Trees of Homotopy Types of Two-Dimensional CW-Complexes

Trees of Homotopy Types of Two-Dimensional CW-Complexes

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1. Introduction

This paper is concerned with the homotopy theory and simple homotopy theory of connected finite 2 dimensional CW complexes with finite cyclic fundamental group. The main theorem presents a complete classification of such complexes up to homotopy type, and this theorem has the corollary that homotopy type and simple homotopy type coincide for these complexes.

This work is motivated by the general problem of describing the sets $\mathbf{HT}(\pi)$ and $\mathbf{SHT}(\pi)$ of homotopy types and simple homotopy types of *π-complexes*, that is, connected finite 2 dimensional CW-complexes with a given fundamental group π. A visually satisfying description of $\mathbf{HT}(\pi)$ or $\mathbf{SHT}(\pi)$ is that of either set as a graph whose edges connect the type of each π-complex X to the type of its sum $X \vee S^2$ with the 2-sphere S^2. These graphs are actually *trees*; they clearly contain no circuits, and they are connected because any two π-complexes have the same type once each is summed with an appropriate number of copies of the 2-sphere S^2. To re establish this latter observation of J. H. C. Whitehead ([20, Theorem 12]), note that each π-complex has the simple homotopy type of one modeled in an obvious fashion on some finite presentation of the fundamental group π (see Proposition 1). But two finite presentations of the same group π differ by a finite sequence of *Tietze* operations, two of which leave the simple homotopy type of the associated topological model unchanged, while two alter the simple homotopy type by an S^2 summand.

Of special interest in each of these trees are the roots and the junctions. The *roots* are the (simple) homotopy types that do not admit a factorization involving an S^2 summand; they generate the rest of the types in the tree under the operation of forming sum with S^2. The *junctions* are the (simple) homotopy types that admit two or more inequivalent factorizations involving an S^2 summand; they determine the shape of the tree. Each junction is a 2-dimensional instance of non-cancellation of the 2-sphere S^2 with respect to the sum operation.

When the group π is a free group F of finite rank or is the finite cyclic group Z_q of prime order q, complete descriptions of the trees $\mathbf{HT}(\pi)$ and $\mathbf{SHT}(\pi)$ can be derived from the literature, as follows.

A result of C. T. C. Wall ([17, Proposition 3.3]) can be specialized to read that for a free group F of finite rank r every F-complex has the homotopy type of a sum of r

[1]) Both authors partially supported by NSF Grant GP 34087.

copies of the 1-sphere S^1 and finitely many copies of the 2-sphere S^2. Moreover, since the Whitehead group $\mathrm{Wh}(F)$ of a free group F is trivial ([6], [15]), homotopy type and simple homotopy type coincide for F-complexes. Thus the trees $\mathbf{HT}(F)$ and $\mathbf{SHT}(F)$ are identical; both are bamboo stalks with no junctions, and with a single root determined by the type of the sum $\vee S^1$ of r copies of the 1-sphere S^1.

In the case where π is the cyclic group Z_q of prime order q, a result of W. H. Cockcroft and R. G. Swan ([4, Theorem 1]) is pertinent. It can be interpreted to say that every Z_q-complex has the homotopy type of a sum of the pseudo-projective plane $P_q = S^1 \cup_q e^2$ and copies of the 2-sphere S^2. Thus the tree $\mathbf{HT}(Z_q)$ is a bamboo stalk with no junctions and with the homotopy type of the pseudo-projective plane P_q as its single root. Furthermore, W. H. Cockcroft and R. M. F. Moss ([3]) have recently observed that, even though the Whitehead group $\mathrm{Wh}(Z_q)$ is not trivial, homotopy type and simple homotopy type coincide for Z_q-complexes, q a prime. Thus, here also, the two trees $\mathbf{HT}(Z_q)$ and $\mathbf{SHT}(Z_q)$ are identical.

The simplest unresolved case involves the cyclic group Z_{pq} of order pq, where p and q are distinct primes. The two descriptions $\pi = Z_{pq}$ and $\pi = Z_p \oplus Z_q$ suggest the two presentations $\mathscr{P} = (c : c^{pq})$ and $\mathscr{Q} = (a, b : a^p, b^q, aba^{-1}b^{-1})$. An obvious question is whether or not the associated topological models $P = S^1 \cup_{pq} e^2$ (the pseudo-projective plane of order pq) and $Q = (S^1 \times S^1) \cup_p e^2 \cup_q e^2$ (the torus with pseudo-projective membranes of order p and order q glued onto its generators) determine distinct roots in either of the trees $\mathbf{HT}(Z_{pq})$ and $\mathbf{SHT}(Z_{pq})$ of homotopy and simple homotopy types. Actually, manipulations with the given group presentations show that the Z_{pq}-complexes $P \vee S^2$ and Q have the same simple homotopy type, hence the presentation $\mathscr{Q}$ does not provide a new root for either tree. This is disappointing, and moreover, our main theorem shows that is is hopeless to look further for new roots modeled on devious presentations of the cyclic group Z_{pq}.

This paper's main result, which we formulate now, is proved in Section 3. Let Z_n be a finite cyclic group of arbitrary order n, and let $P_n = S^1 \cup_n e^2$ denote the pseudo-projective plane of order n.

THEOREM A. *Let X be a Z_n-complex, that is, a connected finite 2-dimensional* CW*-complex with finite cyclic fundamental group Z_n. Then*

(1) *X has the homotopy type of the sum $P_n \vee S^2 \vee \cdots \vee S^2$ of the pseudo-projective plane P_n and rank $H_2(X)$ copies of the 2-sphere S^2.*

(2) *There is a homotopy equivalence $f: X \to P_n \vee S^2 \vee \cdots \vee S^2$ realizing any prescribed Whitehead torsion $\tau(f) \in \mathrm{Wh}(Z_n)$.*

Thus homotopy type and simple homotopy type coincide for connected finite 2-dimensional CW-complexes with finite cyclic fundamental group, and the trees $\mathbf{HT}(Z_n)$ and $\mathbf{SHT}(Z_n)$ of homotopy types and simple homotopy types of Z_n-complexes are again bamboo stalks with no junctions and with their single root determined by the pseudo-projective plane P_n, as in the prime order case.

For a finitely presented group π, the fact that the tree $\mathbf{HT}(\pi)$ or $\mathbf{SHT}(\pi)$ has a single root can be reformulated in a number of equivalent ways. In stating this result we abbreviate the relations of same homotopy type and same simple homotopy type by $\cong$ and $\cong_S$.

THEOREM B. *The following are equivalent statements for a finitely presented group* π.

(1) *The tree* $\mathbf{(S)HT}(\pi)$ *of (simple) homotopy types of* π*-complexes has a single root.*

(2) *For* π*-complexes, there is a cancellation law for* S^2*-summands*:

$$X \vee S^2 \underset{(S)}{\cong} Y \vee S^2 \Rightarrow X \underset{(S)}{\cong} Y.$$

(3) *For* π*-complexes, there is a cancellation law for suspensions*:

$$\Sigma X \cong \Sigma Y \Rightarrow X \underset{(S)}{\cong} Y.$$

(4) For π*-complexes, rank* $H_2(X) = \text{rank} H_2(Y) \Rightarrow X \cong_{(S)} Y$.

A proof of Theorem B is easily based on the observation that the suspension ΣX of a π-complex X has the homotopy type of the sum of the Moore space $M(\pi/[\pi, \pi], 2)$ and rank $H_2(X)$ copies of the 3-sphere S^3. The Moore space $M(\pi/[\pi, \pi], 2)$ is characterized up to homotopy type by the facts that it is simply connected and has trivial homology groups, save H_2 which is isomorphic to the abelianization $\pi/[\pi, \pi]$ of π.

We would like to thank D. Harrison for many conversations concerning the $J[Z_n]$-module cancellation problem involved in the proof of Theorem A and R. Swan for informing us of the work of H. Jacobinski.

2. Cellular Models, Nielsen Transformations, and Chain Complexes

Theorem A is proven in Section 3 modulo some results on homotopy equivalences between π-complexes that are established in Section 4. In this preliminary section we introduce the underlying vocabulary for this paper.

Cellular model of a presentation. Let $\mathscr{P} = (g_1, \ldots, g_k : r_1, \ldots, r_m)$ be a finite presentation. The free group on the generators $g_1, \ldots, g_k$ is denoted by $F = F[g_1, \ldots, g_k]$, and the smallest normal subgroup of F containing the relators $r_1, \ldots, r_m$ is denoted by R.

The *cellular model* P *of the presentation* $\mathscr{P}$ is a CW-complex with a single 0-cell e^0, one 1-cell e_i^1 for each generator g_i, and one 2-cell e_j^2 for each relator r_j. The 2-cell e_j^2 is attached to the 1-skeleton according to the instructions provided by the relator r_j, once the 1-cells are oriented and the relator is made a reduced word.

If we are content to describe P merely up to simple homotopy type, we can proceed in the following manner. We identify the free group F with the group $[S^1, \vee_{i=1}^k S_i^1]$ of based homotopy classes under the isomorphism mapping the i-th generator g_i to the homotopy class of the inclusion of the i-th summand of the sum $\vee_{i=1}^k S_i^1$. Then the m-tuple $(r_1, \ldots, r_m)$ of relators corresponds to a homotopy class $[r] \in [\vee_{j=1}^m S_j^1, \vee_{i=1}^k S_i^1]$. The cells of P are provided with orientations and characteristic maps when we describe P as the adjunction space

$$P = \left(\bigvee_{i=1}^{k} S_i^1\right) \cup_r \left(\bigvee_{j=1}^{m} B_j^2\right),$$

determined by a representative $r\colon \vee_{j=1}^m S_j^1 \to \vee_{i=1}^m S_i^1$. Since different representatives r of the homotopy class $[r]$ determine complexes that differ by elementary deformations of the third kind ([19, Lemma 1]), this gives P up to simple homotopy type.

The presentation $\mathscr{P}$ is a *presentation for the group* π if there exists surjective homomorphism $F \to \pi$ whose kernel is the relator subgroup R. In this case, the cellular model P is a π*-complex*, i.e., a connected finite 2-dimensional CW-complex with fundamental group π. One easily verifies the following converse by the standard process of collapsing a maximal tree.

PROPOSITION 1. *Every connected finite 2-dimensional* CW*-complex with funda mental group* π *has the simple homotopy type of the cellular model* P *of some finite presentation* $\mathscr{P}$ *of* π.

Nielsen transformations of presentations. The following transformations of a set $\{W_1, \ldots, W_m\}$ of freely reduced words in the free group $F = F[g_1, \ldots, g_k]$ are called elementary Nielsen transformations: permuting the W_i and taking inverses of some of them; leaving fixed all W_ϱ, $\varrho \neq r$, and replacing W_r by the freely reduced form of any one of the following $W_r W_s^\eta$, $W_s^\eta W_r W_s^{-\eta}$, where $1 \leqslant r < s \leqslant m$ and $\eta = \pm 1$.

Let $\mathscr{P} = (g_1, \ldots, g_k\colon r_1, \ldots, r_m)$ be a presentation for a group π. If an elementary Nielsen transformation is applied to the set $\{r_1, \ldots, r_m\}$ of relators, it produces a set of relators $\{r_1^*, \ldots, r_m^*\}$ for a new presentation $\mathscr{P}^* = (g_1, \ldots, g_k\colon r_1^*, \ldots, r_m^*)$ of the same group π. If an elementary Nielsen transformation is applied to the set $\{g_1, \ldots, g_k\}$ of generators, it produces an alternative set of free generators $\{g_1', \ldots, g_k'\}$ for the group $F = F[g_1, \ldots, g_k]$, hence the relators $r_1, \ldots, r_m$, when written as reduced words in the new generators, determine relators $r_1', \ldots, r_m'$ for a new presentation $\mathscr{P}' = (g_1', \ldots, g_k'\colon r_1', \ldots, r_m')$ of the same group π.

The two transformations $\mathscr{P} \to \mathscr{P}^*$ and $\mathscr{P} \to \mathscr{P}'$ are *elementary Nielsen transformation of presentations*. Finite compositions of such are refered to as *Nielsen transformations*. One important consequence of the work of Section 4 is this next result.

PROPOSITION 2. *A Nielsen transformation $\mathscr{P} \to \mathscr{Q}$ reducing the presentation $\mathscr{P}$ to the presentation $\mathscr{Q}$ corresponds to a simple homotopy equivalence $P \to Q$ of the associated cellular models.*

The reason for considering Nielsen transformations is that each finite presentation $\mathscr{P}=(g_1,\ldots, g_k\colon r_1,\ldots, r_m)$ with non-negative deficiency $m-k$ can be reduced by a Nielsen transformation to a presentation of the form

$$\mathscr{Q} = (a_1,\ldots, a_k\colon a_1^{\omega_1}W_1,\ldots, a_k^{\omega_k}W_k, W_{k+1},\ldots, W_m)$$

in which each word W_j has zero exponent sum on each generator a_i, and each non-negative integer ω_i divides its successor ω_{i+1} ([8, p. 140]). We refer to $\mathscr{Q}$ as a *pre-Abelian* presentation since the integers $\omega_1,\ldots, \omega_k$ determine the direct product decomposition $Z/(\omega_1)\times\cdots\times Z/(\omega_k)$ of the abelianization of the group presented. Such a decomposition in which ω_i divides ω_{i+1} is unique up to the trivial factors associated with any $\omega_i=1$. Thus every pre-Abelian presentation of the finite cyclic group Z_n of arbitrary order n is of the form

$$\mathscr{Q} = (a_1,\ldots, a_k\colon a_1W_1,\ldots, a_{k-1}W_{k-1}, a_k^nW_k, W_{k+1},\ldots, W_m).$$

We define the *deficiency of a group* π to be the minimum, taken over all finite presentations $\mathscr{P}$ for π, of the number of relators in $\mathscr{P}$ minus the number of generators of $\mathscr{P}$. For example, any finite group π has non-negative deficiency. The following proposition is an immediate consequence of the previous propositions and the existence of Nielsen reductions of presentations with non-negative deficiency.

PROPOSITION 3. *If π is a group with non-negative deficiency, then every connected finite 2-dimensional* CW*-complex with fundamental group π has the simple homotopy type of the cellular model of some pre-Abelian presentation of π.*

Cellular chain complexes. An application of the previous proposition will constitute the first step in our analysis of Z_n-complexes. The second step is best formulated within the framework of their associated cellular chain complexes.

Let P be the cellular model of the finite presentation $\mathscr{P}=(g_1,\ldots, g_k\colon r_1,\ldots, r_m)$ of the group π. The universal covering $\tilde{P}$ of P admits the fundamental group π of P as the group of covering transformations, and there is a natural oriented cellular structure on $\tilde{P}$ with respect to which the covering projection is an orientation preserving cellular map and the covering transformations $x\colon\tilde{P}\to\tilde{P}$, $x\in\pi$, are orientation preserving cellular homeomorphisms. Thus, each group element $x\in\pi$ determines a chain map $x\colon C_*(\tilde{P})\to C_*(\tilde{P})$ of the cellular chain complex $C_*(\tilde{P})$ and this action makes $C_*(\tilde{P})$ into a chain complex of modules over the integral group ring $J[\pi]$.

Moreover, the chain modules $C_0(\tilde{P})$, $C_1(\tilde{P})$, and $C_2(\tilde{P})$ are free $J[\pi]$-modules of rank 1, k, and m, respectively, which we give preferred bases $\{\tilde{e}^0\}$, $\{\tilde{e}_1^1,\ldots, \tilde{e}_k^1\}$, and $\{\tilde{e}_1^2,\ldots, \tilde{e}_m^2\}$, where these cells in the universal covering $\tilde{P}$ are selected in the following manner. Let $\tilde{e}^0$ be any 0-cell over the 0-cell e^0 of P, and label by $\tilde{e}_i^1$ the 1-cell over e_i^1 in P whose boundary is $\partial_1 \tilde{e}_i^1 = x_i \tilde{e}^0 - \tilde{e}^0$, where x_i represents the image of the i-th generator g_i in the group π. Finally, label by $\tilde{e}_j^2$ the lifting of the 2-cell e_j^2 at $\tilde{e}^0$. Notice that with respect to these bases, the boundary operators in the chain complex $C_*(\tilde{P})$: $C_2(\tilde{P}) \xrightarrow{\partial_2} C_1(\tilde{P}) \xrightarrow{\partial_2} C_0(\tilde{P})$ have matrix representations $\partial_1 = (x_i - 1)$ and $\partial_2 = (\partial r_j / \partial x_i)$, the latter matrix being the Jacobian matrix of the presentation described in the free differential calculus of R. H. Fox ([5, p. 198]).

A chain complex $C_*: C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ of free $J\pi$-modules with preferred bases is *realized* by the presentation $\mathscr{P}$ of π if it can be obtained from $\mathscr{P}$ in the fashion just described.

The following result is another conclusion of Section 4.

PROPOSITION 4. *If presentations $\mathscr{Q}$ and $\mathscr{R}$ realize the same chain complex C_* with differing preferred bases in the 2-dimensional module C_2, then there is a homotopy equivalence $f: Q \to R$ between their cellular models that induces the identity chain homomorphism $C_*(\tilde{Q}) \to C_*(\tilde{R})$. Hence, the Whitehead torsion of this homotopy equivalence is the class of the matrix that records the change of basis in the module C_2.*

3. A Proof of Theorem A

We now specialize the preparatory work of Section 2 to the case where π is the finite cyclic group Z_n of arbitrary order n. Given any connected finite 2-dimensional CW-complex P with fundamental group Z_n, Proposition 3 provides a simple homotopy equivalence between P and the cellular model Q of some pre-Abelian presentation

$$\mathscr{Q} = (a_1, \dots, a_k: a_1 W_1, \dots, a_{k-1} W_{k-1}, a_k^n W_k, W_{k+1}, \dots, W_m)$$

of the cyclic group Z_n.

Since the words $W_j \in F$ have zero exponent sum on each generator a_i, these words map to the unit element under the homomorphism $F \to Z_n$. Therefore it follows from the form of the relators of $\mathscr{Q}$ that the first $k-1$ generators $a_1, \dots, a_{k-1}$ map to the unit element in Z_n, while the last generator a_k maps to a generator x of Z_n. Thus, the universal covering projection $\tilde{Q} \to Q$ has the 1-dimensional skeleton sketched in Figure 1.

Then the boundary operators of the chain complex $C_*(\tilde{Q})$ of the universal covering $\tilde{Q}$ of Q are described in terms of the preferred bases $\{z\} = \{\tilde{e}^0\}$ for $C_0(\tilde{Q})$, $\{v_1, \dots, v_k\} = \{\tilde{e}_1^1, \dots, \tilde{e}_k^1\}$ for $C_1(\tilde{Q})$, and $\{u_1, \dots, u_m\} = \{\tilde{e}_1^2, \dots, \tilde{e}_m^2\}$ for $C_2(\tilde{Q})$ as follows:

$$\partial_1(v_1) = 0, \dots, \partial_1(v_{k-1}) = 0, \partial_1(v_k) = (x-1)z$$
$$\partial_2(u_i) = ?v_1 + \dots + ?v_{k-1} + 0v_k \quad (i \neq k)$$
$$\partial_2(u_k) = ?v_1 + \dots + ?v_{k-1} + Nv_k,$$

where the element x is the generator (determined by a_k) of the multiplicative group Z_n,

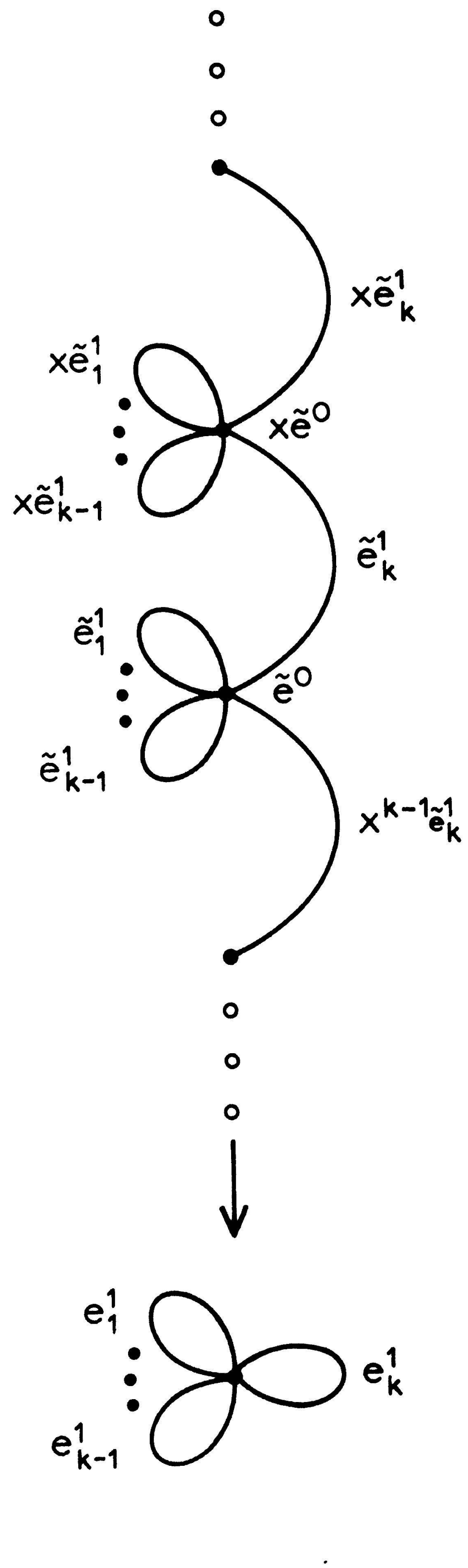

Fig. 1.

and the elements $x-1$ and $N=1+x+\cdots+x^{n-1}$ are in the integral group ring $J[Z_n]$.

Thus, the chain complex with preferred bases $C_*(\tilde{Q})$ takes the form

$$\begin{array}{ccccc} C_2(\tilde{Q}) & \begin{pmatrix} ? & ? & & ? & ? \\ \vdots & & & & \vdots \\ ? & ? & ? & ? & ? \\ 0 \ldots & 0 & N & 0 & \ldots 0 \end{pmatrix} & C_1(\tilde{Q}) & (0 \ldots 0 \; x-1) & C_0(\tilde{Q}) \\ \| & & \| & & \| \\ \langle u_1, \ldots, u_m \rangle & \xrightarrow{\hspace{6em}} & \langle v_1, \ldots, v_k \rangle & \xrightarrow{\hspace{5em}} & \langle z \rangle, \end{array}$$

where $\langle \ldots \rangle$ denotes the free $J[Z_n]$-module with the enclosed elements as preferred basis, and the matrices displayed are those of boundary operators ∂_1 and ∂_2 with respect to the preferred bases in the chain modules.

The remainder of the proof involves the construction of a new basis for the 2-dimensional chain module $C_2(\tilde{Q})$ with respect to which the boundary operator ∂_2 has a matrix of a more convenient and complete form. Since $\tilde{Q}$ is simply connected, the chain complex $C_*(\tilde{Q})$ is exact at $C_1(\tilde{Q})$ and therefore the submodules image ∂_2 and kernel ∂_1 of $C_1 = C_1(\tilde{Q})$ coincide. The latter submodule has the direct sum decomposition $\langle v_1, \ldots, v_{k-1} \rangle \oplus N \langle v_k \rangle$ since multiplication in the ring $J[Z_n]$ by the element $x-1$ has as kernel the ideal generated by N. It follows that the boundary operator is a surjection $\partial_2 : C_2(\tilde{Q}) \to \langle v_1, \ldots, v_{k-1} \rangle \oplus N \langle v_k \rangle$, and therefore there is an internal direct sum decomposition $\langle w_1, \ldots, w_{k-1} \rangle \oplus K$ of the 2-dimensional chain module $C_2 = C_2(\tilde{Q})$ such that $\partial_2(w_i) = v_i (1 \leqslant i \leqslant k-1)$ and $\partial_2(K) = N\langle v_k \rangle$. To construct one such decomposition, let K be the kernel of the surjection obtained by composing the boundary operator $\partial_2 : C_2 \to C_1$ with the projection of the chain module C_1 onto its direct summand $\langle v_1, \ldots, v_{k-1} \rangle$, and label by $w_1, \ldots, w_{k-1}$ the images of the basis elements $v_1, \ldots, v_{k-1}$ under a selected splitting of the resulting exact sequence

$$0 \to K \to C_2 \to \langle v_1, \ldots, v_{k-1} \rangle \to 0$$

of $J[Z_n]$-modules. These elements $w_1, \ldots, w_{k-1}$ of C_2 serve as a basis for a free submodule $\langle w_1, \ldots, w_{k-1} \rangle$ of C_2, and the internal direct sum decomposition $\langle w_1, \ldots, w_{k-1} \rangle \oplus K$ of the free $J[Z_n]$-module C_2 shows that the $J[Z_n]$-module K is *stably free*. Furthermore, H. Jacobinski's cancellation theorem for projective $J[Z_n]$-modules ([7], [14, Theorem 19.8], [16, p. 178]) shows that K is, in fact, *free*.

If $\{w_k, \ldots, w_m\}$ is a basis of this free $J[Z_n]$-module K then the images $\partial_2(w_i) = t_i N v_k$ $(k \leqslant i \leqslant m)$ generate the ideal $N\langle v_k \rangle$ since $\partial_2(K) = N\langle v_k \rangle$. Now the ring elements $t_k, \ldots, t_m$ may be assumed to be integers, since for an arbitrary element $\Sigma \alpha_i x^i$ of the integral group ring $J[Z_n]$ we have the relation $(\Sigma \alpha_i x^i)N = (\Sigma \alpha_i)N$, where the integer $\Sigma \alpha_i$ is called the *augmentation* of the element $\Sigma \alpha_i x^i$. In fact, the ideal in $J[Z_n]$ generated by N is isomorphic to the infinite cyclic group Z with the trivial module structure. Therefore, the integers $t_k, \ldots, t_m$ that determine the generating subset $\{t_k N, \ldots, t_m N\}$ of this ideal must be relatively prime. It follows that there is a unimodular matrix over Z that transforms the basis for K into one whose members $w_k, \ldots, w_m$ have the images $\partial_2(w_k) = N v_k$, $\partial_2(w_{k+1}) = 0, \ldots, \partial_2(w_m) = 0$.

Thus the original chain complex with the basis $\{w_1, \ldots, w_{k-1}, w_k, \ldots, w_m\}$ for C_2 and the original bases for C_1 and C_0 takes the form

$$
\begin{array}{ccccc}
C_2 & & C_1 & & C_0 \\
\| & \begin{pmatrix} 1 \cdot & 0 & 0 & & 0 \\ & \cdot & & & \\ 0 & \cdot 1 & 0 & & 0 \\ 0 \ldots & 0 & N & 0 \ldots & 0 \end{pmatrix} & \| & (0 \ldots 0 \ x-1) & \| \\
\langle w_1, \ldots, w_m \rangle & \xrightarrow{\hspace{6em}} & \langle v_1, \ldots, v_k \rangle & \xrightarrow{\hspace{4em}} & \langle z \rangle .
\end{array}
$$

This chain complex with preferred bases is realized by the presentation

$$\mathcal{R} = (a_1, \ldots, a_k \colon a_1, \ldots, a_{k-1}, a_k^n, 1, \ldots, 1)$$

with $m-k$ trivial relators, and the associated cellular model R has the simple homotopy type of the sum $P_n \vee S^2 \vee \cdots \vee S^2$ of the pseudo-projective plane and $m-k$ copies of the 2-sphere S^2. This proves part (1) of Theorem A, since Proposition 4 provides a homotopy equivalence $f \colon Q \to R$ between the cellular models of the original pre-Abelian presentation $\mathcal{Q}$ and the new presentation $\mathcal{R}$. The Whitehead torsion $\tau(f)$ of this homotopy equivalence is the class of the matrix that records the change of basis of C_2 from $\{u_1, \ldots, u_m\}$ to $\{w_1, \ldots, w_{k-1}, w_k, \ldots, w_m\}$.

The arguments of Cockcroft and Moss [3] show that part (2) of Theorem A can be considered as an immediate consequence of part (1) and the work of P. Olum on the self-equivalences of the pseudo-projective plane ([10], [11]).

To prove part (2) directly, we argue this way. An element in $\mathrm{Wh}(Z_n)$ is represented by a matrix in $GL(J[Z_n])$ and taking determinants gives a unit in $J[Z_n]$. If U_n^1 denotes the group of units of $J[Z_n]$ of augmentation $+1$, and if T_n^1 denotes the subgroup of the trivial units $1, x, \ldots, x^{n-1}$, then the determinant function gives an isomorphism $\det \colon \mathrm{Wh}(Z_n) \approx U_n^1 / T_n^1$ ([1, Ch. XI (7.3)]). For a unit $c = \Sigma \alpha_i x^i$ of augmentation $\Sigma \alpha_i = +1$, we have the relation $cN = N$, hence the form of the matrix of the boundary operator ∂_2 is unchanged when the basis $\{w_1, \ldots, w_{k-1}, w_k, \ldots, w_m\}$ of C_2 is replaced by the basis $\{w_1, \ldots, w_{k-1}, cw_k, \ldots, w_m\}$. It follows that the original chain complex with the new basis $\{w_1, \ldots, w_{k-1}, cw_k, \ldots, w_m\}$ for C_2 is still realized by the presentation $\mathcal{R}$. Therefore, Proposition 4 provides a homotopy equivalence $f_c \colon \mathrm{R} \to R$ whose Whitehead torsion $\tau(f_c)$ is the determinant of the matrix that records the change of basis of C_2 from $\{u_1, \ldots, u_m\}$ to $\{w_1, \ldots, w_{k-1}, cw, \ldots, w_m\}$. The homotopy equivalences $f, f_c \colon Q \to R$ therefore have Whitehead torsions which differ by an arbitrary element $\{c\} \in U_n^1 / T_n^1$ of the Whitehead group $\mathrm{Wh}(Z_n)$. In this way we can construct a homotopy equivalence realizing any prescribed Whitehead torsion element $\tau \in \mathrm{Wh}(Z_n)$.

4. Homotopy Equivalences Between Cellular Models

In the course of the proof of Theorem A, we replaced an arbitrary Z_n-complex P by one Q modeled on a pre-Abelian presentation $\mathcal{Q}$. Then we replaced the Z_n-complex Q by one modeled on a presentation $\mathcal{R}$ that realized the chain complex $C_*(\tilde{Q})$ with a modified basis system. The claim that these replacements can be accomplished without altering (simple) homotopy type is based on Propositions 2 and 4. In this section, we prove these two propositions.

Proposition 2 states that a Nielsen transformation $\mathscr{P} \to \mathscr{Q}$ corresponds to a simple homotopy equivalence $P \to Q$ of the associated cellular models. Since a Nielsen transformation is a finite composition of elementary Nielsen transformation, it suffices to establish the following result.

PROPOSITION 5. *Corresponding to elementary Nielsen transformations of presentations $\mathscr{P} \to \mathscr{P}^*$ and $\mathscr{P} \to \mathscr{P}'$ are simple homotopy equivalences $P \to P^*$ and $P \to P'$ of the associated cellular models.*

Proof. There are three more basic transformations of a presentation that leave unaltered the simple homotopy type of the cellular model. The first process of appending a new generator g and a new relator gW, where W is a word in the old generators, corresponds to a 2-dimensional elementary expansion ([19, p. 345]) of the associated cellular model. The second process of replacing a relator r_j by a relator $\bar{r}_j$ satisfying the condition that $r_j^{-1}\bar{r}_j$ is a consequence of the other relators corresponds to a 2-dimensional elementary deformation of the third kind ([19, Lemma 1]) of the cellular model. The reason is that the condition on r_j and $\bar{r}_j$ is equivalent to the condition that the two attaching maps modeled on these relators are homotopic in the presence of the other 2-cells. The third process of permuting or taking inverses of some of the generators of relators merely amounts to a change in the indexing or orientation of the corresponding cells in the topological model.

To complete the proof of the proposition it remains to observe that the elementary Nielsen transformation can be factored into processes of these three types and their inverses. For example the Nielsen transformation $\mathscr{P} \to \mathscr{P}'$ induced by a Nielsen replacement of the generator g_k by $g_k' = g_k g_j$ factors this way:

$$\begin{aligned}
(g_1, \ldots, g_k : r_1, \ldots, r_m) &\leftrightarrow (g_1, \ldots, g_k, g_k' : r_1, \ldots, r_m, g_k'(g_j^{-1} g_k^{-1})) \\
&\leftrightarrow (g_1, \ldots, g_k, g_k' : r_1', \ldots, r_m', g_k'(g_j^{-1} g_k^{-1})) \\
&\leftrightarrow (g_1, \ldots, g_k, g_k' : r_1', \ldots, r_m', g_k(g_j g_k'^{-1})) \\
&\leftrightarrow (g_1, \ldots, g_k' : r_1', \ldots, r_m').
\end{aligned}$$

These are processes of type 1, 2, 3, and 1 respectively.

In the proof of Theorem A we have occasion to change the preferred basis for the 2-dimensional chain module C_2 of a chain complex

$$\begin{array}{ccccc}
C_2 & & C_1 & & C_0 \\
\| & & \| & & \| \\
\langle u_1, \ldots, u_m \rangle & \xrightarrow{\partial_2} & \langle v_1, \ldots, v_k \rangle & \xrightarrow{\partial_1} & \langle z \rangle
\end{array}$$

realized by a presentation $\mathcal{Q}$ of π. We get a new chain complex of free $J[\pi]$-modules with preferred bases

$$\begin{array}{ccccc} C_2 & & C_1 & & C_0 \\ \| & & \| & & \| \\ \langle w_1, \dots, w_m \rangle & \overset{\partial_2}{\to} & \langle v_1, \dots, v_k \rangle & \overset{\partial_1}{\to} & \langle z \rangle . \end{array}$$

Two questions arise: Is the new chain complex of free $J[\pi]$-modules with preferred bases realized by some presentation $\mathcal{R}$ of π? If so, what is the relationship between the cellular models Q and R of the presentations $\mathcal{Q}$ and $\mathcal{R}$? The first question is related to an unresolved conjecture of C. T. C. Wall ([18, p. 131]), but in the situation encountered in Section 3 the presentation $\mathcal{R}$ obviously exists. We suppose then that the presentation $\mathcal{R}$ realizes the chain complex with new preferred bases, and focus our attention on the second question. Since the identity map between the two chain complexes respects the preferred bases of the chain modules C_0 and C_1, we may identify the generators of the two presentations, the 1-skeltons Q^1 and R^1 of their cellular models, and the fundamental groups of the complete models Q and R. It follows that the identity map on the 1-skeletons extends to some map $f: Q \to R$ between the cellular models. This map $f: Q \to R$ induces a chain homomorphism

$$\begin{array}{ccccc} C_2(\tilde{Q}) & \overset{\partial_2}{\to} & C_1(\tilde{Q}) & \overset{\partial_1}{\to} & C_0(\tilde{Q}) \\ {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1=1}\downarrow & & {\scriptstyle f_0=1}\downarrow \\ C_2(\tilde{R}) & \overset{\partial_2}{\to} & C_1(\tilde{R}) & \overset{\partial_1}{\to} & C_0(\tilde{R}) \end{array}$$

which is the identity on the chain modules C_0 and C_1. Thus the *deviation* $1-f_2$: $C_2(\tilde{Q}) \to C_2(\tilde{R})$ between the identity homomorphism 1 and the homomorphism f_2 takes values in the kernel of $\partial_2: C_2(\tilde{R}) \to C_1(\tilde{R})$. We can identify this kernel with the 2-nd homotopy module $\pi_2(R)$ of R by means of the Hurewicz isomorphism $\pi_2(\tilde{R}) \approx H_2(\tilde{R}) = \text{kernel } \partial_2$ and the covering projection isomorphism $\pi_2(\tilde{R}) \to \pi_2(R)$, and then the deviation $1-f_2$ can be considered as a $J[\pi]$-module homomorphism $D: C_2(\tilde{Q}) \to \pi_2(R)$. Since the module $C_2(\tilde{Q})$ is free of rank m this homomorphism D corresponds in a natural fashion to a homotopy class α: $\bigvee_{j=1}^m S_j^2 \to R$.

Now we can consider the cellular model

$$Q = \left(\bigvee_{i=1}^{k} S_i^1 \right) \cup_r \left(\bigvee_{j=1}^{m} B_j^2 \right)$$

as the mapping cone of the map r: $\bigvee_{j=1}^m S_j^1 \to \bigvee_{i=1}^k S_i^2$, hence there is a *cooperation* $c: Q \to Q \vee (\bigvee_{j=1}^m S_j^2)$ that collapses the midbelt of the cone $C(\bigvee_{j=1}^m S_j^1) = \bigvee_{j=1}^m B_j^2$. This cooperation can be utilized to form the composition

$$f^\alpha = \nabla \circ f \vee \alpha \circ c: Q \to Q \vee \left(\bigvee_{j=1}^{m} S_j^2 \right) \to R \vee R \to R,$$

which is the result of the *Puppe action* ([12]) of $\alpha: \vee_{j=1}^{m} S_j^2 \to R$ on the original map $f: Q \to R$. This new map f^{α} agrees with the original map f on the fundamental group π, and on the chain modules C_0 and C_1, but on the 2-dimensional chain module C_2 it induces the identity homomorphism because of the relations

$$f_2^{\alpha} = f_2 + D = f_2 + (1 - f_2) = 1 : C_2(\tilde{Q}) \to C_2(\tilde{R}).$$

It follows that f_2^{α} restricts to the kernel of the boundary homomorphism ∂_2 to give an isomorphism, which can be identified with the homomorphism $f_{\#}^{\alpha}: \pi_2(Q) \to \pi_2(R)$. This shows that $f^{\alpha}: Q \to R$ is a homotopy equivalence. Proposition 4 is established, since one can verify the additional claim concerning the Whitehead torsion of the homotopy equivalence by invoking the Whitehead torsion properties P1–P5 of [19].

5. Concluding Remarks

The problem of determining the tree $\mathbf{HT}(\pi)$ or $\mathbf{SHT}(\pi)$ is equivalent to that of establishing the extent to which cancellation of S^2-summands is possible in the simple homotopy relation $X \vee (\vee_{i=1}^{s} S_i^2) \cong_S Y \vee (\vee_{j=1}^{t} S_j^2)$ provided for two π-complexes X and Y by J. H. C. Whitehead's simple homotopy theory. It should not be surprising than that an associated algebraic cancellation result is a key factor in each of the homotopy tree determinations discussed in the introduction. The simple homotopy relation above provides a $J[\pi]$-module isomorphism

$$\phi: \pi_2(X) \oplus J[\pi]^{(s)} \approx \pi_2(Y) \oplus J[\pi]^{(t)}.$$

If the homology group $H_2(Y)$ of Y has minimal rank for π complexes, then $t \geqslant s$ and there is the question whether cancellation of the free factor $J[\pi]^{(s)}$ of the isomorphic $J[\pi]$-modules above is valid and ϕ yields a module isomorphism $\theta: \pi_2(X) \approx \approx \pi_2(Y) \oplus J[\pi]^{(t-s)}$.

When π is a free group F of finite rank r, the appropriate choice for Y is the sum $\vee S^1$ of r copies of the 1-sphere S^1. Then the homotopy module $\pi_2(Y)$ is trivial and the $J[F]$-module isomorphism ϕ shows that for an arbitrary F-complex X, the $J[F]$-module $\pi_2(X)$ is stably free. If the algebraic cancellation that was questioned above is possible, then the stably free $J[F]$-module $\pi_2(X)$ is free of rank $t-s$, and then any map $Y \vee (\vee_{j=1}^{t-s} S_j^2) \to X$ that induces an isomorphism of the fundamental groups and also sends the copies of the 2 sphere S^2 onto the free generators of $\pi_2(X)$ is a homotopy equivalence. Thus, the key to establishing the fact that the tree $\mathbf{HT}(F)$ has $Y = \vee S^1$ as its single root is the result of H. Bass ([2]) that projective $J[F]$-modules are free, for this guarantees that the algebraic cancellation of the free factor $J[F]^{(s)}$ is valid in this case.

When π is a finite cyclic group Z_n of arbitrary order n, the pseudo-projective plane

P_n is a π-complex Y with homology $H_2(Y)$ of minimal rank. A trivial extension of the work in [4] shows that for any other Z_n-complex X the existence of a $J[Z_n]$-module isomorphism $\theta: \pi_2(X) \approx \pi_2(P_n) \oplus J[Z_n]^{(t-s)}$ implies that the algebraic 2-types of X and the sum $P_n \vee (\vee_{j=1}^{t-s} S_j^2)$ are isomorphic, and hence, that these complexes have the same homotopy type ([9, Theorem 1]). Thus, the key factor in the determination by Cockcroft and Swan ([4, Theorem 1]) of the homotopy types of Z_q-complexes, q a prime, is the classification by I. Reiner ([13]) of finitely generated, torsion free $J[Z_q]$-modules, q a prime, for this classification establishes the validity of the cancellation of free $J[Z_q]$-factors that was questioned earlier and provides a module isomorphism $\theta: \pi_2(X) \approx \pi_2(P_n) \oplus J[Z_q]^{(t-s)}$.

For arbitrary order n, the theory of $J[Z_n]$-lattices, i.e., finitely generated, torsion free $J[Z_n]$-modulus, is not as manageable, and the cancellation of free $J[Z_n]$-factors appears available only up to the point where a free factor remains ([14, Theorem 19.6]). There is, however, a full Cancellation Theorem for projective $J[Z_n]$-lattices due to H. Jacobinski ([7, Corollary 5.3], [14, Theorem 19.8], [16, p. 178]). In order to employ this cancellation result, the classification of Z_n-complexes in Section 3 deals with the associated cellular chain complexes, and ignores the cancellation problem involving the homotopy modules. In this way, the theory of algebraic 2-types developed by S. MacLane and J. H. C. Whitehead ([9]) does not appear directly, but, in fact, a complete exposition of the theory of algebraic 2-types of 2-dimensional complexes can be based on the construction in Section 4 involving the Puppe action by a deviation.

Here are some open questions that we intend to consider.

The arguments given earlier in this section for the free group F apply to any group π which admits a 2-dimensional Eilenberg-MacLane Space $K(\pi, 1)$ and for which stably free $J[\pi]$-modules are free. Thus a compact connected 2-dimensional manifold $M \neq S^2, P_2$ determines a single root of the tree $\mathbf{HT}(\pi_1(M))$, provided stably free $J[\pi_1(M)]$-modules are free. When is this the case?

Any finite abelian group π has this desirable property employed in the proof of Theorem A that stably free $J[\pi]$-modules are free ([14, Theorem 19.8]). Do such non-cyclic groups nevertheless provide a tree $\mathbf{HT}(\pi)$ that has more than a single root?

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Received October 7, 1972