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by HANS MARTIN REIMANN

This note contains an example of a 2-dimensional surface in 3-space, which is represented by an absolutely continuous (in the sense of Tonelli) homeomorphism f. Although the surface has finite Lebesgue area and f is a mapping "of bounded distortion" with L^2 – integrable partial derivatives, there exists a 2-dimensional zero set which is mapped onto a set of positive 2-dimensional Hausdorff measure.

A real valued continuous function f defined in a bounded domain G^k in k-dimensional Euclidean space E^k is absolutely continuous in the sense of Tonelli if:

(i) Given any closed interval $I^k \subset G^k$, $I^k = \{(x_1, \ldots x_k) | a_i \leqslant x_i \leqslant b_i, i=1, \ldots k\}$ f is absolutely continuous as a function of x_i on a.e. line parallel to the x_i axis; $i=1,\ldots k$;

(ii) The partial derivatives which exist a.e. are integrable in G^k. For mappings $f=(f_1,\ldots f_n): G^k \to E^n$ we write $f \in ACL^p (p>1)$, if all coordinate functions f_i, $i=1,\ldots n$, are absolutely continuous in the sense of Tonelli and furthermore the partial derivatives are integrable to the power p.

Cesari [1952] proved that mapping s $f \in ACL^p, p>2, f: G^2 \to E^2$ have the following property: Every subset of G^2 of zero (2-dim.) measure is mapped onto a set of zero measure. We will refer to this property by saying that f satisfies condition N with respect to 2-dimensional Lebesgue measure $m_2: N(m_2)$. In the same paper Cesari presented examples of mappings $f \in ACL^2, f: G^2 \to E^2$, which do not satisfy condition $N(m_2)$ and give rise to further phenomena. Some of Cesari's examples are based on conformal representations as the one below.

Cesari's result carries over to higher dimensions: Calderon [1951] has shown that mappings $f \in ACL^p, p>k, f: G^k \to E^k$ are generalized Lipschitzian in the sense of Rado-Reichelderfer [1955]. From their results it then follows that f satisfies condition $N(m_k)$. This result still holds if $f \in ACL^p$, $p>k$, is a mapping $f: G^k \to E^n$, $n>k$. Condition N is then satisfied with respect to k-dimensional Hausdorff measure H_k.

If $f \in ACL^k$ is a homeomorphism, $f: G^k \to E^k$, one can also conclude that f satisfies $N(m_k)$. This is well known for $k=2$ (for a proof see e.g. Lehto-Virtanen [1965] p.158). A proof for the case $k>2$ has been given by Reshetnjak [1966].

A mapping $f \in ACL^k, f: G^k \to E^k$ is said to be of bounded distortion if there exists a constant $C \geqslant 1$ such that

$$|df|^k \leqslant C\, Jf$$

holds a.e. in G^k. Here $Jf(x)$ is the (signed) Jacobian and $|df(x)|$ is the norm of the linear transformation $df(x)$, which is given by the partial derivatives of f at x. For mappings $f \in ACL^k$, $f: G^k \to E^n$, $n>k$, we interpret this condition as $|df|^k \leqslant C \|Jf\|$

a.e. in G^k with

$$\|Jf\| = \left(\frac{1}{k!}\sum\left[\frac{\partial(f_{\alpha_1}, \dots f_{\alpha_k})}{\partial(x_1, \dots x_k)}\right]^2\right)^{1/2},$$

where the sum in this expression extends over all multiindices $\alpha=(\alpha_1,\dots \alpha_k)$, $1\leqslant\alpha_i\leqslant n$. (Intuitively $\|Jf\|$ denotes the "surface element".) To guarantee that $f:G^2\to E^3$, $f\in ACL^2$, is of bounded distortion it is sufficient to verify that a.e. in G^2

$$\sum_{i,j}\left(\frac{\partial f_i}{\partial x_j}\right)^2 \leqslant C'\,\|Jf\|$$

for some constant C'.

From Reshetnjak's work [1967] it is known that mappings $f\in ACL^k$, $f:G^k\to E^k$, which are of bounded distortion, satisfy $N(m_k)$. The homeomorphisms of bounded distortion are the quasiconformal mappings (see e.g. Gehring [1962]). The investigation of extremal length properties of quasiconformal mappings leads to the following question: Do homeomorphisms $f:G^k\to E^n$, $n>k$, $f\in ACL^k$, which are of bounded distortion, satisfy condition $N(H_k)$? The following example provides a negative answer to this question.

Let J be an Osgood curve, i.e. a closed Jordan curve in the plane with positive 2-dimensional measure. J separates the plane into a bounded and an unbounded component. We map the unit square $Q=\{(x, y)\,|\,0<x<1, 0<y<1\}$ conformally onto the bounded component J^0. By the Carathéodory extension theorem this mapping h can be extended continuously and one to one to a mapping h_c of the closed square $\bar{Q}$ onto $J^0\cup J$. Furthermore we can choose h in such a way as to have $A=\{(x, y)\,|\,x=0, 0<y<1\}$ mapped onto a set of positive 2-dimensional measure.

We define now the continuous mapping $g=(u, v):R\to J^0\cup J$ by setting $R=\{(x, y)\,|\,0\leqslant|x|<1, 0<y<1\}$ and

$$g(x, y) = \begin{cases} h_c(x, y) & \text{for } (x, y)\in Q\cup A \\ h(-x, y) & \text{otherwise} \end{cases}$$

Next we construct an auxiliary function $w:R\to E^1$ in terms of the bounded positive function $a(x, y)=\min\{1, |h'(x, y)|\}:Q\to E^1$, where $h=(u, v)$, $|h'|^2=|u_xv_y-u_yv_x| =u_x^2+v_x^2=u_y^2+v_y^2>0$. We define

$$w(x, y) = \begin{cases} \inf\limits_{\gamma}\displaystyle\int_{\gamma} a(x, y)\, ds & \text{for } (x, y)\in Q \\ 0 & \text{for } (x, y)\in R\setminus Q \end{cases}$$

where the infimum is taken over all rectifiable curves $\gamma\subset Q$ connecting (x, y) with A. $w(x, y)$ is positive for all $(x, y)\in Q$ since $a(x, y)$ is positive and continuous in Q.

THEOREM. The mapping $f=(u, v, w): R \to E^3$ constructed above has all the properties:

a) $f \in ACL^2$

b) f is a homeomorphism

c) f is of bounded distortion

d) f maps the set A (with $H_2(A)=0$) onto a set B with $H_2(B)>0$.

a) w satisfies a uniform Lipschitz condition with constant 1, hence $w \in ACL^2$. $g=(u, v)$ is conformal in Q and maps Q onto a bounded domain. Therefore

$$\int_R |g'|^2 \, dx \, dy = 2 \int_Q |g'|^2 \, dx \, dy < \infty ,$$

which means that the partial derivatives of u and v are square integrable. In order to show that $g \in ACL^2$ it is sufficient to prove that for a.e. y, $0<y<1$, $g(x, y)$ is absolutely continuous as a function of x. We choose y in such a way that $V(y)=\int_{-1}^{1} |g'(x, y)| \, dx < \infty$. For these values the function $g(x, y)$ is absolutely continuous in x, since it has an integral representation

$$g(x, y) = g(x, 0) + \int_0^x g'(t, y) \, dt$$

and the total variation $V(y)$ is finite.

b) Because $w(x, y) \neq 0$ for $(x, y) \in Q$, f is a homeomorphism.

c) F satisfies the distortion condition $|df|^2 \leqslant C \|Jf\|$ a.e. in R. For $(x, y) \in R \setminus \bar{Q}$ this is clearly true for any constant $C \geqslant 1$. In the case $(x, y) \in Q$ we obtain the following estimates:

$$\|Jf\| \geqslant |u_x v_y - u_y v_x| = |g'|^2$$

and

$$|w_x| \leqslant \left| \lim_{h \to 0} h^{-1} \int_x^{x+h} a(t, y) \, dt \right| \leqslant a(x, y) \leqslant |g'(x, y)|$$

From this we conclude

$$|(u_x, v_x, w_x)|^2 \leqslant |g'|^2 + a^2 \leqslant 2 \, |g'|^2 .$$

An analogous relation holds for the derivatives with respect to y and therefore $|df|^2 \leqslant C \|Jf\|$ for any $C \geqslant 4$. This clearly is not the best estimate. We remark that by replacing the function $a(x, y)$ in the definition for $w(x, y)$ by $c \cdot a(x, y)$, c constant, we obtain $C \to 1$ for $c \to 0$.

d) f does not satisfy condition $N(H_2)$

The set $A=\{(x, y)\,|\,x=0, 0<y<1\}$ has zero 2-dimensional measure ($H_2(A)=0$) and f maps A onto a set B with $H_2(B)>0$. (Observe that $H_2(B)=m_2(B)$, since B lies in the plane $w=0$.)

We add a few remarks:

1) f does not satisfy condition N with respect to 2-dimensional integralgeometric measure I_2. Using the characterization of I_2 given by Federer [1947] p. 145, this statement can easily be verified.

2) Since $f\in ACL^2$, the Lebesgue area of f is given by $L(f)=\int_R \|Jf\|\, dx\, dy$. f therefore is an example of a homeomorphism with the property that $L(f)\neq H_2(f(R))$. A similar example of such a mapping has been constructed by Breckenridge [1970].

3) $g: R\to E^2$ is another example of a mapping of the type described by Cesari: $g\in ACL^2$ does not satisfy condition $N(m_2)$.

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University of Michigan, Ann Arbor.

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