z^*}(z) && \text{for } z \in D \\ &G(z, z^*) = 0 && \text{for } z \in \Gamma_0 \\ &\frac{\partial G(z, z^*)}{\partial n_z} + \sigma G(z, z^*) = 0 && \text{for } z \in \Gamma_1 \end{aligned}\right\} \tag{2.1}$$

where $\sigma(z) \geqslant 0$ is a continuous function on Γ_1. It should be mentioned that in view of the hypothesis concerning Γ_1, Γ_0 cannot be empty. With the help of the Green's function, the solution of the boundary value problem $\Delta\varphi = -F$ in D, $\varphi = 0$ on Γ_0, $\partial\varphi/\partial n + \sigma\varphi = 0$ on Γ_1 can be represented in the form

$$\varphi(z) = \int_D G(z, z^*)\, F(z^*)\, dx^*\, dy^*.$$

Let us write

$$D(t)=\{z\in D;\ G(z, z^*)>t\}$$

and

$$\Gamma(t)=\{z\in D;\ G(z, z^*)=t\}.$$

If $\Gamma(t)$ is not a closed curve, then its extremities lie on Γ_1. It follows from the maximum principle for subharmonic functions that $G(z, z^*)>0$ in D.
Let

$$A(t;\varrho)=\int_{D(t)}\varrho\, dx\, dy.$$

THEOREM 2.1. *Let D, Γ_1, Γ_0 and ϱ satisfy the same assumptions as for Theorem 1.1. Then*

$$m(t)=e^{-2\alpha t}\left(\frac{1}{A(t;\varrho)}-\frac{C}{2\alpha}\right) \tag{2.2}$$

is a non-decreasing function of t.
Proof [1]. By the divergence theorem we have

$$\oint_{\Gamma(t)}\left|\frac{\partial G(z, z^*)}{\partial n}\right| ds=1$$

if $\Gamma(t)$ is a closed contour line, or

$$\int_{\Gamma(t)}\left|\frac{\partial G(z, z^*)}{\partial n}\right| ds\leqslant 1$$

if $\Gamma(t)$ is not a closed level line. If *dn* is the length of the piece of normal to $\Gamma(t)$ between $\Gamma(t)$ and $\Gamma(t+dt)$, then

$$A(t;\varrho)-A(t+dt;\varrho)=\int_{\Gamma(t)}\varrho\, dn\, ds+o(dt).$$

By letting *dt* tend to zero, we get

$$-A'\equiv-\frac{dA}{dt}=\int_{\Gamma(t)}\frac{\varrho\, ds}{|\partial G(z, z^*)/\partial n|}\geqslant\int_{\Gamma(t)}\frac{\varrho\, ds}{|\partial G/\partial n|}\int_{\Gamma(t)}|\partial G/\partial n|\, ds. \tag{2.3}$$

[1]) This version of the proof makes use of a simplification suggested by J. Hersch.

By the Schwarz inequality

$$\int_{\Gamma(t)} \frac{\varrho\, ds}{|\partial G/\partial n|} \cdot \int_{\Gamma(t)} |\partial G/\partial n|\, ds \geqslant \left\{ \int_{\Gamma(t)} \sqrt{\varrho}\, ds \right\}^2 = L^2(\Gamma(t)). \tag{2.4}$$

By Corollary 1.3, $L^2(\Gamma(t)) \geqslant (2\alpha - CA(t; \varrho))\, A(t; \varrho)$.
Observing (2.4) and inserting the inequality for L^2 into (2.3), we get

$$-A' \geqslant 2\alpha A - CA^2. \tag{2.5}$$

Multiplying (2.5) by $e^{-2\alpha t}$, we obtain

$$0 \leqslant \frac{d}{dt}\left[e^{-2\alpha t}\left(\frac{1}{A} - \frac{C}{2\alpha}\right)\right], \tag{2.6}$$

which proves the assertion.

Consider the sector $S(\alpha, R)$ and $\hat{\varrho}(r) = (1 + (Cr^2/4))^{-2}$. Let $\hat{G}(z, 0)$ be the Green's function, $\Delta\hat{G} = -\delta_0$ in $S(\alpha, R)$, $\hat{G}(z, 0) = 0$ on $\hat{\Gamma}_0$ and $\partial\hat{G}/\partial n = 0$ on $\hat{\Gamma}_1$.

$\hat{G}(z, 0)$ can be calculated explicitly; i.e. $\hat{G}(z, 0) = (1/\alpha)\log(R/r)$. In this case the following relation holds

$$0 = \left[e^{-2\alpha t}\left(\frac{1}{A} - \frac{C}{2\alpha}\right)\right]' \quad \text{for all} \quad t \geqslant 0. \tag{2.7}$$

If $\sigma = 0$, then $G(z, z^*)$ is a conformal invariant in the following sense. If $f(w): D' \to D$ is a conformal mapping with $f(\Gamma_0') = \Gamma_0$, then the corresponding Green's function is $G_{D'}(w, w^*) = G_D(f(w), f(w^*))$. (2.7) holds if and only if $G(z, z^*)$ and (D, Γ_0, ϱ) are conformally equivalent to $\hat{G}(z, 0)$ and $(S(\alpha, R), \hat{\Gamma}_0, \hat{\varrho})$.

The next corollary is a consequence of Theorem 2.1 in the special case $\varrho = \text{const.}$ (and corresponds to Corollary 1.1).

COROLLARY 2.1. *Let*

$$\pi - \sup_{\beta \subseteq \Gamma_1} \left\{ \int_\beta \kappa\, ds \right\} = \alpha > 0,$$

$$A(t) = \int_{D(t)} dx\, dy \quad \textit{and} \quad A = \int_D dx\, dy.$$

Then

$$t \leqslant \frac{1}{2\alpha}\left\{\log \frac{A}{A(t)}\right\}. \tag{2.8}$$

Equality is achieved only for $\hat{G}(z, 0)$.
This result was already obtained in [3].

Suppose now that Γ_1 consists of two concave arcs with a corner at $z=0$. The angle β defined in Sec. 1.1 is equal to $\pi-\alpha$, where $\alpha>0$. Near the origin the Green's function corresponding to $\sigma=0$ has the development

$$G(z,0)=\frac{1}{\alpha}\log\frac{a}{r}+h(z)$$

where $a>0$ and $h(z)$ is a regular harmonic function in a sufficiently small neighborhood of the origin with $h(0)=0$.

COROLLARY 2.2. *If $\partial/\partial n \log\varrho\leqslant 0$ on Γ_1, and if $0<\alpha\leqslant\pi$ then*

$$\frac{2}{\alpha a^2\varrho(0)}\geqslant\frac{1}{A(0;\varrho)}-\frac{C}{2\alpha}. \tag{2.9}$$

Proof. For t sufficiently large, we have

$$t=\frac{1}{\alpha}\log\frac{a}{r}+o(1)$$

and

$$A(t;\varrho)=\alpha\varrho(0)\cdot\frac{r^2}{2}+o(r^2).$$

Hence

$$e^{-2\alpha t}=\frac{2A(t;\varrho)}{\alpha\varrho(0)}\cdot\frac{1}{a^2}+o(A)$$

which leads together with Theorem 2.1 to the inequality (2.9).

For applications concerning upper bounds for the solutions of Poisson problems we refer to [5, 4].

2.2. Estimates for Eigenvalues

Let D, Γ_0, Γ_1 and ϱ satisfy the assumptions of Theorem 1.1. Consider the membrane eigenvalue problem

$$\left.\begin{aligned} \Delta\varphi(x,y)+\lambda\varrho(x,y)\,\varphi(x,y)&=0 \quad\text{in}\quad D\\ \varphi&=0 \quad\text{on}\quad \Gamma_0\\ \frac{\partial\varphi}{\partial n}&=0 \quad\text{on}\quad \Gamma_1. \end{aligned}\right\} \tag{2.10}$$

By the classical theory there exist infinitely many positive eigenvalues $0<\lambda_1<\lambda_2\leqslant\ldots.$

The lowest eigenvalue $\lambda_1 = \lambda$ is defined as the minimum of the Rayleigh quotient

$$R[U] = \frac{\int\limits_D (U_x^2 + U_y^2)\, dx\, dy}{\int\limits_D \varrho U^2\, dx\, dy}, \tag{2.11}$$

if $U(x, y)$ ranges over the class of functions which vanish on Γ_0 and which are piecewise continuously differentiable in D. Besides (2.10) consider the problem

$$\left.\begin{aligned} \Delta\Phi + \Lambda\hat{\varrho}(r)\,\Phi &= 0 \quad \text{in} \quad S(\alpha, R) \\ \Phi &= 0 \quad \text{on} \quad \hat{\Gamma}_0 \\ \frac{\partial\Phi}{\partial n} &= 0 \quad \text{on} \quad \hat{\Gamma}_1 . \end{aligned}\right\} \tag{2.12}$$

For the definition of $S(\alpha, R)$, $\hat{\Gamma}_0$, $\hat{\Gamma}_1$ and $\hat{\varrho}$ see Sec. 1.1. We shall denote by Λ the lowest eigenvalue. The radius R of the domain is chosen such that

$$\int\limits_0^R \int\limits_0^\alpha \hat{\varrho}(r)\, r\, d\theta\, dr = \int\limits_D \varrho(x, y)\, dx\, dy \equiv M . \tag{2.13}$$

Hence, an elementary calculation yields

$$R = \sqrt{M}\left(\frac{\alpha}{2} - \frac{MC}{4}\right)^{-1/2} . \tag{2.14}$$

From (2.14) it follows that R is defined only if $2\alpha > MC$.

The next result generalizes the inequality of Rayleigh-Faber-Krahn [12].

THEOREM 2.2. *If $MC < 2\alpha$, then*

$$\lambda \geqslant \Lambda . \tag{2.15}$$

Equality holds if and only if (D, Γ_0, ϱ) is conformally equivalent to $(S(\alpha, R), \hat{\Gamma}_0, \hat{\varrho})$.

Proof. Let $\varphi(x, y)$ be the eigenfunction corresponding to the first eigenvalue λ. It does not change sign and can therefore be taken to be positive. Let $D(t) = \{(x, y) \in D;\ \varphi(x, y) > t\}$ and $\Gamma(t) = \{(x, y) \in \bar{D};\ \varphi(x, y) = t\}$. $\Gamma(t)$ consists of closed lines or of arcs whose endpoints lie on Γ_1. A classical transformation of the Dirichlet integral yields

$$\int\limits_{D(t)} \operatorname{grad}^2 \varphi\, dx\, dy - \int\limits_{D(t+dt)} \operatorname{grad}^2 \varphi\, dx\, dy = \int\limits_{\Gamma(t)} \left|\frac{dt}{dn}\right|^2 dn\, ds + o(dt). \tag{2.16}$$

s denotes the arc-length of the level line $\Gamma(t)$, and $dn = dn(s)$ is the length of the normal between $\Gamma(t)$ and $\Gamma(t+dt)$. By the Schwarz inequality we have

$$\int_{\Gamma(t)} \left|\frac{dt}{dn}\right|^2 dn\, ds = dt \int_{\Gamma(t)} \left|\frac{dt}{dn}\right| ds \geqslant dt \left(\int_{\Gamma(t)} \sqrt{\varrho}\, ds\right)^2 \frac{1}{\int\limits_{\Gamma(t)} \frac{\varrho\, ds}{|dt/dn|}}. \tag{2.17}$$

If we write

$$A(t) = \int_{D(t)} \varrho\, dx\, dy,$$

then

$$-dA(t) = A(t) - A(t+dt) = \int_{\Gamma(t)} \varrho\, dn\, ds + o(dt)$$

and

$$-A'(t) = \lim_{dt \to 0} \frac{A(t) - A(t+dt)}{dt} = \int_{\Gamma(t)} \varrho\, |\mathrm{grad}\, \varphi|^{-1}\, ds.$$

Because of our assumptions, Corollary 1.3 can be applied to estimate $\int_{\Gamma(t)} \sqrt{\varrho}\, ds$. We have therefore

$$\left\{\int_{\Gamma(t)} \sqrt{\varrho}\, ds\right\}^2 \geqslant (2\alpha - CA(t))\, A(t).$$

This inequality together with (2.17) and the expression for $-A'(t)$ yields

$$\int_D \mathrm{grad}^2 \varphi\, dx\, dy \geqslant \int_{t=0}^{\max \varphi(x,y)} \frac{(2\alpha - CA(t))\, A(t)}{-A'(t)}\, dt. \tag{2.18}$$

The Rayleigh quotient $R[\varphi]$ is estimated from below by

$$R[\varphi] \geqslant \frac{\int\limits_0^{\max \varphi} \frac{(2\alpha - CA(t)) \cdot A(t)}{-A'(t)}\, dt}{-\int\limits_0^{\max \varphi} t^2 A'(t)\, dt}. \tag{2.19}$$

We now introduce the new variable

$$r=\sqrt{A(t)}\left(\frac{\alpha}{2}-\frac{CA(t)}{4}\right)^{-1/2}.$$

The right-hand side of (2.19) is then transformed into

$$\frac{\int_0^R \left(\frac{dt}{dr}\right)^2 r\,dr}{\int_0^R t^2 \frac{r\,dr}{(1+Cr^2/4)^2}}. \tag{2.20}$$

In view of the minimum property of Λ, the expression (2.20) is greater than or equal to Λ. Since the eigenfunction Φ corresponding to Λ is radially symmetric, the minimum of (2.20) is achieved for $t(r)=\Phi(r)$. Observing that $R[\varphi]=\lambda$, we have therefore proved the assertion (2.15). The second statement follows immediately if we remember that inequality (1.17) has been used to evaluate $\int_{\Gamma(t)}\sqrt{\varrho}\,ds$ for all t. This theorem extends results obtained in [2, 3].

THEOREM 2.3. *Let the hypothesis of Theorem* 2.1 *be satisfied and suppose that* $C\geqslant 0$ *and* $C\int_D \varrho\,dx\,dy\leqslant\alpha$. *Then we have*

$$\lambda\geqslant 2C. \tag{2.21}$$

Proof. From Theorem 2.2 it follows that $\lambda\geqslant\Lambda$ where Λ is the first eigenvalue of (2.12) with $R=\sqrt{M}(\alpha/2-MC/4)^{-1/2}\leqslant 2/\sqrt{C}$. Because of the minimum property of Λ, Λ is a decreasing function of R. Hence $\lambda\geqslant\hat{\Lambda}$ where $\hat{\Lambda}$ is the first eigenvalue of (2.12) with $R=2/\sqrt{C}$. The corresponding eigenfunction is $\hat{\Phi}(r)=(4-Cr^2)/(4+Cr^2)$. Inserting this expression into (2.12), we get $\hat{\Lambda}=2C$ which establishes the theorem. – When Γ_1 is empty, we have $\alpha=\pi$. Inequality (2.21) then holds if $C\int_D\varrho\,dx\,dy\leqslant\pi$. In [6] we proved that for this particular case the inequality (2.21) remains valid even if $C\int_D\varrho\,dx\,dy\leqslant 2\pi$.

By Theorem 2.2 we have [cf. 3].

COROLLARY 2.3. *Let* $\varrho\equiv 1$ *and*

$$\pi-\sup_{\beta\subseteq\Gamma_1}\left\{\int_\beta \kappa\,ds\right\}=\alpha>0.$$

Then

$$\lambda \geqslant \frac{j_0^2 \cdot \alpha}{2A} \qquad \left(A = \int_D dx\, dy\right)$$

where $j_0 = 2.4048\ldots$ *is the first zero of the Bessel function* $J_0(r)$.

If $\kappa(s) < 0$ on Γ_1, Corollary 2.3 corresponds to Theorem III of Nehari [11].

2.3. A Generalization of the Schwarz Symmetrization

In this section we extend the concept of Schwarz symmetrization of a domain and a function [12].

Let D, Γ_0 and ϱ be defined as in Sec. 1.1. Consider in D a positive function $f(x, y)$ of class $C^\infty(\bar{D})$ vanishing at Γ_0.

We define

$$D^* = S(\alpha, R) \tag{2.22}$$

with

$$R = \sqrt{M}\left(\frac{\alpha}{2} - \frac{MC}{4}\right)^{-1/2} \quad \text{and} \quad M = \int_D \varrho\, dx\, dy.$$

Let

$$D(t) = \{(x, y) \in D;\ f(x, y) \geqslant t\} \quad \text{and} \quad A(t) = \int_{D(t)} \varrho\, dx\, dy.$$

$A(t)$ is a decreasing function of t. Its inverse $t(A)$ exists. On D^* we define the function

$$f^*(r) = t\left(\frac{\alpha r^2}{2\left(1 + \frac{Cr^2}{4}\right)}\right). \tag{2.23}$$

$f^*(r)$ has been constructed in such a way that

$$\int_0^\alpha \int_{\{r;\, f^*(r) \geqslant t\}} \hat{\varrho}\, r\, dr\, d\theta = A(t)$$

where $\hat{\varrho}(r)$ is defined in Sec. 1.1.

As in [12, see also 2, 3] we prove for all continuous functions $g(x)$

$$\int_D g[f(x, y)]\, \varrho(x, y)\, dx\, dy = \int_{D^*} g[f^*(r)]\, \varrho(r)\, dx\, dy. \tag{2.24}$$

Under the hypothesis of Theorem 2.2 we have

$$\int_D \mathrm{grad}^2 f \, dx \, dy \geqslant \int_{D^*} \mathrm{grad}^2 f^* \, dx \, dy. \tag{2.25}$$

The proof of this inequality uses the same type of arguments as in Theorem 2.2. With the help of this symmetrization estimates for the modulus of a domain can be derived. The methods and results resemble those of [2, 3]. Since the generalization is immediate by means of (2.24) and (2.25), it will be omitted.

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Received February 14, 1974