Classification Theorems for Quadratic Forms over Fields

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1. Introduction

In the study of quadratic form theory, the Classification Problem has always occupied a unique and central role. Namely, given a field F[3]), what are the basic invariants which classify (the isometry classes of) quadratic forms over F? The question in this generality has so far defied an answer, as no one has been able to exhibit a complete set of natural invariants which work for *all* fields. However, for *specific* classes of fields, the Classification Problem has been solved in various specific ways. Thus, one way to treat the Classification Problem is to ask the following slightly different question: which are the fields whose quadratic forms are classified by a *prescribed* set of invariants? Let us record from the literature some known answers to this alternative question, in order to lead up to and motivate the main result in this note.

CLASSIFICATION THEOREM 1′ (Triviality). *Quadratic forms over F are classified by* "dim" *iff F is quadratically closed, iff $IF=0$.*

Here, IF denotes the ideal of even dimensional forms in the Witt ring, $W(F)$.

CLASSIFICATION THEOREM 1 (Sylvester-Pfister Law). *Quadratic forms over F are classified by* "dim" *and the total signature (i.e. the totality of signatures with respect to all orderings of F) iff F is pythagorean, iff IF is torsion-free.*

The following is also easy to see:

CLASSIFICATION THEOREM 2′. *The following are equivalent:*

(1) *Quadratic forms over F are classified by* "dim" *and* "det".

(2) $I^2F=0$.

(3) *All F-quaternion algebras split.*

(4) *If an F-quaternion algebra splits over a quadratic extension of F, then it splits over F.*

(5) *All binary forms $\langle 1, a\rangle$ ($a\in\dot{F}$) are universal.*

EXAMPLES (for which the above statements hold): finite fields; algebraic extensions of $\mathbf{C}(x)$; the power series field $\mathbf{C}((x))$.

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[3]) All fields have characteristic different from 2 in this paper.

CLASSIFICATION THEOREM 2. *The following are equivalent:*

(1) *Quadratic forms over F are classified by* "dim", "det" *and the total signature.*

(2) I^2F *is torsion-free.*

(3) *If an F-quaternion algebra splits in every real closure of F, then it splits over F.*

(4) *If an F-quaternion algebra splits over some* $F(\sqrt{w})$ *where w is totally positive, then it splits over F.*

(5) *All binary forms* $\langle 1, a\rangle$ *represent all totally positive elements of F.*

Proofs of these equivalences are covered by [2, Cor. 2.9] and [6, Theorem E]. The latter contains also further statements equivalent to each of the above.

EXAMPLES. Algebraic extensions of $\mathbf{R}(x)$; any formally real field with square classes $\{\pm 1, \pm 2\}$.

CLASSIFICATION THEOREM 3′ [5, Theorem 3.11]. *The following are equivalent:*

(1) *Quadratic forms over F are classified by* "dim", "det" *and the Hasse invariant.*

(2) $I^3F=0$.

(3) *All F-Cayley algebras split.*

(4) *If an F-Cayley algebra splits over a quadratic extension of F, then it splits over F.*

(5) *All quaternionic norm forms* $\langle 1, a, b, ab\rangle$ $(a, b\in \dot{F})$ *are universal.*

EXAMPLES. Algebraic extensions of $\mathbf{C}(x, y)$; $\mathfrak{p}$-adic fields; non-formally real global fields; $\mathbf{C}((t_1))((t_2))$.

Note that Theorems 1′, 2′ are implicitly addressed to non-formally real fields, while Theorems 1, 2 are respectively their generalizations to *arbitrary* fields. This strikes a resonant note to the papers [3, 4, 6], where it is demonstrated that many things said about non-real fields can be appropriately generalized to arbitrary fields. In this perspective, one is naturally led to conjecture that Theorem 3′ can be superceded by the following much broader statement:

CLASSIFICATION THEOREM 3. *The following are equivalent:*

(1) *Quadratic forms over F are classified by* "dim", "det", *Hasse invariant and the* total signature.

(2) I^3F *is torsion-free.*

(3) *If an F-Cayley algebra splits in every real closure of F, then it splits over F.*

(4) *If an F-Cayley algebra splits over some* $F(\sqrt{w})$ *where w is totally positive, then it splits over F.*

(5) *All quaternionic norm forms* $\langle 1, a, b, ab\rangle$ $(a, b\in\dot{F})$ *represent all totally positive elements of F.*

EXAMPLES. Algebraic extensions of $\mathbf{R}(x, y)$; global fields.

The purpose of this note is to render a proof of Theorem 3. Though we restrict ourselves to fields throughout the paper, it seems reasonable to expect that the same theorem works essentially over semi-local rings. This observation is supported by the work of Mandelberg [9], where the non-dyadic semi-local analog of Theorem 3′ has already been obtained (under mild restrictions). On the other hand, Sah [12, Theorem 3] has established the analog of Theorem 3′ for fields of characteristic 2.

A word about notations. For a field F, write $\dot{F}=F-\{0\}$, and $\sigma(F)$=the set of totally positive elements (=non-zero sums of squares, by Artin-Schreier). For $a_i \in \dot{F}$, $\langle\langle a_1, \ldots, a_n\rangle\rangle$ denotes the "n-fold Pfister form" $\otimes_{i=1}^{n}\langle 1, a_i\rangle$. The ideal power I^nF is additively generated by all n-fold Pfister forms in $W(F)$. If a Pfister form φ lies in $W_t(F)$, we say that φ is a *torsion* Pfister form. Standard facts about quadratic forms can be found in [8].

2. Auxiliary Results

For convenience of the reader, we shall recall here a few results from our earlier work [2, 3, 5], to be used in the sequel.

PROPOSITION 1 [5, Section 3]. *Let $K=F(\sqrt{a})$ be a quadratic extension of F. Let $s: K \to F$ be the F-linear functional defined by $s(1)=0$, $s(\sqrt{a})=1$. Let $s_*: W(K) \to W(F)$ be the transfer map induced by s, and $r^*: W(F) \to W(K)$ be the functorial map. Then,*

(1) *We have a zero sequence*

$$0 \to \langle\langle -a\rangle\rangle \cdot I^{n-1}F \to I^nF \xrightarrow{r^*} I^nK \xrightarrow{s_*} I^nF \quad \textit{for all} \quad n \geq 0.$$

(By definition, $I^{-1}F = I^0F = W(F)$.)

(2) *The above sequence is exact for $n=0, 1, 2$.*

(3) *The above sequence is exact for $n=3$, except possibly at the term I^3K.*

(4) *If $\gamma \in I^3K$ is 8-dimensional and $s_*(\gamma)=0$, then there exists $q \in I^3F$ such that $r^*(q)=\gamma$.*

PROPOSITION 2 [3, Cor. 2.3]. *Suppose σ is a $2n$-dimensional form such that $2\sigma=0 \in W(F)$. Then $\sigma \cong \perp_{i=1}^{n}\langle a_i\rangle\langle\langle -w_i\rangle\rangle$ for suitable $a_i \in \dot{F}$, and w_i which are sums of two squares.*

COROLLARY 1. *If σ is a Pfister form, then $2\sigma=0$ iff $\sigma \cong \langle\langle -w, \ldots\rangle\rangle$ where w is a sum of two squares.*

Proof. "*If*" is clear. Assuming $2\sigma=0$, we have $\sigma \cong \langle a\rangle\langle 1, -w\rangle \perp \ldots$, where $a \in \dot{F}$

and $w=b^2+c^2\neq 0$. Using standard facts about Pfister forms, $\sigma\cong\langle a\rangle\cdot\sigma\cong\langle 1, -w,\ldots\rangle$ $\cong\langle\langle -w,\ldots\rangle\rangle$. Q.E.D.[4]).

PROPOSITION 3 [2, Theorem 2.8]. *Every element in* $I^2F\cap W_t(F)$ *is a sum of forms* $\langle\langle a, -w\rangle\rangle$, *where* $a\in\dot{F}$ and $w\in\sigma(F)$.

We shall now prove some lemmas.

LEMMA 1. *Let* $n\geqslant 1$. *Suppose there are no anisotropic n-fold Pfister form* φ *satisfying* $2\varphi=0\in W(F)$. *Then there are no anisotropic torsion m-fold Pfister form for any* $m\geqslant n$.

Proof. Suppose γ is an m-fold Pfister form ($m\geqslant n$) such that $2^{t+1}\gamma=0$ but $2^t\gamma\neq 0$. Consider the $(m+t)$-fold Pfister form $2^t\gamma$ which is killed by 2. According to Corollary 1, $2^t\gamma\cong\langle\langle -w, x_2,\ldots, x_{m+t}\rangle\rangle$ ($w=b^2+c^2\neq 0, x_i\in\dot{F}$). But by hypothesis $\langle\langle -w, x_2,\ldots, x_n\rangle\rangle$ $=0$ since it is killed by 2. Thus $2^t\gamma=0$, a contradiction.

COROLLARY 2 (Pfister: see [8, p. 300]). *Let* $r\geqslant 1$. *If any r-fold Pfister form represents any non-zero sum of* 2 *squares, then any r-fold Pfister form represents all of* $\sigma(F)$.

Proof. Apply Lemma 1 with $n=m=r+1$, using again Corollary 1.

COROLLARY 3. *In Theorem* 3, *we have* (3)$\Leftrightarrow$(4)$\Leftrightarrow$(5)$\Leftrightarrow$:

(A) *There are no anisotropic torsion* 3-*fold Pfister forms* (over F).

Proof. A Cayley algebra splits iff its 3-fold Pfister norm form is hyperbolic [7, p. 371]. Thus, (3)$\Leftrightarrow$(A) follows from Pfister's Local-Global Principle [11]. By Corollary 1 and Lemma 1, we have (A)$\Leftrightarrow$(5). Considering the form $\langle\langle a, b, -w\rangle\rangle$ ($w\in\sigma(F)$), we get (4)$\Leftrightarrow$(5).

LEMMA 2. *Let* $K=F(\sqrt{a})$ *be a quadratic extension of F. Then* I^2K *coincides with the* $\dot{F}$-*module J in* $W(K)$ *generated by* $\langle\langle e, z\rangle\rangle$ *where* $e\in\dot{F}$, $z\in\dot{K}$.

Proof. Let $x, y\in\dot{K}$ and $b\in\dot{F}$. From the equation $\langle\langle xb, y\rangle\rangle=\langle b\rangle\,\langle\langle x, y\rangle\rangle+\langle\langle -b, y\rangle\rangle$ $\in W(K)$, it follows that $\langle\langle x, y\rangle\rangle\in J\Rightarrow\langle\langle xb, y\rangle\rangle\in J$. Thus, we need only show that $\varphi=\langle\langle c+\sqrt{a}, d-\sqrt{a}\rangle\rangle\in J$, where $c, d\in F$. If $c=-d$, φ is hyperbolic. If $c\neq -d$, $\varphi\cong\langle\langle c+d, (c+\sqrt{a})(d-\sqrt{a})\rangle\rangle\in J$. Q.E.D.[5]).

LEMMA 3. *Let* $K=F(\sqrt{w})$ *be a quadratic extension of F, where* $w\in\sigma(F)$. *Assume that Property* (A) (*see Corollary* 3) *holds for F. Then it also holds for K.*

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[4]) Here is a proof which avoids Prop. 2. Write $\sigma=\langle 1\rangle\perp\sigma'$. Since $\sigma\cong\langle -1\rangle\,\sigma$, σ represents -1, and hence σ' represents some $-w=-(b^2+c^2)\neq 0$. We then have $\sigma\cong\langle\langle -w,\ldots\rangle\rangle$.

[5]) As observed by Mandelberg, this Lemma together with Frobenius reciprocity yields a quick inductive proof of the inclusion $s_*(I^nK)\subset I^nF$ asserted in Prop. 1 (1).

Proof. Let $\gamma = \langle\langle x, y, -z\rangle\rangle$, where $x, y \in \dot{K}$ and z is a sum of 2 squares in $\dot{K}$. It suffices to show that γ must be hyperbolic. Let r^*, s_* be as in Proposition 1 (with $a = w$).

Step 1. We claim that $s_*(\gamma) = 0 \in W(F)$. Since $s_*: W(K) \to W(F)$ is a $\dot{F}$-module homomorphism (by Frobenius reciprocity), we may assume that $x \in \dot{F}$, by Lemma 2. Consequently, $s_*(\gamma) = s_*\langle\langle x, y, -z\rangle\rangle = \langle\langle x\rangle\rangle \cdot s_*(\langle\langle y, -z\rangle\rangle)$. The latter lies in $\langle\langle x\rangle\rangle \cdot \cdot (I^2F \cap W_t(F))$, which is zero in view of Proposition 3 and the hypothesis on F.

Step 2. Since $s_*(\gamma) = 0$, there exists an anisotropic form $q \in I^3F$ such that $r^*(q) = \gamma$ (Proposition 1 (4)). *In the following, assume that* γ *is anisotropic.* We may then write $q \cong f \perp \langle\langle -w\rangle\rangle \cdot g$, where f, g are forms over F, $\dim f = 8$ (see [8, p. 200]). If $\dim g = 1$, then $\dim q = 10$, and $q \in I^3F \Rightarrow q$ is isotropic [11, Case 5 on p. 123], a contradiction. If $\dim g \geqslant 2$, $\langle\langle -w\rangle\rangle \cdot g$ contains a subform $\langle b\rangle \langle\langle -w, c\rangle\rangle$, which is universal by hypothesis and hence q is isotropic, again a contradiction. Consequently, g is the zero form, and $\dim q = 8$. This means that $q \cong \langle b_1\rangle \langle\langle b_2, b_3, b_4\rangle\rangle$ where $b_i \in \dot{F}$ [11, Case 4 on p. 123].

Step 3. Our hypothesis for F implies that $\langle\langle -w\rangle\rangle \cdot I^2F = 0$. By Proposition 1 (3), it follows that $r^*: I^3F \to I^3K$ is injective. Since $r^*(q) = \gamma$ is torsion, q must be torsion too. But then $\langle\langle b_2, b_3, b_4\rangle\rangle$ is an anisotropic torsion Pfister form – a final contradiction. Q.E.D.

3. Proof of Theorem 3

We are now ready to complete the proof of Theorem 3. In view of Corollary 3, we need only show (1)$\Rightarrow$(A)$\Rightarrow$(2)$\Rightarrow$(1).

(1)$\Rightarrow$(A). Consider $\langle\langle a, b, -w\rangle\rangle$ ($w \in \sigma(F)$) and $\langle\langle 1, 1, -1\rangle\rangle$. These both have dimension 8, determinant 1, trivial Hasse invariant, and zero total signature. Hence (1) implies that $\langle\langle a, b, -w\rangle\rangle$ is hyperbolic.

(A)$\Rightarrow$(2). Suppose F satisfies (A), but there exists a nonzero anisotropic form $\sigma \in I^3F$ with $2\sigma = 0$. We may suppose F to have been chosen such that $\dim \sigma = 2n$ is as small as possible. By Proposition 2, $\sigma \cong \perp_{i=1}^{n} \langle a_i\rangle \langle\langle -w_i\rangle\rangle$, $a_i \in \dot{F}$, $w_i \in \sigma(F)$. Let $K = F(\sqrt{w_1})$. This field also satisfies (A), by Lemma 3. Since the anisotropic part of σ over K is $< 2n$, the form σ must become hyperbolic over K, by the choice of n. According to Proposition 1 (3) and the Property (A), we get $\sigma \in \langle\langle -w_1\rangle\rangle \cdot I^2F = 0$, a contradiction.

Now that we know (A)$\Leftrightarrow$(2), we may restate Lemma 3 as:

COROLLARY 4. *Let* $K = F(\sqrt{w})$, $w \in \sigma(F)$. *Then,* I^3F *is torsion-free* $\Rightarrow I^3K$ *is torsion-free.*

Remark. The same implication, of course, holds for I^2. The proof is immediate from the I^2-exact sequence in Proposition 1 (2).

It still remains to ascertain one last implication: (2)$\Rightarrow$(1), for the conclusion of the proof of Theorem 3. To do this, we first make some observations about invariants. For a regular quadratic form q over F, the *Clifford invariant*, $\Gamma(q)$, is given by the class of the Clifford algebra of q in the Brauer-Wall group $BW(F)$ (see [8, p. 115]). It is well-known that $\Gamma(q)$ contains exactly the same information as the aggregate of "dim. mod. 2", "det" and the Hasse invariant [8, p. 120–123]. Thus, we have nothing to lose in working with Γ. On the other hand, the single invariant Γ is much nicer to work with, because it is *well-defined* on $W(F)$, and is a *homomorphism* into $BW(F)$. Working with Γ in general avoids many unpleasant calculations. We shall now prove

PROPOSITION 4. *Suppose I^3F is torsion-free, and q is a form such that $q \in W_t(F)$. Then $\Gamma(q)=1 \Rightarrow q$ is hyperbolic.*

Proof. Assume that $\Gamma(q)=1$, but q is non-hyperbolic. Since $\ker(\Gamma: W(F)\to BW(F)) \subset I^2F$, we have $q \in I^2F \cap W_t(F)$, so we can write $q=\sum_{i=1}^{n}\langle\langle a_i, -w_i\rangle\rangle$, $a_i \in \dot{F}$, $w_i \in \sigma(F)$. We may suppose F, q to have been chosen such that n is as small as possible. Let $K=F(\sqrt{w_1})$. Since $q=\sum_{i=2}^{n}\langle\langle a_i, -w_i\rangle\rangle$ in $W(K)$, and I^3K is still torsion-free (Corollary 4!), q must become hyperbolic over K, by the choice of n. According to Proposition 1 (2), $q=\langle\langle -w_1\rangle\rangle\cdot\langle b_1, \dots, b_{2r}\rangle \in W(F)$, for suitable $b_i \in \dot{F}$. Thus,

$$\begin{aligned} q &= \langle b_1\rangle \langle\langle -w_1, b_1b_{r+1}\rangle\rangle + \cdots + \langle b_r\rangle \langle\langle -w_1, b_rb_{2r}\rangle\rangle \\ &\equiv \langle\langle -w_1, b_1b_{r+1}\rangle\rangle + \cdots + \langle\langle -w_1, b_rb_{2r}\rangle\rangle \\ &\equiv \langle\langle -w_1, (-1)^{r+1} b_1 \dots b_{2r}\rangle\rangle \quad (\mathrm{mod}\, I^3F). \end{aligned}$$

Since $\Gamma(q)=1$ and $\Gamma(I^3F)=1$ (see [8, p. 117]), we see that $\Gamma(\langle\langle -w_1, (-1)^{r+1} b_1 \dots b_{2r}\rangle\rangle)=1$. This means that $\langle\langle -w_1, (-1)^{r+1} b_1 \dots b_{2r}\rangle\rangle$ is hyperbolic (see [8, p. 116]). Thus, $q \in I^3F$. But then $q \in I^3F \cap W_t(F)=0$, a contradiction. Q.E.D.

Using Pfister's Local-Global Principle, we obtain:

COROLLARY 5. *Suppose I^3F is torsion-free. Let $s_\alpha: W(F)\to W(F_\alpha)\cong \mathbf{Z}$ be the "signature maps", where $\{F_\alpha\}$ are a complete family of real closures of F. Then*

$$(\Gamma, \textstyle\prod_\alpha s_\alpha): W(F)\to BW(F)\oplus \textstyle\prod_\alpha W(F_\alpha)$$

is a monomorphism. In particular, quadratic forms over F are classified by "dim", *the Clifford invariant and the total signature.*

Since "dim", "det" and the Hasse invariant together determine Γ as observed before, Corollary 5 provides the implication (2)$\Rightarrow$(1) in Theorem 3. *The proof of Theorem* 3 *is now complete.*

We shall now make some remarks about Theorem 3.

Remark 1. The statements (1) to (5) in Theorem 3 are also equivalent to each of the following: (6) *Quadratic forms over F are classified by* "dim" *and Milnor's total*

Stiefel-Whitney class w *in* [10]. (7) *Quadratic forms over* F *are classified by* "dim" *and Delzant's total Stiefel-Whitney class* $\tilde{w}$ in [1]. (Note: w takes its value in the algebraic k-groups of Milnor, while $\tilde{w}$ takes its value in the Galois cohomology of F). In fact, (6)$\Leftrightarrow$(2) has been shown in [2, Theorem 2.15]. (7)$\Rightarrow$(6) is trivial since $\tilde{w}$ is a "specialization" of w. To see that (1)$\Rightarrow$(7), suppose φ and σ have the same "dim" and the same $\tilde{w}=(\tilde{w}_i)$. For $i=1, 2$, this says that φ and σ have the same "det" and the same Hasse invariant. But they also have the same total signature, since $\varphi-\sigma\in W_t(F)$ by [13, Cor. 6.2]. Therefore, $\varphi\cong\sigma$ by (1).

Remark 2. Suppose F, F' are fields for which there exists a ring isomorphism $g: W(F)\cong W(F')$. Then, if the statements in Theorem 3 apply to F, they will likewise apply to F'. This is because IF is the unique maximal ideal in $W(F)$ containing 2, which implies that $g(I^3F)=I^3F'$.

Remark 3. The "hereditary" property in Corollary 4 is peculiar to quadratic extensions of the type $F(\sqrt{w})\supset F\ (w\in\sigma(F))$. In fact, let F_1 be a pythagorean field which has a non-pythagorean algebraic extension $E_1=F_1(\alpha)$. Let $F_2=F_1((x))$, $F_3=F_2((y))$, and $E_i=F_i(\alpha)$. Then, for $i=1, 2, 3$, F_i satisfies Theorem i, but E_i does not. (If F_1 is *formally real* pythagorean, then so are F_2 and F_3 and they even satisfy Theorem 1.)

Remark 4. A number of other properties also share the "hereditary" feature of Corollary 4, under quadratic extensions of the type $K=F(\sqrt{w})\ (w\in\sigma(F))$. For example, it can be shown that, *if every totally positive element of* F *is a sum of* 2^n *squares, then the same holds for* K. If a field satisfies the statements of Theorem 3, then, in particular, $w\in\sigma(F)\Rightarrow\langle\langle 1, 1, -w\rangle\rangle$ is hyperbolic $\Rightarrow w$ is a sum of four squares. However, this latter property (though "hereditary" in the above sense) does not imply the statements in Theorem 3. For example, every totally positive element in $L=\mathbf{Q}((t))$ is a sum of four squares (see [8, p. 315]), but I^3L is *not* torsion-free (e.g. $\langle\langle 1, -3, t\rangle\rangle$ is an anisotropic torsion Pfister form over L).

Appendix: Similarity Factors and a Theorem of Dieudonné

For a quadratic form q of dimension n over F, let $d_\pm(q)$ denote $(-1)^{n(n-1)/2}\cdot\det(q)$ (the "signed determinant"), and let $s(q)$ denote the Hasse invariant of q. Also, let $D(q)$ denote the nonzero values of F represented by q, and let $G(q)$ denote the group of similarity factors of q (i.e. $G(q)=\{a\in\dot{F}: a\cdot q\cong q\}$).

LEMMA 4. *If* $\dim q=n=2r$, *and* $a\in\dot{F}$, *then* $s(a\cdot q)=s(q)$ *iff* $a\in D\langle 1, -d_\pm(q)\rangle$. *In particular,* $G(q)\subset D\langle 1, -d_\pm(q)\rangle$.

Proof. From [8, p. 140, Ex. 8], $s(a\cdot q)$ and $s(q)$ differ by a quaternion algebra $(a, (-1)^{n(n-1)/2}\cdot d^{n-1}/F)$, where $d=\det(q)$. Since $n=2r$, this quaternion algebra is $(a, d_\pm(q)/F)$, which splits iff $a\in D\langle 1, -d_\pm(q)\rangle$. Q.E.D.

In general, $b\in D\langle 1, -d_\pm(q)\rangle$ need not imply $b\in G(q)$. For b to be in $G(q)$, there exists at least one other obvious necessary condition, namely, b must be positive in every ordering of F at which q is non-hyperbolic. Thus, if we write $G'(q)$ for the group

$$\{b \in D\langle 1, -d_{\pm}(q)\rangle : b > 0 \text{ in every ordering of } F \text{ at which } q \text{ is non-hyperbolic}\},$$

we have an inclusion $G(q) \subset G'(q)$, for all even dimensional forms q.

THEOREM. *The conditions* (1) *through* (5) *in Classification Theorem* 3 *are also equivalent to each of the following:*

(8) $G(q) = G'(q)$ *for all even dimensional forms* q *over* F.

(9) $G(q) = G'(q)$ *for all torsion* 2*-fold Pfister forms* q *over* F.

(10) *Torsion* 2*-fold Pfister forms over* F *are universal.*

Proof. (1)⇒(8). If $b \in G'(q)$, then, q and $b \cdot q$ have the same "dim", "det", Hasse invariant (by Lemma 4), and the same total signature (by inspection). Thus, $b \cdot q \cong q$ by (1).

(8)⇒(9) is obvious.

(9)⇒(10). If q is a torsion 2-fold Pfister form, the group $G'(q)$ clearly coindides with $\dot{F}$. Thus, (9) implies that $G(q) = \dot{F}$, i.e., q is universal.

To complete the proof, we shall show that (10) implies the Condition (A) in Corollary 3. By Lemma 1, it is sufficient to show that any 3-fold Pfister form φ satisfying $2\varphi = 0 \in W(F)$ is isotropic. By Corollary 1, $\varphi \cong \langle\langle -w, x, y\rangle\rangle$, where w is a sum of two squares, and $x, y \in \dot{F}$. By (10), $\langle\langle -w, x\rangle\rangle$ is universal, so φ is isotropic.

Q.E.D.

Since global fields satisfy the condition (1) (by the Hasse-Minkowski Theorem), we obtain:

COROLLARY 6. *If F is a global field, then $G(q) = G'(q)$ for any even dimensional form q.*

This result is a theorem of Dieudonné [14, Théorème 3]. However, our proof ((1)⇒(8) above) is a drastic simplification of Dieudonné's long arguments in [14] (which, incidentally, also use the Hasse-Minkowski Theorem). Actually, Dieudonné's proof in [14] seems to contain a gap (in the middle of p. 402), as pointed out by Dan Shapiro. We would like to thank Dan Shapiro who called our attention to Dieudonné's paper [14], and collaborated in this appendix.

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