Comparison Domains for the Problem of the Angular Derivative

Autor(en): **Eke, B.G.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **46 (1971)**

PDF erstellt am: **25.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-35508

Comparison Domains for the Problem of the Angular Derivative

B. G. EKE

1. Introduction

Let S be a simply connected domain in the $w=u+iv$ plane having an accessible boundary point, w_∞, located at $w=\infty$. Suppose $w=w(z)=u(z)+iv(z)$ maps $\Sigma=\{z=x+iy:0<y<\pi\}$ $(1-1)$ and conformally onto S so that $w(x+i\pi/2)\to w_\infty$ as $x\to+\infty$. If in any substrip $\{z:0<\delta\leqq y\leqq\pi-\delta\}$ of Σ the difference $w(z)-z$ tends to a finite limit as $\operatorname{Re}(z)\to+\infty$, then we say that S [or $w(z)$] has a (finite) angular derivative at w_∞ [or $\operatorname{Re}(z)=+\infty$]. The problem of determining geometrical conditions on S which imply or are implied by the existence of an angular derivative has long been studied, (see e.g. [7] Chapter VI for results prior to 1955 and [2], [4], [8], [9]). A necessary and sufficient condition for S to have an angular derivative at $\operatorname{Re}(w)=+\infty$ has been given when S is contained in $0<v<\pi$ ([8] Theorem 6, [4], [7] p. 215) and when S contains $0\leqq v\leqq\pi$ ([7], p. 216). For more general S less is known and in this paper we give a necessary and sufficient condition on a class of strip-like domains which need neither contain nor be contained in a strip of width π. These domains may be useful as interior comparison domains for sufficiency (see e.g. [9], Theorem 2) and as exterior comparison domains for necessity investigations into this aspect of the study of the boundary behaviour for quite general classes of simply connected domains.

Suppose $\{u_n\}_1^\infty$, $\{v_n\}_1^\infty$, $\{v_n'\}_1^\infty$ are sequences of real numbers such that

$$u_{n+1}-u_n\geqq d>0(n=1,2,\ldots);\quad \lim_{n\to\infty} v_n=0;\quad \lim_{n\to\infty} v_n'=\pi;\quad v_n<v_n'(n=1,2,\ldots)$$

and define for $n=1,2,\ldots,$

$$\theta_n=v_n'-v_n,\qquad \lambda_n=\max\left(|v_{n+1}'-v_n'|,\quad |v_{n+1}-v_n|\right).$$

We consider throughout the remainder of the paper simply connected domains S which are the interior of the union of the rectangles $\{w=u+iv:u_n\leqq u\leqq u_{n+1};$ $v_{n+1}\leqq v\leqq v_{n+1}'\}$ $(n=1,2,\ldots)$ and the half strip $\{w=u+iv:u\leqq u_1,\ v_1\leqq v\leqq v_1'\}$, and the maps $w=w(z)=u(z)+iv(z)$ of Σ onto S for which

$$\lim_{x\to+\infty} u(x+i\pi/2)=+\infty,\qquad \lim_{x\to-\infty} u(x+i\pi/2)=-\infty.$$

Recently Warschawski ([9], Theorem 1) has proved that the convergence of

$$\sum_{n=1}^{\infty} |\pi - \theta_{n+1}| (u_{n+1} - u_n) \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n^2 \log 1/\lambda_n$$

is sufficient to ensure S has an angular derivative at $\mathrm{Re}(w) = +\infty$. For the strip-like domains under consideration the difference $w(z) - z$ will tend to a finite limit for unrestricted approach to $\mathrm{Re}(z) = +\infty$ whenever the angular derivative exists. We prove the following

THEOREM. *If S is a strip-like domain for which either*

$$\sum_{n=1}^{\infty} ((\pi - \theta_{n+1})/\theta_{n+1}) (u_{n+1} - u_n) \tag{1}$$

$$\textit{or} \quad \sum_{n=1}^{\infty} \lambda_n^2 \log 1/\lambda_n \tag{2}$$

is convergent, then a necessary and sufficient condition for S to have an angular derivative at $\mathrm{Re}(w) = +\infty$ *is the convergence of the other sum.*

The special case $v_n' = v_n + \pi$ $(n = 1, 2, \ldots)$ was considered by Ferrand [5] and with Dufresnoy [6] and they showed $\sum_{n=1}^{\infty} \lambda_n^2$ convergent necessary and $\sum_{n=1}^{\infty} \lambda_n^{3/2}$ convergent sufficient for an angular derivative. Warschawski's result indicates that the convergence of (2) is sufficient and we can now assert that this is also necessary.

Also the convergence of (2) may already be implied by the convergence of (1).

If this is so then consideration of equation (24) enables us to assert that the convergence of (1) is necessary and sufficient for an angular derivative. For example, suppose S is contained in $0 < v < \pi$, then $\pi - \theta_{n+1} \geqq \lambda_{n+1}$ and so the convergence of (1) implies that of $\sum_{n=1}^{\infty} \lambda_n$ (and hence that of (2)). In this case the theorem is contained in [8], Theorem 6 and [4].

The author is grateful to Professor S. E. Warschawski for the opportunity of reading [9] prior to publication.

2. Some deductions from the Poisson integral

Let $A_n (D_n)$ and $B_n (C_n)$ be the two vertices on the lower (upper) boundary of S which have abscissa u_n and where the interior angles are $\pi/2$, $3\pi/2$ respectively. Suppose $z = \alpha_n, \beta_n, \gamma_n + i\pi, \delta_n + i\pi$ ($\alpha_n, \beta_n, \gamma_n, \delta_n$ real) are the pre-images of A_n, B_n, C_n, D_n respectively under $w = w(z)$. (The boundaries of S and Σ are in $(1-1)$ correspondence).

Denote by I_n $(n = 1, 2, \ldots)$ whichever of the intervals $\alpha_n < x < \beta_n$ or $\beta_n < x < \alpha_n$ is not empty and by I_n' $(n = 1, 2, \ldots)$ either $\gamma_n < x < \delta_n$ or $\delta_n < x < \gamma_n$. Then

$$\left.\begin{aligned}
&\lim_{\xi\to+\infty} v(\xi)=0,\ \lim_{\xi\to+\infty} v(\xi+i\pi)=\pi,\ \lim_{\xi\to-\infty} v(\xi)=v_1,\ \lim_{\xi\to-\infty} v(\xi+i\pi)=v_1',\\
&\frac{dv(\xi)}{d\xi}=0\left(\xi\notin\bigcup_{n=1}^{\infty} I_n\right),\ \frac{dv(\xi+i\pi)}{d\xi}=0\left(\xi\notin\bigcup_{n=1}^{\infty} I_n'\right),\\
&\int_{I_n}\frac{dv(\xi)}{d\xi}\,d\xi=v_{n+1}-v_n,\ \int_{I'_n}\frac{dv(\xi+i\pi)}{d\xi}\,d\xi=v_{n+1}'-v_n'(n=1,2,\ldots).
\end{aligned}\right\}\quad(3)$$

Now $v(z)$ is harmonic in Σ, has continuous boundary values except at the two infinite boundary points, and remains bounded in Σ. So, from the Poisson integral representation,

$$\pi v(z)=\pi v(x+iy)=\int_{-\infty}^{\infty} v(\xi)\,d\arctan(e^{\xi}-e^{x}\cos y)/e^{x}\sin y$$

$$+\int_{-\infty}^{\infty} v(\xi+i\pi)\,d\arctan(e^{\xi}+e^{x}\cos y)/e^{x}\sin y.$$

Integrating by parts,

$$\pi v(z)=\frac{\pi}{2}(\pi-\theta_1)+\theta_1 y-\int_{-\infty}^{\infty}\frac{dv(\xi)}{d\xi}\arctan(e^{\xi}-e^{x}\cos y)/e^{x}\sin y\,d\xi$$

$$-\int_{-\infty}^{\infty}\frac{dv(\xi+i\pi)}{d\xi}\arctan(e^{\xi}+e^{x}\cos y)/e^{x}\sin y)\,d\xi,$$

whence

$$\left.\begin{aligned}
\pi\frac{\partial v(z)}{\partial x}&=\int_{-\infty}^{\infty}\frac{dv(\xi)}{d\xi}\,\frac{e^{\xi+x}\sin y}{e^{2x}+e^{2\xi}-2e^{\xi+x}\cos y}\,d\xi\\
&+\int_{-\infty}^{\infty}\frac{dv(\xi+i\pi)}{d\xi}\,\frac{e^{\xi+x}\sin y}{e^{2x}+e^{2\xi}+2e^{\xi+x}\cos y}\,d\xi,
\end{aligned}\right\}\quad(4)$$

$$\left.\begin{aligned}
\pi\frac{\partial v(z)}{\partial y}&=\theta_1-\int_{-\infty}^{\infty}\frac{dv(\xi)}{d\xi}\,\frac{e^{2x}-e^{x+\xi}\cos y}{e^{2x}+e^{2\xi}-2e^{\xi+x}\cos y}\,d\xi\\
&+\int_{-\infty}^{\infty}\frac{dv(\xi+i\pi)}{d\xi}\,\frac{e^{2x}+e^{x+\xi}\cos y}{e^{2x}+e^{2\xi}+2e^{\xi+x}\cos y}\,d\xi.
\end{aligned}\right\}\quad(5)$$

For $x_1<x_2$, the Cauchy-Riemann equations, (3) and (5) give, interchanging the orders of integration,

$$\pi u(x_2 + i\pi/2) - \pi u(x_1 + i\pi/2) = \pi \int_{x_1}^{x_2} \frac{\partial v(x + i\pi/2)}{\partial y}\, dx$$

$$= \theta_1(x_2 - x_1) - \int_{x_1}^{x_2} dx \int_{-\infty}^{\infty} \frac{dv(\xi)}{d\xi} \frac{e^{2x}}{e^{2x} + e^{2\xi}}\, d\xi$$

$$+ \int_{x_1}^{x_2} dx \int_{-\infty}^{\infty} \frac{dv(\xi + i\pi)}{d\xi} \frac{e^{2x}}{e^{2x} + e^{2\xi}}\, d\xi$$

$$= \theta_1(x_2 - x_1) - \sum_{n=1}^{\infty} \int_{I_n} \frac{1}{2} \frac{dv(\xi)}{d\xi} \log\left(\frac{e^{2x_2} + e^{2\xi}}{e^{2x_1} + e^{2\xi}}\right) d\xi$$

$$+ \sum_{n=1}^{\infty} \int_{I'_n} \frac{1}{2} \frac{dv(\xi + i\pi)}{d\xi} \log\left(\frac{e^{2x_2} + e^{2\xi}}{e^{2x_1} + e^{2\xi}}\right) d\xi.$$

Since $\lim_{x_1 \to -\infty}(\pi u(x_1 + i\pi/2) - \theta_1 x_1)$ exists finitely (k say) we may let $x_1 \to -\infty$ and obtain

$$\left.\begin{aligned} \pi u(x_2 + i\pi/2) &= k + \theta_1 x_2 - \frac{1}{2} \sum_{n=1}^{\infty} \int_{I_n} \frac{dv(\xi)}{d\xi} \log(1 + e^{2(x_2 - \xi)})\, d\xi \\ &+ \frac{1}{2} \sum_{n=1}^{\infty} \int_{I'_n} \frac{dv(\xi + i\pi)}{d\xi} \log(1 + e^{2(x_2 - \xi)})\, d\xi. \end{aligned}\right\} \qquad (6)$$

If x_2 exceeds the values in I_n and I'_n, then

$$-\frac{1}{2} \int_{I_n} \frac{dv(\xi)}{d\xi} \log(1 + e^{2(x_2 - \xi)})\, d\xi + \frac{1}{2} \int_{I'_n} \frac{dv(\xi + i\pi)}{d\xi} \log(1 + e^{2(x_2 - \xi)})\, d\xi$$

$$= -\int_{I_n} \frac{dv(\xi)}{d\xi} (x_2 - \xi)\, d\xi + \int_{I'_n} \frac{dv(\xi + i\pi)}{d\xi} (x_2 - \xi)\, d\xi$$

$$-\frac{1}{2} \int_{I_n} \frac{dv(\xi)}{d\xi} \log(1 + e^{2(\xi - x_2)})\, d\xi + \frac{1}{2} \int_{I'_n} \frac{dv(\xi + i\pi)}{d\xi} \log(1 + e^{2(\xi - x_2)})\, d\xi$$

$$= ((x_2 - \beta_n) + O(|\alpha_n - \beta_n|))(v_n - v_{n+1}) + ((x_2 - \gamma_n) + O(|\gamma_n - \delta_n|))(v'_{n+1} - v'_n)$$

$$-\frac{1}{2} \int_{I_n} \frac{dv(\xi)}{d\xi} \log(1 + e^{2(\xi - x_2)})\, d\xi + \frac{1}{2} \int_{I'_n} \frac{dv(\xi + i\pi)}{d\xi} \log(1 + e^{2(\xi - x_2)})\, d\xi,$$

since $dv(\xi)/d\xi$, $dv(\xi+i\pi)/d\xi$ have constant sign on each I_n, I_n' respectively, and where $O(1)$ denotes a quantity which is bounded independently of n. Thus if x_2 exceeds the values in I_n, I_n' $(1 \leqq n \leqq \nu)$, we obtain from (6),

$$\left.\begin{aligned} \pi u(x_2 + i\pi/2) - \pi x_2 = k &+ (\theta_{\nu+1} - \pi)\, x_2 \\ &+ \sum_{n=1}^{\nu} [\beta_n(v_{n+1} - v_n) - \gamma_n(v_{n+1}' - v_n')] \\ &+ O\left(\sum_{n=1}^{\nu} (|\alpha_n - \beta_n| + |\gamma_n - \delta_n|)\,\lambda_n\right) \\ &- \frac{1}{2}\sum_{n=1}^{\nu} \int_{I_n} \frac{dv(\xi)}{d\xi} \log(1 + e^{2(\xi - x_2)})\, d\xi \\ &- \frac{1}{2}\sum_{n=\nu+1}^{\infty} \int_{I_n} \frac{dv(\xi)}{d\xi} \log(1 + e^{2(x_2 - \xi)})\, d\xi \\ &+ \frac{1}{2}\sum_{n=1}^{\nu} \int_{I'_n} \frac{dv(\xi + i\pi)}{d\xi} \log(1 + e^{2(\xi - x_2)})\, d\xi \\ &+ \frac{1}{2}\sum_{n=\nu+1}^{\infty} \int_{I'_n} \frac{dv(\xi + i\pi)}{d\xi} \log(1 + e^{2(x_2 - \xi)})\, d\xi . \end{aligned}\right\} \quad (7)$$

3. Preliminary results and estimate of $\beta_n - \gamma_n$

Since S is semi-conformal (for definition, see e.g. [8]) at $\mathrm{Re}(w) = +\infty$, we know that

$$w'(z) \to 1 \,(\mathrm{Re}(z) \to +\infty \quad \text{with} \quad 0 < \sigma \leqq y \leqq \pi - \sigma: \qquad (8)$$

see e.g. [8], p. 87)

$$\sup_{0 \leqslant y_1, y_2 \leqslant \pi} |u(x + iy_1) - u(x + iy_2)| \to 0 \qquad (9)$$

$(x \to +\infty$: this is implied by e.g. [3], p. 629 or [8], p. 92).

From (8), (9) and $u_{n+1} - u_n \geqq d > 0$ (all n), it follows that $(\alpha_{n+1} - \alpha_n)/(u_{n+1} - u_n)$ tends to 1 as $n \to \infty$ and so we eventually have $\alpha_{n+1} - \alpha_n \geqq d/2$ whence

$$\sum_{n=1}^{\infty} e^{-|x - \alpha_n|} < A(d) \qquad (10)$$

where $A(d)$ denotes some positive constant depending only on d. Let $\hat{\alpha}_n$ be the real number for which $|\alpha_n - \hat{\alpha}_n| = \lambda_n$ and no points of I_n lie between α_n and $\hat{\alpha}_n$. Define $\hat{\beta}_n$, $\hat{\gamma}_n$, $\hat{\delta}_n$ analogously. Then $\hat{\alpha}_{n+1} - \hat{\alpha}_n \geqq d/2$ eventually and so

$$\sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{1}{\cosh(\alpha_j - \hat{\alpha}_n) - 1} < A(d). \quad (n = 1, 2, \ldots) \tag{11}$$

Similar results to (11) holding for the β_j, γ_j, δ_j.

We also need ([9], Lemma 1)

$$\lambda_n^{-1} |\alpha_n - \beta_n|, \lambda_n^{-1} |\gamma_n - \delta_n| < A \quad (n = 1, 2, \ldots). \tag{12}$$

where A is some absolute constant.

LEMMA 1. (i) $u(\beta_n + i\pi/2) - u(\beta_n) = \frac{1}{\pi}(v_{n+1} - v_n) \log \lambda_n$

$$+ O\left(\lambda_n + \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \lambda_j \left(\frac{1}{\cosh(\beta_j - \hat{\beta}_n) - 1} + \frac{1}{\cosh(\alpha_j - \hat{\alpha}_n) - 1}\right)\right) \quad (n \to \infty),$$

(ii) $u(\gamma_n + i\pi) - u(\gamma_n + i\pi/2) = \frac{1}{\pi}(v'_{n+1} - v'_n) \log \lambda_n$

$$+ O\left(\lambda_n + \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \lambda_j \left(\frac{1}{\cosh(\gamma_j - \hat{\gamma}_n) - 1} + \frac{1}{\cosh(\delta_j - \hat{\delta}_n) - 1}\right)\right) \quad (n \to \infty).$$

Proof. We prove (i), (ii) being similar. From (4), we obtain

$$\left.\begin{aligned} \pi u\left(\hat{\beta}_n + \frac{i\pi}{2}\right) - \pi u(\hat{\beta}_n) &= \int_0^{\pi/2} \frac{\partial v(\hat{\beta}_n + iy)}{\partial x} dy \\ &= -\frac{1}{2} \sum_{j=1}^{\infty} \int_{I_j} \frac{dv(\xi)}{d\xi} \log\left(\frac{e^{2\beta_n} + e^{2\xi}}{e^{2\beta_n} + e^{2\xi} - 2e^{\beta_n + \xi}}\right) d\xi \\ &\quad + \frac{1}{2} \sum_{j=1}^{\infty} \int_{I_j'} \frac{dv(\xi + i\pi)}{d\xi} \log\left(\frac{e^{2\beta_n} + e^{2\xi}}{e^{2\beta_n} + e^{2\xi} + 2e^{\xi + \beta_n}}\right) d\xi. \end{aligned}\right\} \tag{13}$$

Now $$-\frac{1}{2}\int_{I_r}\frac{dv(\xi)}{d\xi}\log\left(\frac{e^{2\beta_n}+e^{2\xi}}{(e^{\xi}-e^{\beta_n})^2}\right)d\xi=\int_{I_n}\frac{dv(\xi)}{d\xi}\log|e^{\xi-\beta_n}-1|\,d\xi$$

$$-\frac{1}{2}\int_{I_n}\frac{dv(\xi)}{d\xi}\log(1+e^{2(\xi-\beta_n)})\,d\xi$$

$$=(v_{n+1}-v_n)\log\lambda_n+\int_{I_n}\frac{dv(\xi)}{d\xi}\log\left(\frac{|e^{\xi-\beta_n}-1|}{\lambda_n}\right)d\xi$$

$$-\frac{1}{2}\int_{I_n}\frac{dv(\xi)}{d\xi}\log(1+e^{2(\xi-\beta_n)})\,d\xi.$$

If $\xi\in I_n$, then (12) shows that $\lambda_n^{-1}\,|e^{(\xi-\beta_n)}-1|$ takes values lying between two absolute positive constants. Also $1<1+e^{2(\xi-\beta_n)}<1+e^{A}$ independently of n and so the contribution to the nth term in the first sum in (13) is

$$(v_{n+1}-v_n)\log\lambda_n+O(\lambda_n).\quad (n\to\infty).$$

If $j\neq n$, then for $\xi\in I_n$, and as $n\to\infty$,

$$\log\left(\frac{e^{2\beta_n}+e^{2\xi}}{e^{2\beta_n}+e^{2\xi}-2e^{\xi+\beta_n}}\right)=\log\left(1+\frac{1}{\cosh(\xi-\hat{\beta}_n)-1}\right)<\frac{1}{\cosh(\xi-\hat{\beta}_n)-1}$$
$$=\frac{O(1)}{\cosh(\beta_j-\hat{\beta}_n)-1}.$$

The logarithm in the integrand of the second sum of (13) can be similarly estimated and combining our remarks we obtain

$$u\left(\hat{\beta}_n+\frac{i\pi}{2}\right)-u(\hat{\beta}_n)=\frac{1}{\pi}(v_{n+1}-v_n)\log\lambda_n+O\left(\sum_{\substack{j=1\\ j\neq n}}^{\infty}\frac{\lambda_j}{\cosh(\beta_j-\hat{\beta}_n)-1}+\lambda_n\right).\qquad (n\to\infty).$$

Similarly

$$u\left(\hat{\alpha}_n+\frac{i\pi}{2}\right)-u(\hat{\alpha}_n)=\frac{1}{\pi}(v_{n+1}-v_n)\log\lambda_n+O\left(\sum_{\substack{j=1\\ j\neq n}}^{\infty}\frac{\lambda_j}{\cosh(\alpha_j-\hat{\alpha}_n)-1}+\lambda_n\right).\qquad (n\to\infty).$$

These two relations enable us to deduce (i) since the difference between $u(\hat{\beta}_n+i\pi/2)$ and $u(\hat{\alpha}_n+i\pi/2)$ is approximately that between $\hat{\alpha}_n$ and $\hat{\beta}_n$ (by (8)) which is $O(\lambda_n)$ by (12).

LEMMA 2. *As* $n \to \infty$,

$$\beta_n - \gamma_n = \frac{\mu_n}{\pi}\{v_{n+1} - v_n + v'_{n+1} - v'_n\} \log \lambda_n + O(\lambda_n + \Sigma'_n),$$

where Σ'_n *denotes the four sums which appear in Lemma* 1, *and* $\lim_{n\to\infty} \mu_n = 1$.

Proof. Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where $\Gamma_k (1 \leqq k \leqq 3)$ are straight line segments in Σ joining respectively β_n to $\beta_n + i\pi/2$; $\beta_n + i\pi/2$ to $\gamma_n + i\pi/2$; $\gamma_n + i\pi/2$ to $\gamma_n + i\pi$. The change in $u(z)$ as z describes Γ is 0. From (8),

$$u\left(\gamma_n + \frac{i\pi}{2}\right) - u\left(\beta_n + \frac{i\pi}{2}\right) = \mu_n^{-1}(\gamma_n - \beta_n)$$

where $\mu_n \to 1$ as $n \to \infty$. Now Lemma 2 follows from Lemma 1.

4. Continuation of proof of Theorem

Suppose x_2 satisfies $\max(\alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu) < x_2 < \min(\alpha_{\nu+1}, \beta_{\nu+1}, \gamma_{\nu+1}, \delta_{\nu+1})$. Then all the integrals in (7) may be estimated by

$$O\left(\sum_{n=1}^{\infty} \frac{\lambda_n}{\exp(|x_2 - \alpha_n|)}\right)$$

which is bounded independently of x_2, by (10). Using (12) we find

$$\left.\begin{aligned} \pi u(x_2 + i\pi/2) - \pi x_2 = (\theta_{\nu+1} - \pi)\, x_2 + \sum_{n=1}^{\nu} \{\beta_n(v_{n+1} - v_n) \\ - \gamma_n(v'_{n+1} - v'_n)\} + O(1 + \sum_{n=1}^{\nu} \lambda_n^2). \end{aligned}\right\} \tag{14}$$

Now $\beta_n(v_{n+1} - v_n) - \gamma_n(v'_{n+1} - v'_n)$ is equal to

$$(\beta_n - \gamma_n)(v_{n+1} - v_n) - \gamma_n(\theta_{n+1} - \theta_n) \quad \text{and} \quad (\beta_n - \gamma_n)(v'_{n+1} - v'_n) - \beta_n(\theta_{n+1} - \theta_n).$$

We substitute the first of these expressions if $|v_{n+1} - v_n| = \lambda_n$ and the second otherwise. Noting that $(v_{n+1} - v_n + v'_{n+1} - v'_n)$ has, if not zero, the same sign as the v-difference with absolute value λ_n, we obtain from (14) and Lemma 2,

$$\begin{aligned} \pi u(x_2 + i\pi/2) - \pi x_2 = (\theta_{\nu+1} - \pi)\, x_2 + \sum_{n=1}^{\nu} \frac{\mu_n}{\pi} |v_{n+1} - v_n + v'_{n+1} - v'_n|\, \lambda_n \log \lambda_n \\ - \sum_{n=1}^{\nu} t_n(\theta_{n+1} - \theta_n) + O(1 + \sum_{n=1}^{\nu} \lambda_n^2 + \sum_{n=1}^{\nu} (\lambda_n \Sigma'_n)), \end{aligned}$$

where t_n is β_n if $|v'_{n+1}-v'_n|<\lambda_n$ and γ_n otherwise. Thus

$$\left.\begin{aligned}\pi u(x_2+i\pi/2)-\pi x_2 &= (x_2-t_\nu)(\theta_{\nu+1}-\pi)\\ &\quad+\sum_{n=1}^{\nu}\frac{\mu_n}{\pi}|v_{n+1}-v_n+v'_{n+1}-v'_n|\,\lambda_n\log\lambda_n\\ &\quad+\sum_{n=1}^{\nu-1}(t_{n+1}-t_n)(\theta_{n+1}-\pi)\\ &\quad+O\Big(1+\sum_{n=1}^{\nu}\lambda_n^2+\sum_{n=1}^{\nu}(\lambda_n\Sigma'_n)\Big).\end{aligned}\right\} \tag{15}$$

5. Estimation of $t_{n+1}-t_n$.

Let T_n be either β_n or $\gamma_n+i\pi$ according to whether t_n is β_n or γ_n. Then

$$\left.\begin{aligned}u_{n+1}-u_n &= \int_{t_n}^{t_{n+1}}\frac{\partial u(x+i\pi/2)}{\partial x}\,dx\\ &\quad+u(t_n+i\pi/2)-u(T_n)+u(T_{n+1})-u(t_{n+1}+i\pi/2).\end{aligned}\right\} \tag{16}$$

From (5),

$$\left.\begin{aligned}\pi\int_{t_n}^{t_{n+1}}\frac{\partial u(x+i\pi/2)}{\partial x}\,dx &= \int_{t_n}^{t_{n+1}}dx\Big\{\theta_1-\sum_{j=1}^{n}\int_{I_j}\frac{dv(\xi)}{d\xi}\Big(1-\frac{e^{2\xi}}{e^{2x}+e^{2\xi}}\Big)d\xi\\ &\quad-\sum_{j=n+1}^{\infty}\int_{I_j}\frac{dv(\xi)}{d\xi}\,\frac{e^{2x}}{e^{2x}+e^{2\xi}}\,d\xi\\ &\quad+\sum_{j=1}^{n}\int_{I_j'}\frac{dv(\xi+i\pi)}{d\xi}\Big(1-\frac{e^{2\xi}}{e^{2x}+e^{2\xi}}\Big)d\xi\\ &\quad+\sum_{j=n+1}^{\infty}\int_{I_j'}\frac{dv(\xi+i\pi)}{d\xi}\Big(\frac{e^{2\xi}}{e^{2x}+e^{2\xi}}\Big)d\xi\Big\}\\ &= \theta_{n+1}(t_{n+1}-t_n)+\frac{1}{2}\sum_{j=1}^{n}\Big\{\int_{I_j}\frac{dv(\xi)}{d\xi}\\ &\quad\times\log\Big(\frac{1+e^{2(\xi-t_n)}}{1+e^{2(\xi-t_{n+1})}}\Big)d\xi\end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned}&-\int_{I_j'}\frac{dv(\xi+i\pi)}{d\xi}\log\left(\frac{1+e^{2(\xi-t_n)}}{1+e^{2(\xi-t_{n+1})}}\right)d\xi\Bigg\}\\&+\frac{1}{2}\sum_{j=n+1}^{\infty}\Bigg\{\int_{I_j}\frac{dv(\xi)}{d\xi}\log\left(\frac{1+e^{2(t_n-\xi)}}{1+e^{2(t_{n+1}-\xi)}}\right)d\xi\\&-\int_{I_j'}\frac{dv(\xi+i\pi)}{d\xi}\log\left(\frac{1+e^{2(t_n-\xi)}}{1+e^{2(t_{n+1}-\xi)}}\right)d\xi\Bigg\}.\end{aligned}\right\}\tag{17}$$

If $1\leqq j<n$, then

$$\left.\begin{aligned}&\int_{I_j}\frac{dv(\xi)}{d\xi}\log\left(\frac{1+e^{2(\xi-t_n)}}{1+e^{2(\xi-t_{n+1})}}\right)d\xi-\int_{I_j'}\frac{dv(\xi+i\pi)}{d\xi}\log\left(\frac{1+e^{2(\xi-t_n)}}{1+e^{2(\xi-t_{n+1})}}\right)d\xi\\&=\log\left(\frac{1+e^{2(\alpha_j-t_n)}}{1+e^{2(\alpha_j-t_{n+1})}}\right)\cdot(\theta_j-\theta_{j+1})\\&\quad+O\left(\lambda_j\sup_{\xi\in I_j\cup I_j'}\log\left(\frac{1+e^{2(\xi-t_n)}}{1+e^{2(\alpha_j-t_n)}}\cdot\frac{1+e^{2(\alpha_j-t_{n+1})}}{1+e^{2(\xi-t_{n+1})}}\right)\right)\\&=(\omega_{nj}-\omega_{n+1,j})(\theta_j-\theta_{j+1})+O\left(\lambda_j\max\left(\frac{\lambda_j\log\lambda_j^{-1}+\Sigma_j'}{\exp(2|\alpha_j-t_n|)},\right.\right.\\&\quad\left.\left.\frac{\lambda_{j+1}\log\lambda_{j+1}^{-1}+\Sigma_{j+1}'}{\exp(2|\alpha_j-t_{n+1}|)}\right)\right),\end{aligned}\right\}\tag{18}$$

where $\omega_{nj}=\log(1+e^{-2|\alpha_j-t_n|})$, since $\alpha_j<t_n$ if $j<n$ and n is large enough. If $j=n$ and $\alpha_n\leqq t_n$, then (18) holds before and if $\alpha_n>t_n$ we introduce an error of order of magnitude at most $\lambda_n^2\log\lambda_n^{-1}+\lambda_n\Sigma_n'$ in the first term in (18) which can be accommodated by the second term. Thus (18) is valid if $1\leqq j\leqq n$. A similar calculation shows that the jth term in the second sum in (17) can also be estimated by the expression in (18).

Thus, from (16), (17), (18), we find as $n\to\infty$,

$$\left.\begin{aligned}t_{n+1}-t_n&=\frac{\pi}{\theta_{n+1}}\{u_{n+1}-u_n+u(t_{n+1}+i\pi/2)-u(t_n+i\pi/2)\\&\quad-u(T_{n+1})+u(T_n)\}-\frac{1}{2\theta_{n+1}}\sum_{j=1}^{\infty}(\omega_{nj}-\omega_{n+1,j})(\theta_j-\theta_{j+1})\\&\quad+O\left(\sum_{j=1}^{\infty}\lambda_j\max\left(\frac{\lambda_j\log\lambda_j^{-1}+\Sigma_j'}{\exp(2|\alpha_j-t_n|)},\frac{\lambda_{j+1}\log\lambda_{j+1}^{-1}+\Sigma_{j+1}'}{\exp(2|\alpha_j-t_{n+1}|)}\right)\right).\end{aligned}\right\}\tag{19}$$

6. Completion of proof of Theorem

From (15), (19) we obtain for $\max(\alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu) < x_2 < \min(\alpha_{\nu+1}, \beta_{\nu+1}, \gamma_{\nu+1}, \delta_{\nu+1})$,

$$\left.\begin{aligned}
\pi u(x_2 + i\pi/2) - \pi x_2 &= (x_2 - t_\nu)(\theta_{\nu+1} - \pi) + \frac{1}{\pi}\sum_{n=1}^{\nu} \mu_n |v_{n+1} \\
&\quad - v_n + v'_{n+1} - v'_n| \lambda_n \log \lambda_n - \pi \sum_{n=1}^{\nu-1} \left(\frac{\pi - \theta_{n+1}}{\theta_{n+1}}\right)(u_{n+1} - u_n) \\
&\quad - \frac{1}{2}\sum_{n=1}^{\nu-1}\sum_{j=1}^{\infty} \left(\frac{\theta_{n+1} - \pi}{\theta_{n+1}}\right)(\omega_{nj} - \omega_{n+1,j})(\theta_j - \theta_{j+1}) \\
&\quad + \pi \sum_{n=1}^{\nu-1} \left(\frac{\theta_{n+1} - \pi}{\theta_{n+1}}\right)(u(t_{n+1} + i\pi/2) - u(t_n + i\pi/2) \\
&\quad - u(T_{n+1}) + u(T_n)) + O\left(1 + \sum_{n=1}^{\nu} \lambda_n^2 + \sum_{n=1}^{\nu} (\lambda_n \Sigma'_n)\right) \\
&\quad + O\left(\sum_{n=1}^{\nu-1}\sum_{j=1}^{\infty} \lambda_j (\theta_{n+1} - \pi) \max\left(\frac{\lambda_j \log \dfrac{1}{\lambda_j} + \Sigma'_j}{\exp(2|\alpha_j - t_n|)},\right.\right. \\
&\quad \left.\left.\frac{\lambda_{j+1} \log \dfrac{1}{\lambda_{j+1}} + \Sigma'_{j+1}}{\exp(2|\alpha_j - t_{n+1}|)}\right)\right).
\end{aligned}\right\} \quad (20)$$

Now

$$\left|\sum_{n=1}^{\nu-1}\sum_{j=1}^{\infty} (\omega_{nj} - \omega_{n+1,j})(\theta_j - \theta_{j+1})\left(\frac{\theta_{n+1} - \pi}{\theta_{n+1}}\right)\right|$$

$$\begin{aligned}
&= \left|\sum_{j=1}^{\infty} (\theta_j - \theta_{j+1})\left\{\sum_{n=1}^{\nu-1} \omega_{nj}\left(\frac{\theta_{n+1} - \pi}{\theta_{n+1}} - \frac{\theta_n - \pi}{\theta_n}\right)\right.\right. \\
&\quad \left.\left. + \omega_{1j}\left(\frac{\theta_1 - \pi}{\theta_1}\right) - \omega_{\nu j}\left(\frac{\theta_\nu - \pi}{\theta_\nu}\right)\right\}\right| \\
&\leqslant \sum_{j=1}^{\infty}\sum_{n=1}^{\nu-1} \frac{4\pi\lambda_n\lambda_j}{\theta_n\theta_{n+1}} \omega_{nj} + \left|\frac{\theta_1 - \pi}{\theta_1}\right| \sum_{j=1}^{\infty} 2\lambda_j\omega_{1j} + \left|\frac{\theta_\nu - \pi}{\theta_\nu}\right| \sum_{j=1}^{\infty} 2\lambda_j\omega_{\nu j},
\end{aligned}$$

since $|\theta_j-\theta_{j+1}|\leqslant 2\lambda_j$. But $\omega_{nj}\sim\exp(-2|\alpha_j-t_n|)$ $(|n-j|\to\infty)$ and so the single sums converge. A simple calculation on the sum over j (using $2\lambda_n\lambda_j\leqq\lambda_n^2+\lambda_j^2$) shows that $\sum_{n=1}^{\nu-1}\sum_{j=1}^{\infty}\lambda_n\lambda_j\omega_{nj}$ is $O\left(\sum_{n=1}^{\nu-1}\lambda_n^2\right)+o(1),(\nu\to\infty)$. Similarly, using (11), $\sum_{n=1}^{\nu-1}(\lambda_n\Sigma_n')$ is $O\left(\sum_{n=1}^{\nu-1}\lambda_n^2\right)+o(1),(\nu\to\infty)$.

Next

$$\begin{aligned}
&\sum_{n=1}^{\nu-1}\left(\frac{\theta_{n+1}-\pi}{\theta_{n+1}}\right)(u(t_{n+1}+i\pi/2)-u(t_n+i\pi/2)-u(T_{n+1})+u(T_n))\\
&\qquad=\pi\sum_{n=1}^{\nu-1}\{\theta_{n+1}^{-1}(u(t_n+i\pi/2)-u(t_{n+1}+i\pi/2)+u(T_{n+1})-u(T_n))\}\\
&\qquad\quad+[u(t_\nu+i\pi/2)-u(T_\nu)]+[u(T_1)-u(t_1+i\pi/2)]\\
&\qquad=\pi\sum_{n=1}^{\nu-1}\left(\frac{1}{\theta_n}-\frac{1}{\theta_{n+1}}\right)(u(T_n)-u(t_n+i\pi/2))+O(1)\quad(\nu\to\infty)\\
&\qquad=-\sum_{n=1}^{\nu-1}\frac{|\theta_{n+1}-\theta_n|}{\theta_n\theta_{n+1}}\lambda_n\log\frac{1}{\lambda_n}\\
&\qquad\quad+O\left(1+\sum_{n=1}^{\nu-1}\lambda_n^2+\sum_{n=1}^{\nu-1}(\lambda_n\Sigma_n')\right)\quad(\nu\to\infty),
\end{aligned}$$

taking the part of Lemma 1 for which the v-difference is λ_n.

Finally in our discussion of (20) we look at the last term. This is bounded by

$$\left.\begin{aligned}
&\sum_{j=1}^{\infty}\max\left(\lambda_j^2\log\frac{1}{\lambda_j},\lambda_{j+1}^2\log\frac{1}{\lambda_{j+1}}\right)\sum_{n=1}^{\nu-1}\frac{|\theta_{n+1}-\pi|}{\exp(2\min(|\alpha_j-t_n|,|\alpha_j-t_{n+1}|))}\\
&\qquad+O\left(\sum_{n=1}^{\nu-1}\sum_{j=1}^{\infty}\lambda_j\left(\frac{\Sigma_j'}{e^{2|\alpha_j-t_n|}}+\frac{\Sigma_{j+1}'}{e^{2|\alpha_j-t_{n+1}|}}\right)\right)
\end{aligned}\right\}\qquad(21)$$

If $j\leqslant\nu$, the coefficient of $\lambda_j^2\log 1/\lambda_j$ in the first term of (21) is at most

$$2\sum_{n=1}^{\infty}\frac{|\theta_{n+1}-\pi|}{\exp(2\min(|\alpha_j-t_n|,|\alpha_{j-1}-t_n|,|\alpha_j-t_{n+1}|,|\alpha_{j-1}-t_{n+1}|))}$$

a function of j which tends to 0 as $j\to\infty$.

The total contribution to the first term in (21) from the terms with $j > \nu$ is bounded independently of ν. A typical triple sum in the second term of (21) is

$$\sum_{n=1}^{\nu-1} \sum_{j=1}^{\infty} \sum_{\substack{m=1 \\ m \neq j}}^{\infty} \frac{\lambda_j \lambda_m}{\exp(2|\alpha_j - t_n|) \cdot (\cosh(\alpha_m - \hat{\alpha}_j) - 1)} \tag{22}$$

and we now show that this is $O(\sum_{n=1}^{\nu-1} \lambda_n^2) + O(1)$, $O(1)$ independent of ν as $\nu \to \infty$.

By using (8), (9) as in section 3 we find that

$$\exp(2|\alpha_j - t_n|) \geqq A \exp(|j-n|\, d)$$
$$\cosh(\alpha_m - \hat{\alpha}_j) - 1 \geqq A \exp(|m-j|\, d/2) \quad (m \neq j)$$

where A denotes a positive absolute constant which need not necessarily be the same each time it occurs.

Thus (22) is of order of magnitude at most

$$\sum_{n=1}^{\nu-1} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \frac{\lambda_j^2}{e^{|j-n|d} e^{|m-j|d/2}} + \sum_{n=1}^{\nu-1} \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \frac{\lambda_m^2}{e^{|j-n|d} e^{|m-j|d/2}} . \tag{23}$$

The first triple sum in (23) is bounded by

$$A(d) \sum_{n=1}^{\nu-1} \sum_{j=1}^{\infty} \frac{\lambda_j^2}{e^{|j-n|d}} = A(d) \left(\sum_{n=1}^{\nu-1} \sum_{j=1}^{\nu-1} + \sum_{n=1}^{\nu-1} \sum_{j=\nu}^{\infty} \right) \leqslant A(d) \sum_{n=1}^{\nu-1} \lambda_n^2 + A(d).$$

If $q \geqq \nu$, then the coefficient of λ_q^2 in the second triple sum in (23) is

$$\left(\sum_{n=1}^{\nu-1} \sum_{j=1}^{\nu-1} + \sum_{n=1}^{\nu-1} \sum_{j=\nu}^{\infty} \right) \frac{1}{e^{|j-n|d} e^{|q-j|d/2}} \leqq A(d) \sum_{j=1}^{\nu-1} \frac{1}{e^{|q-j|d/2}}$$

$$+ A \sum_{j=\nu}^{\infty} \frac{1}{e^{|j-\nu+1|d} e^{|q-j|d/2}} \leqq \frac{A(d)}{e^{(q-\nu+1)d/2}} + \frac{A(q-\nu+1)}{e^{(q-\nu+1)d/2}} + \frac{A(d)}{e^{(q-\nu+1)d}},$$

since $|q-j| + |j-\nu+1| = q-\nu+1$ if $\nu \leqq j \leqq q$.

If $q < \nu$, λ_q^2 has coefficient

$$\sum_{n=1}^{\nu-1} \sum_{j=1}^{\infty} \frac{1}{e^{|q-j|d/2 + |j-n|d}} \leqq A(d) \sum_{j=1}^{\infty} \frac{1}{e^{|q-j|d/2}} \leqq A(d).$$

Thus the triple sum in (23) (and so expression (22)) is bounded by

$$A(d)\left(\sum_{n=1}^{\nu-1}\lambda_n^2+\sum_{n=\nu}^{\infty}\frac{(n-\nu+1)\lambda_n^2}{e^{(n-\nu+1)d/2}}\right)\leqq A(d)\sum_{n=1}^{\nu-1}\lambda_n^2+A(d).$$

Similarly for each of the triple sums in the second term of (21).

Thus (20) gives

$$\begin{aligned}\pi u(x_2+i\pi/2)-\pi x_2 &= (x_2-t_\nu)(\theta_{\nu+1}-\pi)\\ &-\sum_{n=1}^{\nu-1}\lambda_n\log\frac{1}{\lambda_n}\left\{\frac{\mu_n}{\pi}|v_{n+1}-v_n+v'_{n+1}-v'_n|\right.\\ &\left.+\pi\left|\frac{\theta_{n+1}-\theta_n}{\theta_n\theta_{n+1}}\right|\right\}+\frac{\mu_\nu}{\pi}|v_{\nu+1}-v_\nu+v'_{\nu+1}-v'_\nu|\,\lambda_\nu\log\lambda_\nu\\ &-\pi\sum_{n=1}^{\nu-1}\left(\frac{\pi-\theta_{n+1}}{\theta_{n+1}}\right)(u_{n+1}-u_n)+O\left(1+\sum_{n=1}^{\nu-1}{}''\right)+o\left(\sum_{n=1}^{\nu}\varepsilon_n\lambda_n^2\log\frac{1}{\lambda_n}\right).\end{aligned}$$

where $\varepsilon_n\to 0\,(n\to\infty)$ and $\Sigma''^{\nu-1}_{n=1}$ denotes a sum which is bounded by

$$A(d)\sum_{n=1}^{\nu-1}\lambda_n^2+A(d).$$

Either $|v_{n+1}-v_n+v'_{n+1}-v'_n|$ or $|\theta_{n+1}-\theta_n|$ is at least λ_n and both are less than $2\lambda_n$. Also $\theta_n\theta_{n+1}\to\pi^2\,(n\to\infty)$, so we obtain

$$\left.\begin{aligned}u(x_2+i\pi/2)-x_2 &= (x_2-t_\nu)\left(\frac{\theta_{\nu+1}-\pi}{\pi}\right)-\frac{1}{\pi^2}\sum_{n=1}^{\nu-1}\tilde{\omega}_n\lambda_n^2\log\frac{1}{\lambda_n}\\ &-\sum_{n=1}^{\nu-1}\frac{(\pi-\theta_{n+1})}{\theta_{n+1}}(u_{n+1}-u_n)+O\left(1+\sum_{n=1}^{\nu}{}''\right),\end{aligned}\right\}\qquad(24)$$

where $\frac{1}{2}<\tilde{\omega}_n<\frac{5}{2}$ for all n sufficiently large. Also (24) is valid whenever $\alpha_\nu<x_2\leqq\alpha_{\nu+1}$ using (9), and the first term on the r.h.s. of (24) is, in absolute value, no greater than $2|\pi-\theta_{\nu+1}|\theta_{\nu+1}^{-1}(u_{\nu+1}-u_\nu)$ if ν is large enough. Suppose now that S has an angular derivative at $\mathrm{Re}(w)=+\infty$. In particular, this implies that $u(x_2+i\pi/2)-x_2$ has a finite limit as $x_2\to\infty$. Thus if either (1) or (2) is a convergent series then it is necessary that the other series also converges.

To show the convergence of (1) or (2) implies that the convergence of the other sum is sufficient for S to have an angular derivative at $\mathrm{Re}(w)=+\infty$ we note that in either case (24) implies that

$$u(x_2 + i\pi/2) - x_2 = O(1) \quad (x_2 \to \infty). \tag{25}$$

Since we are assuming the convergence of (1) we may apply theorem 3 of [1] to deduce from (25) that

$$\lim_{x \to \infty} (u(x+iy) - x)$$

exists finitely for $0 < \sigma \leqq y \leqq \pi - \sigma$, i.e. S has an angular derivative at $\mathrm{Re}(w) = +\infty$. This completes the proof of the theorem.

REFERENCES

[1] B. G. Eke, *Remarks on Ahlfors' distortion theorem*, J. Analyse Math. *19* (1967), 97–134.
[2] B. G. Eke, *On the angular derivative of regular functions*, Math. Scand. *21* (1967), 122–127.
[3] B. G. Eke and S. E. Warschawski, *On the distortion of conformal maps at the boundary*, J. London Math. Soc. *44* (1969), 625–630.
[4] B. G. Eke, *On the differentiability of conformal maps at the boundary*, Nagoya Math. J. (to appear in Vol. 41).
[5] J. Ferrand, *Extension d'une inégalité de M. Ahlfors*, Comptes rendus, Acad. de Paris *220* (1945), 873–874.
[6] J. Ferrand and J. Dufresnoy, *Extension d'une inégalité de M. Ahlfors et application au problème de la dérivée angulaire*, Bull. des Sci. math. 2e serie, *69* (1945), 165–174.
[7] J. Lelong-Ferrand, *Représentation conforme et transformations à intégrale de Dirichlet bornée* (Gauthier-Villars, Paris 1955).
[8] S. E. Warschawski, *On the boundary behaviour of conformal maps*, Nagoya Math. J. *30* (1967), 83–101.
[9] S. E. Warschawski, *Remarks on the angular derivative*, Nagoya Math. J. (to appear in Vol. 41).

Department of Pure Mathematics,
Queen's University,
Belfast BT7 1NN, Northern Ireland.

Received July 21, 1970