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A Characterization of the Veblen-Schütte Functions by Means of Functionals[1)]

HILBERT LEVITZ

1. Introduction

Let Ω denote the first uncountable ordinal. A function ϕ which maps the set $\{x \mid x<\Omega\}$ into itself is called a *normal* function if it is continuous and strictly increasing. In [3] it is shown that a normal function has fixed points; i.e., solutions of the equation $\phi(x)=x$. In fact, the ordering function of these fixed points is itself a normal function. One calls the function arrived at in this way the *derived* function of ϕ. It is also shown there that the process of taking the derived function can be iterated into the transfinite, and a family $\{\phi_\alpha\}_{\alpha<\Omega}$ of normal functions can be obtained with the property that $\phi_0=\phi$ and ϕ_α is the ordering function of the fixed points of ϕ_β for all β such that $0\leqslant\beta<\alpha$. If $\phi(0)>0$, one does not, however, obtain a normal function ϕ_Ω in this way because $\{x \mid \phi_\beta(x)=x \text{ all } \beta<\Omega\}$ can be shown to be empty.

In [5], Schütte extends this family by introducing Klammersymbols. A Klammersymbol is an expression

$$\begin{pmatrix} a_0, a_1, \ldots, a_n \\ \alpha_0, \alpha_1, \ldots, \alpha_n \end{pmatrix}$$

where the α_i, a_i are ordinals less than Ω and $0\leqslant\alpha_0<\alpha_1<\cdots<\alpha_n$. Equality among Klammersymbols is governed by the rule that two Klammersymbols are equal if one can be obtained from the other by the insertion or deletion of columns of the form $\begin{matrix} 0 \\ \alpha_i \end{matrix}$. To each normal function ϕ and to each Klammersymbol A an ordinal ϕA is assigned so that

$$\phi\begin{pmatrix} x \\ 0 \end{pmatrix} = \phi(x) \quad \text{and for} \quad a_1 \neq 0, \quad \phi\begin{pmatrix} a_0, a_1, \ldots, a_n \\ 0, \alpha_1, \ldots, \alpha_n \end{pmatrix}$$

is the a_0-th solution of the set of equation

$$\phi\begin{pmatrix} x, a^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = x \quad \text{where} \quad \alpha^* < \alpha_1, a^* < a_1 .$$

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1) These results have been extracted from the author's thesis at Pennsylvania State University. The idea of using functionals was suggested by W. A. Howard. He also conjectured Theorem 1.1.

The functions

$$\lambda x \phi \begin{pmatrix} x, a_1, \ldots, a_n \\ \alpha_0, \alpha_1, \ldots, \alpha_n \end{pmatrix} \quad \text{and} \quad \lambda x \phi \begin{pmatrix} 1 \\ x \end{pmatrix}$$

are shown to be normal functions[2]). The fixed points of $\lambda x \phi \begin{pmatrix} 1 \\ x \end{pmatrix}$ are referred to by Veblen [6] as E-number (relative to the initial function ϕ). Using properties of the functions

$$\lambda x \phi \begin{pmatrix} x, a_1, \ldots, a_n \\ \alpha_0, \alpha_1, \ldots, \alpha_n \end{pmatrix},$$

Schütte shows how to obtain a recursive well ordering of the natural numbers that has the least E-number (relative to the function $1+x$) as its order type.

2. The Main Results

In this section we give a new characterization of

$$\phi \begin{pmatrix} a_0, a_1, \ldots, a_n \\ \alpha_0, \alpha_1, \ldots, \alpha_n \end{pmatrix}.$$

For this purpose we introduce the notion of *type*. Objects of type 0 are ordinals. Objects of type 1 are functions from ordinals to ordinals. Objects (functionals) of type 2 are functions from objects of type 1 to objects of type 1.

With respect to notation, we use the following conventions:

1) We use $\circ$ to denote the operation of composition of mappings. $\phi \circ \psi$ is the result of applying first ψ, then ϕ.

2) $\mathscr{R}\phi$ will denote the range of ϕ.

3) If T is a functional then $T(x)$ denotes the image of x under the functional T. Parentheses will be omitted when there is no danger of ambiguity.

DEFINITION 1.1. For $\alpha > 0$, $a \geqslant 0$ we define functionals T^a_α of type 2 as follows:

$$T^a_\alpha(\phi) = \lambda x \phi \begin{pmatrix} x, a \\ 0, \alpha \end{pmatrix}.$$

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[2]) We are using the familiar Church lambda symbol notation. $\lambda x \phi \begin{pmatrix} x, a_1, \ldots, a_n \\ \alpha_0, \alpha_1, \ldots, \alpha_n \end{pmatrix}$ is the function of x abstracted from the expression $\phi \begin{pmatrix} x, a_1, \ldots, a_n \\ \alpha_0, \alpha_1, \ldots, \alpha_n \end{pmatrix}$.

Since $\phi\begin{pmatrix} x, a \\ 0, \alpha \end{pmatrix}$ is defined only for normal functions, our functionals T^a_α are defined only for such ϕ. However in the discussion following Theorem 1.2, we shall extend the domain of T^a_α to arbitrary functions ϕ.

We also remark that since $\lambda x\phi\begin{pmatrix} x, a \\ 0, \alpha \end{pmatrix}$ is again a normal function by [5, (3.2), (3.5)][3]), the composition of the functionals T^a_α is well defined.

LEMMA 1.1. *If* $a_1 \neq 0$, $b \neq 0$, $\alpha_1 \neq 0$ *and*

$$\mathscr{A} = \left\{ y \mid \phi\begin{pmatrix} y, a_1 + b^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1 \quad\quad , \alpha_2, \ldots, \alpha_n \end{pmatrix} = y; \quad \text{all} \quad b^* < b, \quad \alpha^* < \alpha_1 \right\}$$

and

$$\mathscr{B} = \left\{ y \mid \phi\begin{pmatrix} y, a^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = y; \quad \text{all} \quad a^* < a_1, \quad \alpha^* < \alpha_1 \right\},$$

then $\mathscr{A} \subseteq \mathscr{B}$.

Proof. Let $y_0 \in \mathscr{A}$, then in particular for $b^* = 0$, $\alpha^* = 0$ we get

$$\phi\begin{pmatrix} y_0, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = y_0$$

so by definition [5, (2.3)], y_0 is the y_0-th solution of the equations:

$$\phi\begin{pmatrix} x, a^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = x; \quad \text{all} \quad a^* < a_1, \quad \alpha^* < \alpha_1$$

therefore $y_0 \in \mathscr{B}$.

LEMMA 1.2.

$$\left[\lambda x\phi\begin{pmatrix} x, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}\right]\begin{pmatrix} b_0, b_1, \ldots, b_k, b \\ 0, \beta_1, \ldots, \beta_k, \alpha_1 \end{pmatrix}$$
$$= \phi\begin{pmatrix} b_0, b_1, \ldots, b_k, a_1 + b, a_2, \ldots, a_n \\ 0, \beta_1, \ldots, \beta_k, \alpha_1 \quad\quad , \alpha_2, \ldots, \alpha_n \end{pmatrix}.$$

Proof. The proof is by transfinite induction on $\begin{pmatrix} b_0, b_1, \ldots, b_k, b \\ 0, \beta_1, \ldots, \beta_k, \alpha_1 \end{pmatrix}$ over the lexicographical well ordering of the Klammersymbols described in [5, p. 17].

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[3]) [5, (3.2, (3.5)] refers to the article listed in the bibliography under [5]; the numbers (3.2) and (3.5) are used in that article to denote various results.

Case 1. $b_1, b_2, \ldots, b_k, b$ are all zero. Then the desired result follows trivially from [5, (2.1)].

Case 2. $b_1, b_2, \ldots, b_k$ are zero and b is not zero. Let γ denote

$$\left[\lambda x \phi \begin{pmatrix} x, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}\right] \begin{pmatrix} b_0, b \\ 0, \alpha_1 \end{pmatrix}.$$

Then γ is the b_0-th member of

$$\left\{ y \mid \left[\lambda x \phi \begin{pmatrix} x, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}\right] \begin{pmatrix} y, b^* \\ \alpha^*, \alpha_1 \end{pmatrix} = y; \quad \text{all} \quad b^* < b, \quad \alpha^* < \alpha_1 \right\}.$$

By induction hypothesis γ is the b_0-th member of

$$\mathscr{A} = \left\{ y \mid \phi \begin{pmatrix} y, a_1 + b^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \qquad \alpha_2, \ldots, \alpha_n \end{pmatrix} = y; \quad \text{all} \quad b^* < b, \quad \alpha^* < \alpha_1 \right\}.$$

Case 2.1. $a_1 = 0$. Then γ is the b_0-th member of

$$\left\{ y \mid \phi \begin{pmatrix} y, 0 + b^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \qquad \alpha_2, \ldots, \alpha_n \end{pmatrix} = y; \quad \text{all} \quad b^* < b, \quad \alpha^* < \alpha \right\}$$

so by definition [5, (2.3)] we get

$$\gamma = \phi \begin{pmatrix} b_0, 0 + b, a_2, \ldots, a_n \\ 0, \alpha_1, \qquad \alpha_2, \ldots, \alpha_n \end{pmatrix}.$$

This is the desired result.

Case 2.2. $a_1 \neq 0$. Let $\mathscr{B}$ denote

$$\left\{ y \mid \phi \begin{pmatrix} y, a^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = y; \quad \text{all} \quad a^* < a_1, \quad \alpha^* < \alpha_1 \right\}$$

By Lemma 1.1 $\mathscr{A} \subseteq \mathscr{B}$, so $\mathscr{A} = \mathscr{A} \cap \mathscr{B}$. Therefore γ is the b_0-th solution to both sets of equations:

$$\phi \begin{pmatrix} y, a^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = y; \quad \text{all} \quad \alpha^* < \alpha_1, \quad \text{all} \quad a^* < a_1$$

and

$$\phi \begin{pmatrix} y, a_1 + b^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \qquad \alpha_2, \ldots, \alpha_n \end{pmatrix} = y; \quad \text{all} \quad \alpha^* < \alpha_1, \quad b^* < b.$$

But

$$\{a^* \mid a^* < a_1\} \cup \{a^* \mid a^* = a_1 + b^* \quad \text{for some} \quad b^* < b\} = \{a^* \mid a^* < a_1 + b\}$$

so γ is the b_0-th solution of the set of equations

$$\phi\begin{pmatrix} y, a^*, a_2, \ldots, a_n \\ \alpha^*, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix} = y; \quad \text{all} \quad \alpha^* < \alpha_1, \quad \text{all} \quad a^* < a_1 + b$$

so by definition [5, (2.3)] we get

$$\gamma = \phi\begin{pmatrix} b_0, a_1 + b, a_2, \ldots, a_n \\ 0, \alpha_1, \quad \alpha_2, \ldots, \alpha_n \end{pmatrix}.$$

This is the desired result.

Case 3. $b_i \neq 0$ for some i, $1 \leqslant i \leqslant k$. Assume t is the smallest such i. Let δ denote

$$\left[\lambda x\phi\begin{pmatrix} x, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}\right]\begin{pmatrix} b_0, b_t, \ldots, b_k, b \\ 0, \beta_t, \ldots, \beta_k, \alpha_1 \end{pmatrix},$$

then δ is the b_0-th solution of the set of equations:

$$\left[\lambda x\phi\begin{pmatrix} x, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}\right]\begin{pmatrix} y, b^*, \ldots, b_k, b \\ \beta^*, \beta_t, \ldots, \beta_k, \alpha_1 \end{pmatrix} = y$$

where $b^* < b_t$, $\beta^* < \beta_t$. By the induction hypothesis, δ is the b_0-th solution of the set of equations:

$$\phi\begin{pmatrix} y, b^*, \ldots, b_k, a_1 + b, a_2, \ldots, a_n \\ \beta^*, \beta_t, \ldots, \beta_k, \alpha_1, \quad \alpha_2, \ldots, \alpha_n \end{pmatrix} = y$$

where $b^* < b_t$, $\beta^* < \beta_t$. So the desired result follows from [5, (2.3)]. This completes the proof.

THEOREM 1.1. *For* $\alpha_1 > 0$,

$$(T_{\alpha_1}^{a_1} \circ T_{\alpha_2}^{a_2} \circ \cdots \circ T_{\alpha_n}^{a_n})\,\phi = \lambda y\phi\begin{pmatrix} y, a_1, a_2, \ldots, a_n \\ 0, \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}.$$

Proof. The proof is by transfinite induction on $\begin{pmatrix} a_1, a_2, \ldots, a_n \\ \alpha_1, \alpha_2, \ldots, \alpha_n \end{pmatrix}$ over the lexicographical well ordering of the Klammersymbols.

By associativity

$$(T_{\alpha_1}^{a_1} \circ T_{\alpha_2}^{a_2} \circ \cdots \circ T_{\alpha_n}^{a_n})\,\phi = T_{\alpha_1}^{a_1}((T_{\alpha_2}^{a_2} \circ T_{\alpha_3}^{a_3} \circ \cdots \circ T_{\alpha_n}^{a_n})\,\phi)$$

by induction hypotheis

$$= T_{\alpha_1}^{a_1}\left(\lambda x \phi \begin{pmatrix} x, & a_2, \ldots, & a_n \\ 0, & \alpha_2, \ldots, & \alpha_n \end{pmatrix}\right)$$

by Definition 1.1

$$= \lambda y \left[\left[\lambda x \phi \begin{pmatrix} x, & a_2, \ldots, & a_n \\ 0, & \alpha_2, \ldots, & \alpha_n \end{pmatrix}\right] \begin{pmatrix} y, & a_1, & 0 \\ 0, & \alpha_1, & \alpha_2 \end{pmatrix}\right]$$

by Lemma 1.2.

$$= \lambda y \left[\phi \begin{pmatrix} y, & a_1, & a_2, \ldots, & a_n \\ 0, & \alpha_1, & \alpha_2, \ldots, & \alpha_n \end{pmatrix}\right].$$

This completes the proof.

The purpose of the remainder of this work is to show that T_α^a can be defined directly by transfinite induction. Thus Theorem 1.1 will provide a new *definition* of Schütte's functions.

LEMMA 1.3. *If* $\alpha > 0$ *then* $T_\alpha^1 \circ T_\alpha^a = T_\alpha^{a+1}$.
Proof.

$$\begin{aligned}
(T_\alpha^{a+1}\phi)\, b &= \left[\lambda x \phi \begin{pmatrix} x, & a+1 \\ 0, & \alpha \end{pmatrix}\right] b \quad \text{by Definition 1.1} \\
&= \phi \begin{pmatrix} b, & a+1 \\ 0, & \alpha \end{pmatrix} \\
&= \left[\lambda x \phi \begin{pmatrix} x, & a \\ 0, & \alpha \end{pmatrix}\right] \begin{pmatrix} b, & 1 \\ 0, & \alpha \end{pmatrix} \quad \text{by Lemma 1.2} \\
&= \left(\lambda y \left[\lambda x \phi \begin{pmatrix} x, & a \\ 0, & \alpha \end{pmatrix} \begin{pmatrix} y, & 1 \\ 0, & \alpha \end{pmatrix}\right]\right) b \quad \text{by } \lambda\text{-abstraction} \\
&= \left(T_\alpha^1 \left[\lambda x \phi \begin{pmatrix} x, & 1 \\ 0, & \alpha \end{pmatrix}\right]\right) b \quad \text{by Definition 1.1} \\
&= (T_\alpha^1 (T_\alpha^a \phi))\, b \quad \text{by Definition 1.1} \\
&= ((T_\alpha^1 \circ T_\alpha^a)\, \phi)\, b.
\end{aligned}$$

LEMMA 1.4. $\phi\begin{pmatrix} x, & 1 \\ 0, & \alpha+1 \end{pmatrix}$ *is the x-th member of* $\mathscr{B} = \left\{ y \mid \phi \begin{pmatrix} y \\ \alpha \end{pmatrix} = y \right\}$.

Proof. By definition [5, (2.3)], $\phi\begin{pmatrix} x, & 1 \\ 0, & \alpha+1 \end{pmatrix} = x$-th member of $\mathscr{A}$ where

$$\mathscr{A} = \left\{ y \mid \phi \begin{pmatrix} y, & 0 \\ \alpha^*, & \alpha+1 \end{pmatrix} = y; \quad \text{all} \quad \alpha^* < \alpha + 1 \right\}.$$

Case 1. $\alpha=0$. Then our result is immediate since trivially $\mathscr{A}=\mathscr{B}$.

Case 2. $0<\alpha$. Then $\mathscr{A}=\mathscr{B}\cap\mathscr{B}'$ where

$$\mathscr{B}'=\left\{y \mid \phi\begin{pmatrix} y, & 0 \\ \alpha^*, & \alpha+1\end{pmatrix}=y \quad \text{all} \quad \alpha^*<\alpha\right\}$$

and

$$\mathscr{B}=\left\{y \mid \phi\begin{pmatrix} y, 0 \\ \alpha, \alpha+1\end{pmatrix}=y\right\}.$$

What we need to show is $\mathscr{A}=\mathscr{B}$.

$\mathscr{A}\subseteq\mathscr{B}$ is trivial so we need only show $\mathscr{B}\subseteq\mathscr{A}$.

Let $y\in\mathscr{B}$, then $\phi\begin{pmatrix} y \\ \alpha\end{pmatrix}=y$.

By [5, (3.4)], $0<y$; so we get that y is the 0-th member of

$$\left\{z \mid \phi\begin{pmatrix} z, & y^* \\ \alpha^*, & \alpha\end{pmatrix}=z \quad \text{all} \quad \alpha^*<\alpha, \quad y^*<y\right\}.$$

In particular, y is the 0-th member of

$$\left\{z \mid \phi\begin{pmatrix} z, & 0 \\ \alpha^*, & \alpha\end{pmatrix}=z \quad \text{all} \quad \alpha^*<\alpha\right\}.$$

So $y\in\mathscr{B}'$, from which it follows that $y\in\mathscr{A}$.

LEMMA 1.5. *$(T_1^1\phi)a$ is the a-th solution of the equation $\phi(x)=x$.*

Proof.

$$(T_1^1\phi)\,a=\left[\lambda x\phi\begin{pmatrix} x, & 1 \\ 0, & 1\end{pmatrix}\right]a=\phi\begin{pmatrix} a, & 1 \\ 0, & 1\end{pmatrix}$$

by Definition 1.1 and λ-abstraction. Our result follows from [5, (2.3)].

LEMMA 1.6. *Let R_α be defined by $R_\alpha\phi=\lambda y[(T_\alpha^y\phi)0]$, then $T_{a+1}^1=T_1^1\circ R_\alpha$.*

Proof.

Case 1. $\alpha=0$. Then the following shows that R_α is the identity functional:

$$(R_0\phi)\,a=(\lambda y[(T_0^y\phi)\,0])\,a=(\lambda y[(\lambda x\phi(y))\,0])\,a=\phi(a).$$

From this the desired result is immediate.

Case 2. $0<\alpha$.

$$(T^1_{\alpha+1}\phi)\, b = \left[\lambda x\phi\begin{pmatrix}x, 1\\ 0, \alpha+1\end{pmatrix}\right] b = \phi\begin{pmatrix}b, 1\\ 0, \alpha+1\end{pmatrix}$$

$= b$-th fixed point of $\lambda y\phi\begin{pmatrix}y\\ \alpha\end{pmatrix}$ by Lemma 1.4

$= \left[T^1_1\left(\lambda y\phi\begin{pmatrix}y\\ \alpha\end{pmatrix}\right)\right] b$ by Lemma 1.5

$= \left[T^1_1\left(\lambda y\phi\begin{pmatrix}0, y\\ 0, \alpha\end{pmatrix}\right)\right] b$

$= \left[T^1_1\left(\lambda y\left[\lambda x\phi\begin{pmatrix}x, y\\ 0, \alpha\end{pmatrix}0\right]\right)\right] b$ by λ-abstraction

$= [T^1_1\lambda y[(T^y_\alpha\phi)\, 0]]\, b$ by definition of T^y_α

$= [T^1_1(R_\alpha\phi)]\, b$ by definition of R_α

$= [(T_1\circ R_\alpha)\,\phi]\, b$.

LEMMA 1.7. *Assume* $\bar{\alpha}$, $\bar{a}$ *are limit ordinals, and* $0<\alpha$, *then*

(i) $\mathscr{R}\lambda x\phi\begin{pmatrix}x, \bar{a}\\ 0, \alpha\end{pmatrix} = \bigcap_{a<\bar{a}} \mathscr{R}\lambda x\phi\begin{pmatrix}x, a\\ 0, \alpha\end{pmatrix}$

(ii) $\mathscr{R}\lambda x\phi\begin{pmatrix}x, 1\\ 0, \bar{\alpha}\end{pmatrix} = \bigcap_{\alpha<\bar{\alpha}} \mathscr{R}\lambda x\phi\begin{pmatrix}x, 1\\ 0, \alpha\end{pmatrix}$

Proof. (i) That the left side is included in the right side is immediate from the definition of $\phi\begin{pmatrix}x, \bar{a}\\ 0, \alpha\end{pmatrix}$.

To show that the right side is included in the left side, let y be a member of the right side; we must show that

$$\phi\begin{pmatrix}y, & a^*\\ \alpha^*, & \alpha\end{pmatrix} = y \quad \text{all} \quad \alpha^* < \alpha, \quad a^* < \bar{a}.$$

Let $\alpha^* < \alpha$, $a^* < \bar{a}$ be given, then since $\bar{a}$ is a limit ordinal, $a^* + 1 < \bar{a}$. Now by hypothesis y is a member of $\mathscr{R}\lambda x\phi\begin{pmatrix}x, a^*+1\\ 0, \alpha\end{pmatrix}$. From [5, (2.3)] we get

$$\phi\begin{pmatrix}y, & a^*\\ \alpha^*, & \alpha\end{pmatrix} = y.$$

(ii) To show that the left side is included in the right side: Let $y\in\mathscr{R}\lambda x\phi\begin{pmatrix}x, 1\\ 0, \bar{\alpha}\end{pmatrix}$ be given; to show that y is a member of the right side we must show that if $\alpha<\bar{\alpha}$ then $y\in\mathscr{R}\lambda x\phi\begin{pmatrix}x, 1\\0, \alpha\end{pmatrix}$. Let $\alpha<\bar{\alpha}$ be given. Now $y\in\mathscr{R}\lambda x\phi\begin{pmatrix}x, 1\\0, \bar{\alpha}\end{pmatrix}$ means that $y=\phi\begin{pmatrix}y\\\alpha^*\end{pmatrix}$ for all $\alpha^*<\bar{\alpha}$; in particular $y=\phi\begin{pmatrix}y\\\alpha^*\end{pmatrix}$ for all $\alpha^*<\alpha$; so by [5, (2.3)], $y\in\mathscr{R}\lambda x\phi\begin{pmatrix}x, 1\\0, \alpha\end{pmatrix}$. To show that the right side is included in the left side, let y be a member of the right side, we must show that

$$\phi\begin{pmatrix}y\\\alpha^*\end{pmatrix}=y \quad \text{all} \quad \alpha^*<\bar{\alpha}.$$

Let $\alpha^*<\bar{\alpha}$ be given, then since $\bar{\alpha}$ is a limit ordinal $\alpha^*+1<\bar{\alpha}$. Now by hypothesis y is a member of $\mathscr{R}\lambda x\phi\begin{pmatrix}x, 1\\0, \alpha^*+1\end{pmatrix}$ so by [5, (2.3)] $\phi\begin{pmatrix}y\\\alpha^*\end{pmatrix}=y$.

COROLLARY 1. *Assume $\alpha>0$, and $\bar{a}$ is a limit. If*

$$\phi\begin{pmatrix}x, \bar{a}\\0, \alpha\end{pmatrix}<t\leqslant\phi\begin{pmatrix}x+1, \bar{a}\\0, \quad \alpha\end{pmatrix},$$

then

$$\phi\begin{pmatrix}x+1, \bar{a}\\0, \quad \alpha\end{pmatrix}=\sup_{a<\bar{a}}\phi\begin{pmatrix}t, a\\0, \alpha\end{pmatrix}.$$

To see this let $\{f_a\}_{a<\bar{a}}$ be the sequence of functions defined by

$$f_a\equiv\lambda x\phi\begin{pmatrix}x, a\\0, \alpha\end{pmatrix}.$$

One can easily show that this sequence satisfies the hypotheses of [3, Th. 1, p. 43]. In view of part (i) of our lemma, the function $\lambda x\phi\begin{pmatrix}x, \bar{a}\\0, \alpha\end{pmatrix}$ is the function ψ referred to in that theorem. The conclusion of that theorem is the desired result.

COROLLARY 2. *Assume $\bar{\alpha}$ is a limit ordinal. If*

$$\phi\begin{pmatrix}x, 1\\0, \bar{\alpha}\end{pmatrix}<t\leqslant\phi\begin{pmatrix}x+1, 1\\0, \quad \bar{\alpha}\end{pmatrix},$$

then

$$\phi\begin{pmatrix}x+1, 1\\0, \quad \bar{\alpha}\end{pmatrix}=\sup_{\alpha<\bar{\alpha}}\phi\begin{pmatrix}t, 1\\0, \alpha\end{pmatrix}.$$

To see this let $\{f_\alpha\}_{\alpha<\bar{\alpha}}$ be the sequence of functions defined by

$$f_\alpha \equiv \lambda x \phi \begin{pmatrix} x, & 1 \\ 0, & \alpha \end{pmatrix}.$$

One can easily show that this sequence satisfies the hypotheses of [3, Th. 1, p. 43]. In view of part (ii) of our lemma, the function $\lambda x \phi \begin{pmatrix} x, & 1 \\ 0, & \bar{\alpha} \end{pmatrix}$ would play the role of ψ in that theorem.

DEFINITION 1.2. $\phi^{(0)}(x) = \phi(x)$

$$\phi^{(n+1)}(x) = \phi(\phi^{(n)}(x)).$$

THEOREM 1.2. *Let $\bar{\alpha}$, $\bar{a}$, $\bar{x}$ denote limit ordinals, then the following relations hold:*

1) $T_\alpha^0 \phi = \phi$ if $\alpha > 0$

2a) $(T_1^1 \phi)\, 0 = \sup_{n<\omega} \phi^{(n)}(0)$

2b) $(T_1^1 \phi)(x+1) = \sup_{n<\omega} \phi^{(n)}([(T_1^1 \phi)\, x] + 1)$

2c) $(T_1^1 \phi)\, \bar{x} = \sup_{x<\bar{x}} [(T_1^1 \phi)\, x]$

3) $T_\alpha^{a+1} = T_\alpha^1 \circ T_\alpha^a$ for $\alpha > 0$

3a) $(T_\alpha^{\bar{a}} \phi)\, 0 = \sup_{a<\bar{a}} [(T_\alpha^a \phi)\, 0]$ if $\alpha > 0$

3b) $(T_\alpha^{\bar{a}} \phi)(x+1) = \sup_{a<\bar{a}} [(T_\alpha^a \phi)([(T_\alpha^{\bar{a}} \phi)\, x] + 1)]$ if $\alpha > 0$

3c) $(T_\alpha^{\bar{a}} \phi)\, \bar{x} = \sup_{x<\bar{x}} [(T_\alpha^{\bar{a}} \phi)\, x]$ if $\alpha > 0$

4) $T_{\alpha+1}^1 = T_1^1 \circ R_\alpha$ where $R_\alpha \phi = \lambda x [(T_\alpha^x \phi)\, 0]$; $\alpha > 0$

4a) $(T_{\bar{\alpha}}^1 \phi)\, 0 = \sup_{\alpha<\bar{\alpha}} [(T_{\bar{\alpha}}^1 \phi)\, 0]$

4b) $(T_{\bar{\alpha}}^1 \phi)(x+1) = \sup_{\alpha<\bar{\alpha}} [(T_\alpha^1 \phi)([(T_{\bar{\alpha}}^1 \phi)\, x] + 1)]$

4c) $(T_{\bar{\alpha}}^1 \phi)\, \bar{x} = \sup_{x<\bar{x}} [(T_{\bar{\alpha}}^1 \phi)\, x]$

Proof.

1) This follows from the fact that $\phi \begin{pmatrix} x \\ 0 \end{pmatrix} = \phi \begin{pmatrix} x, & 0 \\ 0, & \alpha \end{pmatrix}$; [5, (2.2)].

2a) By Lemma 1.5 and [3, p. 44].

2b) By Lemma 1.5 and [3, p. 44].

2c) By continuity of $\lambda x \phi \begin{pmatrix} x, & 1 \\ 0, & 1 \end{pmatrix}$; [5, (3.5)].

3) This is our Lemma 1.3.

3a) By continuity of $\lambda x \phi \begin{pmatrix} 0, & x \\ 0, & \alpha \end{pmatrix}$; [5, (3.5)].

3b) By Corollary 1 to Lemma 1.7.

3c) By continuity of $\lambda x \phi \begin{pmatrix} x, & \bar{a} \\ 0, & \alpha \end{pmatrix}$; [5, (3.5)].

4) This is our Lemma 1.6.

4a) By continuity of $\lambda x \phi \begin{pmatrix} 0, & 1 \\ 0, & x \end{pmatrix}$; [5, (4.3)].

4b) By Corollary 2 to Lemma 1.7.

4c) By continuity of $\lambda x \phi \begin{pmatrix} x, & 1 \\ 0, & \alpha \end{pmatrix}$; [5, (3.5)].

Discussion. Let ϕ run over arbitrary functions (rather than just normal functions). Then 1) defines T_α^0 outright, for $\alpha > 0$. The remaining equations 2a)–4c) define T_α^a, for $a > 0$ and $\alpha > 0$, by transfinite induction. Indeed 2a)–2c) define $(T_1^1 \phi)x$ by transfinite induction on x, so 2a)–2c) define T_1^1. Suppose, for fixed α, that T_α^a is known for all $a < \bar{a}$; then 3a)–3c) define $(T_\alpha^{\bar{a}} \phi)x$ by transfinite induction on x. Thus 3a)–3c) define T_α^a from T_α^1 by transfinite induction on a. T_α^a for $\alpha > 0$ can be regarded as a kind of a-th iterate of the functional T_α^1; indeed, $T_\alpha^{a+1} = T_\alpha^1 \circ T_\alpha^a$ and more generally by our Lemma 1.2, $T_\alpha^{a+b} = T_\alpha^a \circ T_a^\alpha$.

Similarly for $\beta > 0$, 4)–4c) define T_β^b from T_α^a restricted to pairs (a, α) such that $\alpha < \beta$. Thus for $(b, \beta) \neq 1$, the equations 3)–4c) define T_β^b in terms of T_α^a restricted to pairs (a, α) which precede (b, β) in the lexicographic ordering in which α dominates. As already mentioned, 2a)–2c) define T_1^1. Thus 1)–4c) define T_α^a by transfinite induction over the set of pairs (a, α) in the lexicographic ordering.

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