A Problem of Mapping a Finite Set into a Set of Positive Measure

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1. Let E^n be the Euclidean n-dimensional space and m_n the Lebesgue measure in E^n. Let G be a group of transformations acting on E^n. Let $A \subset E^n$ be a measurable set having positive measure and $P_1, P_2, \ldots, P_k \in E^n$. One can ask the following question: Does there exist $g \in G$ such that $g(P_i) \in A$ for $i = 1, 2, \ldots, k$?

We shall denote by h_λ the homothety $h_\lambda : E^n \to E^n$ defined by $h_\lambda(P) = \lambda P$ for all $P \in E^n$. The group of all homotheties h_λ $(\lambda > 0)$ will be denoted by H and the group of translations by T. By SO_n we shall denote the special orthogonal group of degree n. The elements of SO_n are the proper rotations of E^n fixing the origin O.

It follows from a result of Hadwiger [3] that the answer to the question mentioned above is positive when $G = HT$. In this note we shall prove that the answer is positive in the following situation: the origin $O \in E^n$ is a density point of A, $P_i \neq O$ $(i = 1, 2, \ldots, k)$ and $G = H \times SO_n$.

2. Let us recall some definitions. $B(P, r)$ will denote the open ball in E^n with center P and radius $r > 0$. A point $P \in E^n$ is a *density point* of a measurable set $A \subset E^n$ if:

$$\lim_{r \to 0+} \frac{m_n(B(P, r) \cap A)}{m_n(B(P, r))} = 1.$$

The Lebesgue density theorem ([4], Lemma 9, p. 194) asserts that almost every point of A is a density point of A.

The sphere in E^n with center O and radius $r > 0$ will be denoted by $S(r)$. In particular, $S = S(1)$ is the unit sphere in E^n. Each point $P \in E^n \backslash \{O\}$ can be written uniquely in the form $P = r(P)\,\sigma(P)$ where $r(P) > 0$ and $\sigma(P) \in S$. The number $r(P)$ and the point $\sigma(P)$ are sometimes called the polar coordinates of P (see, for instance, [5], p. 149). If $A \subset E^n$, we define

$$A_r = \{\sigma(P) \mid P \in A \cap S(r)\}$$

for every $r > 0$. Evidently, $A_r \subset S$.

Every $g \in SO_n$ is a linear transformation of E^n. The matrix of g with respect to a fixed orthonormal basis is an orthogonal matrix with determinant one. Hence, SO_n can be indentified with a compact subset of E^{n^2}. With respect to the induced topology, SO_n is a compact topological group. A subgroup of SO_n fixing a point other than O

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is closed and isomorphic to SO_{n-1}. The homogeneous space SO_n/SO_{n-1} is homeomorphic to the unit sphere S if $n \geqslant 2$ ([2], p. 33).

It is known ([1], p. 116–117) that there exists a Borel measure m on S which is invariant under the rotations. This measure is uniquely determined up to a multiplicative constant. If m is suitably normalized, then:

$$m_n(A) = \int_0^\infty m(A_r) r^{n-1}\, dr \tag{1}$$

holds for every Borel set $A \subset E^n$ (see, for instance, [5], p. 149–150).

3. Our main result is contained in the following:

THEOREM: *Let $A_1, A_2, \ldots, A_k \subset E^n$ be measurable sets having the origin O as a common density point. Let $P_1, P_2, \ldots, P_k$ be points in E^n* (not necessarily distinct) *such that $P_i \neq O$ ($i = 1, 2, \ldots, k$). Then there exists $g \in G = H \times SO_n$ such that $g(P_i) \in A_i$ ($i = 1, 2, \ldots, k$).*

We need first to prove two lemmas.

LEMMA 1: *Let $A \subset E^n$ be a Borel set such that the origin O is a density point of A. Let $r_1, r_2, \ldots, r_k$ be positive reals. Then, given $\varepsilon > 0$, there exists $\lambda > 0$ such that*

$$m(A_{\lambda r_i}) > m(S)(1-\varepsilon) \quad (i = 1, 2, \ldots, k).$$

Proof: Let, for instance, $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_k$ and

$$F = \{ r > 0 \mid m(A_r) \leqslant m(S)(1-\varepsilon) \}.$$

If the assertion of the lemma is false, then

$$\bigcup_{i=1}^k r_1 r_i^{-1} F = (0, \infty).$$

To each $r > 0$ there corresponds at least one i such that

$$m_1(r_1 r_i^{-1} F \cap (0, r)) \geqslant \frac{r}{k}$$

where m_1 is the Lebesgue measure on the real line. We infer that:

$$m_1\Big(F \cap (0, r)\Big) \geqslant m_1\left(F \cap \left(0, \frac{r r_i}{r_1}\right)\right) \geqslant \frac{r r_i}{k r_1} \geqslant \frac{r r_k}{k r_1}. \tag{2}$$

By (1) we have

$$m_n(B(O, r_0)\backslash A) = \int_0^{r_0} m(S\backslash A_r)\, r^{n-1}\, dr \geqslant \varepsilon\, m(S) \int_{F \cap (0, r_0)} r^{n-1}\, dr$$

for every $r_0 > 0$. If $a = m_1(F \cap (0, r_0))$ then

$$\int_{F \cap (0, r_0)} r^{n-1}\, dr \geqslant \int_0^a r^{n-1}\, dr = \frac{a^n}{n}.$$

Using (2) we get the estimate

$$m_n(B(O, r_0)\backslash A) > \frac{1}{n}\, \varepsilon\, m(S) \left(\frac{r_0 r_k}{k r_1}\right)^n.$$

This contradicts our hypothesis that O is a density point of A.

LEMMA 2: *Let $B_1, B_2, \ldots, B_k$ be Borel subsets of S such that*

$$m(B_i) > m(S)\left(1 - \frac{1}{k^2}\right) \quad (i = 1, 2, \ldots, k).$$

If $Q_1, Q_2, \ldots, Q_k \in S$ then there exists $g_0 \in SO_n$ such that

$$g_0(Q_i) \in B_i \quad (i = 1, 2, \ldots, k).$$

Proof: For $B = B_1 \cap B_2 \cap \ldots \cap B_k$ we have the following inequality:

$$m(B) > m(S)\left(1 - \frac{1}{k}\right).$$

We shall show that in fact there exists $g_0 \in SO_n$ such that

$$g_0(Q_i) \in B \quad (i = 1, 2, \ldots, k).$$

Let $Q \in S$ be fixed. The mapping $\phi: SO_n \to S$ defined by

$$\phi(g) = g(Q) \quad \text{for all} \quad g \in SO_n$$

is continuous. If $C \subset S$ is a Borel set, than also $\phi^{-1}(C)$ is a Borel set in SO_n. We define

$$m'(C) = \mu(\phi^{-1}(C))$$

where μ is the Haar measure on SO_n normalized so that $\mu(SO_n) = m(S)$. It is immediate that m' is a Borel measure on S and that m' is invariant under rotations. Therefore we must have $m' = m$.

In particular, we take $Q=Q_1, Q_2, \ldots, Q_k$ successively to get

$$\mu(U_i) = m(B) > m(S)\left(1 - \frac{1}{k}\right)$$

where

$$U_i = \{g \in SO_n \mid g(Q_i) \in B\}.$$

It follows that

$$\mu\left(\bigcap_{i=1}^{k} U_i\right) > 0,$$

and we can choose $g_0 \in \bigcap_{i=1}^{k} U_i$ arbitrarily.

Lemma 2 is proved.

Proof of the Theorem: Since O is a common density point of $A_1, A_2, \ldots, A_k$ it is also a density point of $A = A_1 \cap A_2 \cap \ldots \cap A_k$ (see [4], Lemma 11, p. 194). Therefore we can assume that $A_1 = A_2 = \cdots = A_k = A$. For any measurable set $C \subset E^n$, there exists a Borel set B such that $B \subset C$ and $m_n(C \setminus B) = 0$ (see [4], Lemma 8, p. 194). Using this fact, we can assume (without loss of generality) that the set A is a Borel set.

By Lemma 1 choose $\lambda > 0$ so that

$$m(A_{\lambda r_i}) > m(S)\left(1 - \frac{1}{k^2}\right) \quad (i = 1, 2, \ldots, k)$$

where $r_i = r\,(P_i)$ $(i = 1, 2, \ldots, k)$. Let $Q_i = \sigma(P_i)$ and choose $g_0 \in SO_n$ so that

$$g_0(Q_i) \in A_{\lambda r_i} \quad (i = 1, 2, \ldots, k).$$

Such g_0 exists by Lemma 2.

It is clear that $g = h_\lambda g_0$ satisfies

$$g(P_i) \in A \quad (i = 1, 2, \ldots, k).$$

The theorem is proved.

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