The Central Connection Problem at Turning Points of Linear Differential Equations

The Central Connection Problem at Turning Points of Linear Differential Equations

WOLFGANG WASOW (University of Wisconsin and Eidgenössische technische Hochschule, Zürich)

Abstract

A system of linear differential equations of the vectorial form $\varepsilon\, dy/dx = A(x, \varepsilon)\, y$ is considered, where ε is a positive parameter, and the matrix $A(x, \varepsilon)$ is holomorphic in $|x| \leqslant x_0$, $0 < \varepsilon \leqslant \varepsilon_0$, with an asymptotic expansion $A(x, \varepsilon) \sim \sum_{r=0}^{\infty} A_r(x)\, \varepsilon^r$, as $\varepsilon \to 0$. The eigenvalues of $A_0(x)$ are supposed to coalesce at $x = 0$ so as to make this point a simple turning point. With the help of refinements of the representations for the inner and outer asymptotic solutions, as $\varepsilon \to 0$, that were introduced in the articles [9] and [10] by the author (see the references at the end of the paper), explicit connection formulas between these solutions are calculated. As part of this derivation it is shown that only the diagonal entries of the connection matrix are asymptotically relevant.

1. Introduction

In the neighborhood of turning points the asymptotic evaluation of solutions of linear differential equations involves, in most problems, some "matching" of two different solutions that have known asymptotic properties in different but overlapping regions. Only for a very small subclass of such problems has it been possible to avoid the matching by calculating uniformly valid approximations to the solutions. When one of the two solutions to be matched is asymptotically known in a region that is bounded away from the turning point, while the other has a known expansion in a domain that includes the turning point, the term *central connecting problem* will be used for this matching question. The expressions *inner* (*or interior*) and *outer* (*or exterior*) solutions are sometimes found in the literature.

The first task in the analysis of the central connection problem is to find inner and outer solutions with expansions whose domains of validity actually do overlap. For a fairly general class of turning points I have done this in two previous papers [9], [10]. Nishimoto has generalized my results considerably in his articles [4,] [5], [6]. Still more general turning points have been analyzed in Iwano and Sibuya [3] and in Iwano [1], [2].

Even with the explicit knowledge of expansions with overlapping regions of validity the actual matching is not a trivial task if one desires analytic insight into the structure of the linear relation that connects the inner and outer solutions. In the present paper this matching will be carried out for the system of differential equation treated in my earlier articles [9] and [10].

The matching of an inner with an outer fundamental system of solutions amounts to the calculation of the linear transformation with constant coefficients which takes one of these fundamental systems into the other. One result of this paper (Theorem 5.1) is that for the two fundamental systems calculated in [9], [10] *only the diagonal entries of the transformation matrix are asymptotically significant.* In § 7 *explicit series for these diagonal entries are calculated.*

I had hoped that the considerable simplification of the differential equation in question which I achieved in [12] and [14] would also facilitate the matching procedure This does. however, not seem to be the case. The formulation of the problem here is therefore essentially the same as in [9], [10]. Some non-trivial refinements of the results of those papers are needed. They are discussed in §§ 8, 9.

2. Description of the Problem

We consider the differential equation

$$\varepsilon \frac{dz}{dx} = A(x, \varepsilon)\, z \tag{2.1}$$

for an n-dimensional vector z. The small parameter ε will be taken *positive*. The more general assumption that the domain of ε is a – possibly narrow – sector of the ε-plane would involve only superficial technical complications of the arguments of this paper. The matrix $A(x, \varepsilon)$ is to be holomorphic in $|x| \leqslant x_0$, $0 < \varepsilon \leqslant \varepsilon_0$ and to have an asymptotic expansion

$$A(x, \varepsilon) \sim \sum_{r=0}^{\infty} A_r(x)\, \varepsilon^r, \quad \text{as} \quad \varepsilon \to 0+, \tag{2.2}$$

valid uniformly in $|x| \leqslant x_0$.

If all eigenvalues of $A_0(0)$ are distinct the local asymptotic nature of system (2.1) is completely known (see, e.g., [13]). In the contrary case we shall call $x=0$ a *turning point* of the differential equation (2.1). With the help of a fundamental theorem of Sibuya [7] (see also [13], §§ 25–27) such turning point problems can be substantially simplified. In particular, it can be shown that the hypotheses

$$A_0(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \ldots 0 & 0 \\ 0 & 0 & 1 & 0 \ldots 0 & 0 \\ . & . & . & . \ldots . & . \\ 0 & 0 & 0 & 0 \ldots 0 & 1 \\ 0 & 0 & 0 & 0 \ldots 0 & 0 \end{bmatrix} \tag{2.3}$$

and

$$\frac{d}{dx}[\det A_0(x)]\Big|_{x=0} = (-1)^{n+1} \tag{2.4}$$

are only mildly restrictive, in as much as problems of the form (2.1) with a turning point at $x=0$ can "in general" be transformed into problems satisfying these assumptions. "In general" is to mean that turning point problems not reducible to equations for which (2.2) and (2.3) hold are characterized by additional identities among the entries of $A(x, \varepsilon)$. (See [9].) The hypotheses (2.3), (2.4) will be adopted throughout this paper. A result of Wasow [11] implies that no further generality is lost by assuming that $A_0(x)$ is a companion matrix. This will be done.

The formulation in [9] does not use condition (2.4), but the equivalent Assumption II of [9], § 2. The preliminary transformation of the linear part of $A_0(x)$ in [9] § 2 can be avoided. When this is done the eigenvalues of $A_0(x)$ must be used, as is done below, instead of those of the linear part of $A_0(x)$, as in [9].

The eigenvalues of $A_0(x)$ are decisive for all asymptotic theories of systems such as (2.1). Because of our hypotheses the characteristic polynomial of $A_0(x)$ has the form

$$(-1)^n[\lambda^n + xa_{n-1}(x)\lambda^{n-1} + \cdots + xa_1(x)\lambda - x + O(x^2)],$$

with the $a_j(x), j=1, 2, \ldots, n-1$, holomorphic in $|x| \leqslant x_0$. Hence, the eigenvalues have the form

$$\lambda_j(x) = \omega^{j-1}x^{1/n} + x^{2/n}\tilde{\lambda}_j(x^{1/n}), \quad j = 1, 2, \ldots, n, \tag{2.5}$$

where

$$\omega = e^{2\pi i/n} \tag{2.6}$$

and the $\tilde{\lambda}_j(t)$ are holomorphic in a region which we may take as $|t| \leqslant x_0^{1/n}$ by choosing x_0 small enough.

3. An Outer Solution

It is a classical result that in the neighborhood of every point in $0<|x|\leqslant x_0$ one can find fundamental matrix solutions of (2.1) of the form

$$\check{U}(x, \varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^x \Lambda(\xi)\, d\xi\right\},$$

where

$$\Lambda(x) = \operatorname{diag}(\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)),$$

and the matrix $\check{U}(x, \varepsilon)$ has an asymptotic series in powers of ε:

$$\check{U}(x, \varepsilon) \sim \sum_{r=0}^{\infty} \check{U}_r(x)\, \varepsilon^r, \quad \varepsilon \to 0+.$$

As $x \to 0$, the functions $\check{U}_r(x)$ become unbounded. In [9], [10] I have strengthened this result by analyzing the singularities of $\check{U}_r(x)$ at $x=0$ in detail. Theorem 3.1 is a modification of Theorem 1.1 of [10]. Let

$$\Omega(t) = \operatorname{diag}(1, t, \ldots, t^{n-1})$$

and denote by $S(\theta_1, \theta_2)$ the sector

$$S(\theta_1, \theta_2) = \left\{ x \mid \theta_1 \leqslant \arg x \leqslant \theta_2,\ |x| \leqslant x_0,\ \theta_2 - \theta_1 < \frac{n\pi}{n+1} \right\}. \tag{3.2}$$

THEOREM 3.1. (An Outer Solution.) *Corresponding to the sector $S(\theta_1, \theta_2)$ there exists a fundamental matric solution $U(x, \varepsilon)$ of* (2.1) *of the form*

$$U(x, \varepsilon) = \Omega(x^{1/n})\,(\varepsilon^{-1/(n+1)} x^{1/n})^{(1-n)/2}\, \hat{U}(x, \varepsilon) \exp\left\{ \frac{1}{\varepsilon} \int_0^x \Lambda(\xi)\, d\xi \right\} \tag{3.3}$$

with $\hat{U}(x, \varepsilon)$ bounded for $|\varepsilon x^{-(n+1)/n}| \leqslant \xi_0$, $x \in S(\theta_1, \theta_2)$, $0 < \varepsilon \leqslant \varepsilon_0$. Furthermore,

$$\hat{U}(x, \varepsilon) = \tilde{U}(x, \varepsilon)\, x^{B(\varepsilon)}, \quad \varepsilon \to 0+, \tag{3.4}$$

where $B(\varepsilon)$ is a diagonal matrix with the expansion

$$B(\varepsilon) \sim \sum_{r=1}^{\infty} B_r \varepsilon^r, \tag{3.5}$$

and

$$\tilde{U}(x, \varepsilon) \sim \sum_{r=0}^{\infty} \tilde{U}_r(x^{1/n})\,(\varepsilon x^{-(n+1)/n})^r, \quad \text{as} \quad \varepsilon x^{-(n+1)/n} \to 0. \tag{3.6}$$

The matrices $\tilde{U}_r(t)$ are holomorphic for $|t| \leqslant x_0^{1/n}$, and

$$\tilde{U}_0(0) = \begin{bmatrix} 1 & 1 & \ldots 1 \\ 1 & \omega & \ldots \omega^{n-1} \\ 1 & \omega^2 & \ldots \omega^{2(n-1)} \\ \cdot & \cdot & \ldots \ldots\ldots\ldots \\ 1 & \omega^{n-1} & \ldots \omega^{(n-1)(n-1)} \end{bmatrix}. \tag{3.7}$$

Remarks:

1) The precise meaning of (3.6) is that

$$\left[\tilde{U}(x, \varepsilon) - \sum_{r=C}^{m} \tilde{U}_r(x^{1/n})\,(\varepsilon x^{-(n+1)/n})^r \right] (\varepsilon x^{-(n+1)/n})^{-(m+1)} \tag{3.8}$$

is for every $m \geqslant 0$ a bounded function in the domain of the (x, ε)-space defined by the inequalities $|\varepsilon x^{-(n+1)/n}| \leqslant \xi_0$, $x \in S(\theta_1, \theta_2)$, $0 < \varepsilon \leqslant \varepsilon_0$. The constant ξ_0 is arbitrary, but the bound depends on ξ_0.

2) By definition, $\arg x^k = k \arg x$.

3) The solution $U(x, \varepsilon)$ is not uniquely characterized by its asymptotic representation. This is a universal feature of asymptotic expansions for linear differential equations, as a parameter ε tends to zero, since one may always add another solution that has been multiplied by a scalar function of ε alone which tends to zero with sufficient rapidity. Moreover, there are other solutions to which Theorem 3.1 applies literally, but for which $\tilde{U}_r(x^{1/n})$, $r = 1, 2, \ldots$, are different functions, also holomorphic in the variable x. Such a solution can for instance, be obtained by multiplying $U(x, \varepsilon)$ to the right by a diagonal matrix function of ε alone which has an asymptotic expansion in powers of ε with the identity matrix as leading term.

The relation of Theorem 3.1 to Theorem 1.1 of [10] will be discussed in § 8.

4. An Inner Solution

As always in the asymptotic theory of differential equations, the inner solutions are obtained with the help of suitable "stretching" and "shearing" transformations. In the present problem it is appropriate to set

$$s = \varepsilon^{-n/(n+1)} x \tag{4.1}$$

and

$$z = \Omega(\varepsilon^{1/(n+1)}) V. \tag{4.2}$$

Then equation (2.1) becomes

$$\frac{dv}{ds} = H(s, \varepsilon) v \tag{4.3}$$

with

$$H(s, \varepsilon) \sim \sum_{r=0}^{\infty} H_r(s) \varepsilon^{r/(n+1)}, \qquad \varepsilon \to 0+. \tag{4.4}$$

The precise meaning of (4.4) when s is large is explained in [9] p. 667.

It is easy to show that the equation (4.3) possesses matrix solutions with asymptotic expansions of the form

$$\sum_{r=0}^{\infty} V_r(s) \varepsilon^{r/(n+1)}, \tag{4.5}$$

as $\varepsilon \to 0+$, valid in any bounded disk $|s| \leqslant s_0$. In [9], [10] I have extended the validity of this expansion to domains that expand indefinitely, as $\varepsilon \to 0$, by analyzing the functions $V_r(s)$ near $s=\infty$. Theorem 4.1 is a more precise, modified version of Theorem 1.2 of [10].

THEOREM 4.1. (An Inner Solution.) *Let*

$$\kappa(s)=\begin{cases}0, & |s|\leqslant s_0\\ 1, & |s|>s_0,\end{cases} \tag{4.6}$$

$$Q(s)=\frac{n}{n+1}\,\Omega(\omega)\,s^{(n+1)/n}. \tag{4.7}$$

Equation (2.1) *admits a fundamental matrix solution* $Y(x,\varepsilon)$ *of the form*

$$Y(x,\varepsilon)=\Omega(\varepsilon^{1/(n+1)}s^{\kappa(s)/n})\,s^{\kappa(s)(1-n)/2n}\hat{Y}(s,\varepsilon)\exp Q(s) \tag{4.8}$$

with $\hat{Y}(s,\varepsilon)$ *bounded for* $|\varepsilon^{1/(n+1)}S^{\kappa(s)(n+2)/n}|\leqslant\eta_0$, $0<\varepsilon\leqslant\varepsilon_0$, $S\in\Sigma$, *where* Σ *is the sector*

$$\Sigma=\left.\begin{array}{ll}\{s\mid|\arg s|\leqslant n\pi/2(n+1)\}, & \text{if } n \text{ is odd}\\ \{s\mid-\pi/2\leqslant\arg s\leqslant(n-1)\,\pi/2(n+1)\}, & \text{if } n \text{ is even}.\end{array}\right\} \tag{4.9}$$

Furthermore,

$$\hat{Y}(s,\varepsilon)=\tilde{Y}(s,\varepsilon)\,s^{D(\varepsilon)}, \tag{4.10}$$

where $D(\varepsilon)$ *is a diagonal matrux with the expansion*

$$D(\varepsilon)\sim\sum_{r=1}^{\infty}D_r\varepsilon^{r/(n+1)},\qquad \varepsilon\to0+, \tag{4.11}$$

and, for $|s|\geqslant s_0$, $s\in\Sigma$,

$$\tilde{Y}(s,\varepsilon)\sim\sum_{r=0}^{\infty}\tilde{Y}_r(s)\,(\varepsilon^{1/(n+1)}s^{(n+2)/n})^r,\quad \text{as}\quad \varepsilon^{1/(n+1)}s^{(n+2)/n}\to0. \tag{4.12}$$

The matrices $\tilde{Y}_r(s)$ *have asymptotic expansions*

$$\tilde{Y}_r(s)\sim\sum_{\nu=0}^{\infty}\tilde{Y}_{r\nu}s^{-\nu/n},\quad \text{as}\quad s\to\infty \text{ in } \Sigma, \tag{4.13}$$

and

$$\tilde{Y}_{00}=\tilde{U}_0(0). \tag{4.14}$$

Remarks:

1) The precise meaning of (4.12) is that

$$\left[\tilde{Y}(s,\varepsilon)-\sum_{r=0}^{m}\tilde{Y}_r(s)\,(\varepsilon^{1/(n+1)}s^{(n+2)/n})^r\right](\varepsilon^{1/(n+1)}s^{(n+2)/n})^{-(m+1)} \tag{4.15}$$

is for every $m \geqslant 0$ a bounded function in the domain of the (s, ε)-space defined by the inequalities

$$|\varepsilon^{1/(n+1)} s^{(n+2)/n}| \leqslant \eta_0, \quad |s| \geqslant s_0, \quad s \in \sum, \quad 0 < \varepsilon \leqslant \varepsilon_0 .$$

2) The solution $Y(s, \varepsilon)$ is not uniquely characterized by its asymptotic representation. Moreover, there are other solutions for which Theorem 4.1 is true with different functions $\tilde{Y}_r(s)$. One may, for instance multiply $Y(x, \varepsilon)$ to the right by an arbitrary diagonal matrix function of ε alone which has an asymptotic series in powers of $\varepsilon^{1/(n+1)}$ with the identity matrix as leading term.

3) By symmetry considerations, explained in [8] analogous theorems can be stated for each of a number of sectors, which, together cover all directions in the s-plane.

The proof of Theorem 4.1 on the basis of Theorem 1.2 of [10] will be discussed in § 9. There, one will also find a description of methods by means of which the series expansions for $\tilde{Y}(s, \varepsilon)$ can be calculated.

5. The Structure of the Matching Matrix

The matrix solutions U and Y are related by an identity of the form

$$U(x, \varepsilon) = Y(x, \varepsilon)\, \Gamma(\varepsilon) \tag{5.1}$$

with a matrix $\Gamma(\varepsilon)$ that is independent of x. Since the regions of validity of our expansions for U and Y overlap, $\Gamma(\varepsilon)$ can be calculated asymptotically by substituting any fixed point of that common domain into the relation

$$\Gamma(\varepsilon) = Y^{-1}(x, \varepsilon)\, U(x, \varepsilon) \tag{5.2}$$

and applying Theorems 3.1 and 4.1. The fact that all values of x must yield the same matrix $\Gamma(\varepsilon)$ will now be exploited to get more precise information on the structure of $\Gamma(\varepsilon)$.

We shall write $\gamma_{jk}(\varepsilon)$ for the (j, k)-entry of $\Gamma(\varepsilon)$ and use an analogous notation for all matrices designated by capital letters.

Let θ_1 and θ_2 be chosen so that the bounding rays of the sector $S(\theta_1, \theta_2)$ in (3.2) are inside the sector $\sum$ of (4.9), i.e.,

$$\left.\begin{aligned} \theta_1 &= -\frac{n\pi}{2(n+1)} + \delta, \quad & \theta_2 &= \frac{n\pi}{2(n+1)} - \delta, \quad \text{if } n \text{ is odd},\\ \theta_1 &= -\frac{\pi}{2} + \delta, & \theta_2 &= \frac{(n-1)\pi}{2(n+1)} - \delta, \quad \text{if } n \text{ is even}, \end{aligned}\right\} \tag{5.3}$$

with $\delta > 0$ and sufficiently small.

We shall have to distinguish repeatedly between the regions of applicability of Theorem 3.1 and Theorem 4.1. The definition below is therefore useful.

DEFINITION 5.1. *Let $\xi_0>0$, $\eta_0>0$ begiven. Consider the set of the (x, ε)-space where, with $\arg x=\theta$, the inequalities* (3.2), (5.3) *and* $0<\varepsilon\leqslant\varepsilon_0$ *are true. We donete by* S_i (interior domain) *its subset defined by the inequality*

$$|\varepsilon^{1/(n+1)}s^{\kappa(s)(n+2)/n}|\leqslant\eta. \tag{5.4}$$

and by S_e (exterior domain) *the subset defined by the inequality*

$$|\varepsilon x^{-(n+1)/n}|\leqslant\xi_0. \tag{5.5}$$

The set $S_m=S_i\cap S_e$ will be called the intermediate domain.

It is convenient to set

$$s_0=\xi_0^{-n/(n+1)}, \tag{5.6}$$

for then $|s|\geqslant s_0$, i.e., $\kappa(s)=1$, precisely when (5.5) holds. We also take

$$\varepsilon_0<\xi_0^{n+2}\eta_0^{n+1}, \tag{5.7}$$

which has the advantage that then there are points in S_m for every ε in $0<\varepsilon\leqslant\varepsilon_0$. This follows from the observation that, given ε, the inequalities (5.4), (5.5) together are equivalent to

$$(\xi_0^{-1}\varepsilon)^{n/(n+1)}\leqslant|x|\leqslant(\eta_0\varepsilon)^{n/(n+2)}.$$

LEMMA 5.1:

$$\Gamma(\varepsilon)=\exp[-Q(s)]\,\hat{\Gamma}(x,\varepsilon)\exp Q(s) \tag{5.8}$$

with $\hat{\Gamma}(x,\varepsilon)$ bounded in S_m.

Proof: Substitution of formulas (3.3) and (4.8) into (5.2) yields, after a few cancelations

$$\Gamma(\varepsilon)=\exp[-Q(s)]\,\hat{Y}^{-1}(s,\varepsilon)\,\hat{U}(x,\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^x\Lambda(\xi)\,\mathrm{d}\xi\right\}, \tag{5.9}$$

which is of the form (5.8) with

$$\hat{\Gamma}(x,\varepsilon)=\hat{Y}^{-1}(s,\varepsilon)\,\hat{U}(x,\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^x\Lambda(\xi)\,d\xi-\frac{n}{n+1}\,\Omega(\omega)\,s^{(n+1)/n}\right\}. \tag{5.10}$$

The first two factors in the right member are bounded in S_m, because of Theorems 3.1 and 4.1. To appraise the last factor we re-write formula (2.5) in the form

$$\Lambda(x)=x^{1/n}\Omega(\omega)+x^{2/n}\tilde{\Lambda}(x^{1/n}),$$

where $\tilde{\Lambda}(t)$ is a diagonal matric holomorphic for $|t| \leqslant x_0{}^{1/n}$. Hence,

$$\int_0^x \Lambda(\xi)\, d\xi = \frac{n}{n+1} \Omega(\omega)\, x^{(n+1)/n} + x^{(n+2)/n} \tilde{\tilde{\Lambda}}(x^{1/n}),$$

with $\tilde{\tilde{\Lambda}}(t)$ diagonal and holomorphic in $|t| \leqslant x_0^{1/n}$ Therefore, the last facor in (5.10) is equal to

$$\exp\{\varepsilon^{-1} x^{(n+2)/n} \tilde{\tilde{\Lambda}}(x^{1/n})\}. \tag{5.11}$$

As

$$|\varepsilon^{-1} x^{(n+2)/n}| = |\varepsilon^{1/(n+1)} s^{(n+2)/n}| \leqslant |\varepsilon^{1/(n+1)} s^{\kappa(s)(n+2)/n}| s_0^{(n+2)/n} \leqslant \eta_0 s_0 \quad \text{in} \quad S_i,$$

the lemma is proved.

COROLLARY 5.1: $\hat{\gamma}_{kk}(x, \varepsilon) = \gamma_{kk}(\varepsilon)$, $k = 1, 2, \ldots, n$.

LEMMA 5.2. *If ξ_0 and ε_0 are taken sufficiently small then $y_{jk}(x, \varepsilon) \neq 0$ in S_m.*

Proof: By (4.13), (4.14) and (3.7) all entries of $\tilde{Y}_0(s)$ are different from zero in the part of S_m in which s in sufficiently large. As $|\varepsilon x^{-(n+1)/n}| \leqslant \xi_0$ is equivalent to $|s| \geqslant \xi_0^{-1}$, we may make ξ_0 so small that no entry of $\tilde{Y}_0(s)$ vanishes in S_m. By Theorem 4.1 the matrix $\hat{Y}(s, \varepsilon)$, and hence $Y(x, \varepsilon)$, has then the same property for all sufficiently small ε.

THEOREM 5.1. *If ξ_0 and ε_0 are sufficiently small then, for $j, k = 1, 2, \ldots, n$,*

$$u_{jk}(x, \varepsilon) = \begin{cases} y_{jk}(x, \varepsilon)\,(\gamma_{kk}(\varepsilon) + \mu_{jk}(x, \varepsilon)), & (x, \varepsilon) \in S_m \\ y_{jk}(x, \varepsilon)\,\gamma_{kk}(\varepsilon) + \mu_{jk}(x, \varepsilon), & (x, \varepsilon) \in S_i - S_m \end{cases} \tag{5.12}$$

with

$$\mu_{jk}(x, \varepsilon) \sim 0, \quad as \quad \varepsilon \to 0+. \tag{5.13}$$

Remark: Relation (5.13) is to mean that for every $m \geqslant 0$ there is a constant c_m such that $|\mu_{jk}(x, \varepsilon)| \leqslant c_m \varepsilon^m$, for $(x, \varepsilon) \in S_i$. The two formulas cannot be combined into one, because $y_{jk}(x, \varepsilon)$ may have zeros in $S_i - S_m$.

Proof of Theorem 5.1. Let $x_{jk} = x_{jk}(\varepsilon)$ be such that $(x_{jk}, \varepsilon) \in S_m$ for all ε in $0 < \varepsilon \leqslant \varepsilon_0$ and denote by s_{jk} the corresponding value of s. If $(x, \varepsilon) \in S_m$ then by Theorem 4.1 and Lemmas 5.1, 5.2,

$$\left.\begin{aligned} u_{jk}(x, \varepsilon) - y_{jk}(x, \varepsilon)\,\gamma_{kk}(\varepsilon) &= \sum_{l \neq k} y_{jl}(x, \varepsilon)\,\gamma_{lk}(\varepsilon) \\ &= y_{jk}(x, \varepsilon)\,[\hat{y}_{jk}(s, \varepsilon)]^{-1} \sum_{l \neq k} \hat{y}_{jl}(s, \varepsilon)\,\hat{\gamma}_{lk}(x_{lk}, \varepsilon) \\ &\times \exp\left\{\frac{n}{n+1}(\omega^{k-1} - \omega^{l-1})(s_{lk}^{(n+1)/n} - s^{(n+1)/n})\right\}. \end{aligned}\right\} \tag{5.14}$$

We choose for x_{lk} a value for which

$$\exp\left\{\frac{n}{n+1}\left(\omega^{k-1}-\omega^{l-1}\right)s_{lk}^{(n+1)/n}\right\}$$

is as small as possible. To that end, let

$$\beta_{lk}=\min_{\theta_1\leqslant\theta\leqslant\theta_1}\operatorname{Re}\left\{\frac{n+1}{n}\left(\omega^{k-1}-\omega^{l-1}\right)e^{i\theta(n+1)/n}\right\}\tag{5.15}$$

and denote by θ_{lk} a value of θ for which the minimum is assumed. Observe that

$$\beta_{lk}<0,\qquad l,k=1,2,\ldots,n,\qquad l\neq k,\tag{5.16}$$

in consequence of (5.3), provided δ is taken small enough. For given ε the maximum of $|s|$ in S_m is $\eta^{n/(n+2)}\varepsilon^{-n/(n+1)(n+2)}$. Hence, we choose

$$s_{lk}=\eta_0^{n/(n+2)}\varepsilon^{-n/(n+1)(n+2)}e^{i\theta_{lk}}.\tag{5.17}$$

Let us restrict (x,ε) temporarily to the subset of S_m obtained by replacing the bound η_0 with a smaller one, $\eta_1<\eta_0$. For such values one has

$$\left.\begin{aligned}&\operatorname{Re}\left\{-\frac{n}{n+1}\left(\omega^{k-1}-\omega^{l-1}\right)s^{(n+1)/n}\right\}\\&\quad=-\operatorname{Re}\left\{\frac{n}{n+1}\left(\omega^{k-1}-\omega^{l-1}\right)e^{i\theta(n+1)/n}\right\}|s|^{(n+1)/n}\\&\quad\leqslant-\beta_{lk}\eta_1^{(n+1)/(n+2)}\varepsilon^{-1/(n+2)}.\end{aligned}\right\}\tag{5.18}$$

On the other hand,

$$\operatorname{Re}\left\{\frac{n}{n+1}\left(\omega^{k-1}-\omega^{l-1}\right)s_{lk}^{(n+1)/n}\right\}=\beta_{lk}\eta_0^{(n+1)/(n+2)}\varepsilon^{-1/(n+2)}.\tag{5.19}$$

Returning to (5.14) one concludes from (5.16), (5.18) and (5.19 that the right member of (5.14) tends to zero with exponential order of magnitude, as $\varepsilon\to0+$, uniformly in the subdomain of S_m obtained by replacing η_0 with η_1. Since η_0 was arbitrary, the proof of the first relation in (5.12) is complete.

To prove the relation (5.12) that pertains to $(x,\varepsilon)\in S_i-S_m$, we replace (5.14) by

$$\begin{aligned}&u_{jk}(x,\varepsilon)-y_{jk}(x,\varepsilon)\,\gamma_{kk}(\varepsilon)\\&\quad=\sum_{l\neq k}y_{jl}(x,\varepsilon)\,\hat{\gamma}_{lk}(x_{lk},\varepsilon)\exp\left\{\frac{n}{n+2}\left(\omega^{k-1}-\omega^{l-1}\right)s_{lk}^{(n+1)/n}\right\}.\end{aligned}$$

Since $y_{jl}(x,\varepsilon)$ is bounded in S_i-S_m, as is $\hat{\gamma}_{lk}(x_{lk},\varepsilon)$, the conclusion of Theorem 5.1 follows from the fact, proved before, that the exponential factor in the right member decays, as $\varepsilon\to0+$.

6. Digression on Asymptotic Series of Several Variables

The notation in this section is independent of that of the rest of the paper. The customary multiindex symbolism will be used:

Let $x=(x_1, x_2, \ldots, x_d)$ be a vector with complex components. A multiindex is defined as a vector with nonnegative integral components. By definition, if r is a multiindex,

$$x^r = \prod_{j=1}^{d} x_j^{r_j}.$$

The norm $|x|$ is defined by $|x| = \sum_{j=1}^{d} |x_j|$.

DEFINITION 6.1. *Let f be a complex valued scalar function of x. Let R be a point set in the x-space whose closure contains the origin and which is in the domain of f. If for every nonnegative integer k,*

$$f(x) = \sum_{|r| \leqslant k} f_r x^r + \omega_k(x)\,|x|^k, \quad \textit{with} \quad \lim_{\substack{x\to 0\\ x\in R}} \omega_k(x) = 0, \tag{6.1}$$

the function $f(x)$ is said to have the asymptotic expansion $\sum_r f_r x^r$, as $x \to 0$ in R. This fact will be expressed by writing

$$f(x) \sim \sum_r f_r x^r, \quad \textit{as} \quad x \to 0 \quad \textit{in} \quad R. \tag{6.2}$$

THEOREM 6.1: *Let*

$$f(x) \sim \sum_r f_r x^r, \quad \textit{as} \quad x\to 0 \quad \textit{in} \quad R, \qquad g(x) \sim \sum_s f_s x^s, \quad \textit{as} \quad x \to 0 \quad \textit{in} \quad S.$$

Then

$$f(x)\,g(x) \sim \sum_r \Big(\sum_{s+t=r} f_s g_t \Big) x^r, \quad \textit{as} \quad x \to 0 \quad \textit{in} \quad R\cap S. \tag{6.3}$$

The proof is immediate and is therefore omitted. Theorem 6.1 is vacuous unless $R \cap S$ has the origin as accumulation point.

THEOREM 6.2: *Let f be a function of two vectors x, y defined for $x\in R$, $y \in S$. If*

$$f(x, y) \sim \sum_r f_r(y)\, x^r, \quad \textit{as} \quad x \to 0 \quad \textit{in} \quad R, \tag{6.4}$$

uniformly for $y \in S$ and

$$f_r(y) \sim \sum_s g_{rs} y^s, \quad \textit{as} \quad y \to 0 \quad \textit{in} \quad S, \tag{6.5}$$

then

$$f(x, y) \sim \sum_{r,s} g_{rs} x^r y^s, \quad \textit{as} \quad (x, y) \to 0 \quad \textit{in} \quad R \times S. \tag{6.6}$$

Proof: Formula (6.6) is, of course, understood as an asymptotic relation in the sense of definition 6.1 for the $2d$-dimensional product space of the vectors $(x_1, x_2, \ldots, x_d, y_1, y_2, \ldots, y_d)$. To prove it we write

$$f(x, y) = \sum_{|r| \leqslant k} f_r(y)\, x^r + \omega_k(x, y)\, |x|^k$$

$$f_r(y) = \sum_{|s| \leqslant k} g_{rs} y^s + \omega_{rk}(y)\, |y|^k$$

with $\omega_k(x, y) \rightrightarrows 0$, as $x \to 0$ in R, uniformly for $y \in S$, and $\omega_{rk}(y) \to 0$, as $y \to 0$ in S. It follows that

$$f(x, y) = \sum_{|r| \leqslant k} \Bigl(\sum_{|s| \leqslant k} g_{rs} y_s + \omega_{rk}(y)\, |y|^k \Bigr) x^r + \omega_k(x, y)\, |x|^k$$

$$= \sum_{|r+s| \leqslant k} g_{rs} x^r y^s + \sum_{\substack{|r|,\, |s| \leqslant k \\ |r+s| \leqslant k}} g_{rs} x^r y^s + \Bigl(\sum_{|r| \leqslant k} \omega_{rk}(y)\, x^r \Bigr) |y|^k + \omega_k(x, y)\, |x|^k.$$

It must be shown that each of the last three terms after being divided by $(|x| + |y|)^k$ tends to zero as $(x, y) \to 0$ in RxS. For the last two terms this is obvious. To prove it for the third-to-last term it suffices to observe that the expansion of $(|x| + |y|)^k = = (\sum_{j=1}^{d} (|x_j| + |y_j|))^k$ by the multinomial theorem implies $(|x| + |y|)^k \geqslant |x^a y^b|$ for any two multiindices a, b with $|a| + |b| = j$. It follows that for given multiindices r, s with $|r+s| > k$,

$$\frac{|x^r y^s|}{(|x| + |y|)^k} \leqslant |x|^{r-a} |y|^{s-b},$$

where a, b can be chosen such that $r-a$, $s-b$ are multiindices, not both zero. This completes the proof of Theorem 6.2.

COROLLARY: *Under the assumptions of Theorem* 6.2

$$f(x, x) \sim \sum_r \Bigl(\sum_{a+b=r} g_{ab} \Bigr) x^r, \quad \textit{as} \quad x \to 0 \quad \textit{in} \quad R \cap S$$

7. Calculation of the Connecting Coefficients

The principal importance of Theorem 5.1 is that it reduces the asymptotic calculation of the outer solution in the inner domain to the calculation of the n products $y_{kk}^{-1}(x, \varepsilon)\, u_{kk}(x, \varepsilon)$ at one point. This is immensely simpler than the calculation of the whole matric $Y^{-1}(x, \varepsilon)\, U(x, \varepsilon)$. For simplicity of notation the subscript "kk" will be dropped in this section.

By means of Theorems 3.1 and 4.1 as well as formulas (5.10), (5.11) one obtains the relation

$$y^{-1}(x, \varepsilon)\, u(x, \varepsilon) = x^{b(\varepsilon)} s^{-d(\varepsilon)} \tilde{y}^{-1}(s, \varepsilon)\, \tilde{u}(x, \varepsilon) \exp\left\{\frac{1}{\varepsilon} x^{(n+2)/n} \tilde{\tilde{\lambda}}(x^{1/n})\right\}. \tag{7.1}$$

Here, $b(\varepsilon)$, $d(\varepsilon)$, $\tilde{\tilde{\lambda}}(x^{1/n})$ stand for $b_k(\varepsilon)$, $d_k(\varepsilon)$, $\tilde{\tilde{\lambda}}_k(x^{1/n})$, the kth entries in the diagonals of the matrices $B(\varepsilon)$, $D(\varepsilon)$, $\tilde{\tilde{\Lambda}}(x^{1/n})$ respectively.

For the asymptotic evaluation of (7.1) in the set S_m we introduce the auxiliary vector

$$q = q(x, \varepsilon) = (q_1(x, \varepsilon), q_2(x, \varepsilon), q_3(x, \varepsilon)) \tag{7.2}$$

where

$$q_1 = x^{1/n}, \qquad q_2 = \varepsilon^{1/(n+1)} x^{-1/n} = s^{-1/n}, \qquad q_3 = \varepsilon^{-1} x^{(n+2)/n} = \varepsilon^{1/(n+1)} s^{(n+2)/n}. \tag{7.3}$$

Then

$$\exp\{\varepsilon^{-1} x^{(n+2)/n} \tilde{\tilde{\lambda}}(x^{1/n})\} = \exp\{q_3 \tilde{\tilde{\lambda}}(q_1)\}. \tag{7.4}$$

This is a function of the vector q with a convergent – and hence asymptotic – series in multiindex powers of q in the subset of three-dimensional q-space defined by $|q_1| \leqslant x_0^{1/n}$. Thus

$$\exp\left\{\frac{1}{\varepsilon} x^{(n+2)/n} \tilde{\tilde{\lambda}}(x^{1/n})\right\} \sim \sum_{r,\nu} \lambda_{r\nu} q_1^r q_3^\nu, \quad \text{as} \quad q \to 0. \tag{7.5}$$

The formulas (7.2), (7.3) map the domain S_m of (x, ε)-space into a set R of the q-space whose closure contains the point $q=0$. From Theorem 3.1 we see that

$$\tilde{u}(x, \varepsilon) \sim \sum_{r=0}^{\infty} \tilde{u}_r(q_1)\, q_2^{nr}, \quad \text{as} \quad q \to 0 \text{ in } R \tag{7.6}$$

$$\tilde{u}_r(x^{1/n}) \sim \sum_{\nu=0}^{\infty} \tilde{u}_{r\nu} q_1^\nu, \quad \text{as} \quad q \to 0 \text{ in } R. \tag{7.7}$$

The Corollary to Theorem 6.2 implies then that

$$\tilde{u}(x, \varepsilon) \sim \sum_{r,\nu} \tilde{u}_{r\nu} q_1^\nu q_2^{rn}, \quad \text{as} \quad q \to 0 \text{ in } R. \tag{7.8}$$

The terms of this series have to be ordered according to increasing values of $\nu + rn$.

Similarly, one shows that, as a consequence of Theorem 4.1,

$$\tilde{y}(s, \varepsilon) \sim \sum_{r,\nu} \tilde{y}_{r\nu} q_3^r q_2^\nu, \quad \text{as} \quad q \to 0 \text{ in } R. \tag{7.9}$$

Formulas (3.7) and (4.14) imply that $\tilde{y}_{00} \neq 0$. By an argument that resembles the proof in the one-dimensional case (see, e.g., [13] Ch. III) one can show that $\tilde{y}^{-1}(s, \varepsilon)$ also has an asymptotic expansion, say

$$\tilde{y}^{-1}(s, \varepsilon) \sim \sum_{r, v} \tilde{w}_{rv} q_3^r q_2^v, \quad \text{as} \quad q \to 0 \text{ in } R. \tag{7.10}$$

Multiplication of the series in (7.5), (7.8) and (7.10) leads to a multiple asymptotic of the form

$$\tilde{y}^{-1}(s, \varepsilon)\, \tilde{u}(x, \varepsilon) \exp\left\{\frac{1}{\varepsilon} x^{(n+2)/n} \tilde{\tilde{\lambda}}(x^{1/n})\right\} \sim \sum_p c_p q^p(x, \varepsilon) \text{ in } S_m \tag{7.11}$$

where $p=(p_1, p_2, p_3)$ is a multiindex. Returning to (7.1) and remembering formula (5.12) we conclude that

$$\gamma_{kk}(\varepsilon) = \gamma(\varepsilon) \sim x^{b(\varepsilon)} s^{-d(\varepsilon)} \sum_p c_p q^p(x, \varepsilon), \quad \text{as} \quad \varepsilon \to 0, \tag{7.12}$$

for $(x, \varepsilon) \in S_m$. The coefficients c_p can be calculated explicitly from the λ_{rv}, $\tilde{u}_{rv}$ and $\tilde{y}_{rv}$.

Now we take advantage of the fact that $\gamma(\varepsilon)$ in (7.12) is independent of x by setting

$$x = x(\varepsilon) = \varepsilon^{\alpha}, \quad \text{i.e.,} \quad s = \varepsilon^{-n/(n+1)} x = \varepsilon^{\alpha - n/(n+1)},$$

and vary α in the interval

$$\frac{n}{n+2} < \alpha < \frac{n}{n+1}. \tag{7.13}$$

As one verifies immediately, $(x(\varepsilon), \varepsilon) \in S_m$, and $\lim_{\varepsilon \to 0} q(x(\varepsilon), \varepsilon) = 0$. With this value of x formula (7.12) becomes

$$\gamma(\varepsilon) \sim \varepsilon^{nd(\varepsilon)/(n+1)} \varepsilon^{\alpha f(\varepsilon)} \sum_p c_p \varepsilon^{A_p + B_p \alpha/n}, \quad \text{as} \quad \varepsilon \to 0, \tag{7.14}$$

where

$$b(\varepsilon) - d(\varepsilon) \sim \sum_{r=1}^{\infty} f_r \varepsilon^{r/(n+1)}, \quad \text{as} \quad \varepsilon \to 0, \tag{7.15}$$

$$f_r = \begin{cases} -d_r, & \text{if } r \neq k(n+1) \\ b_k - d_r, & \text{if } r = k(n+1) \end{cases}, \quad k \text{ an integer},$$

$$A_p = \frac{1}{n+1} p_2 - p_3, \qquad B_p = p_1 - p_2 + (n+2)\, p_3. \tag{7.16}$$

It is very plausible that, as $\gamma(\varepsilon)$ is independent of α, the right member of (7.14) must be termwise independent of α. One has, indeed, the following lemma, whose proof will be postponed to the end of this section to avoid an interruption of the main argument.

LEMMA 7.1:
(a) $f(\varepsilon)\sim 0$. *i.e.*, $b(\varepsilon)\sim d(\varepsilon)$, *as* $\varepsilon \to 0+$;
(b) $c_p=0$, *whenever* $B_p \neq 0$.

Note that, in consequence of this Lemma, D_r differs from zero only if r is a multiple of $n+1$.

When $B_p=0$, the p-term of the summation in (7.14) becomes, in a more explicit notation, $c_{p_1,\, p_2,\, p_3}\varepsilon^{(p_2-(n+1)p_3)/(n+1)}$ with $p_1=p_2-(n+2)p_3$. As $p_1 \geqslant 0$ this shows that also $p_2-(n+1)p_3 \geqslant 0$ for all nonzero terms of the summation in (7.14). Collecting the finitely many terms for which $p_2-(n+1)p_3=r$ and steting $p_3=\nu$, we can write

$$\sum_p c_p \varepsilon^{A_p} \sim \sum_{r=0}^{\infty} \check{\gamma}_r \varepsilon^{r/(n+1)}, \quad \text{as} \quad \varepsilon \to 0+$$

with

$$\check{\gamma}_r = \sum_{\nu=0}^{r} c_{r-\nu,\, r+(n+1)\nu,\, \nu}.$$

Summarizing the arguments of this section we can state the theorem below, in which the letter k has again been introduced to indicate that there are n such formulas, one for each diagonal element $\gamma_{kk}(\varepsilon)$ of $\Gamma(\varepsilon)$.

THEOREM 7.1. *The connecting coefficients* $\gamma_{kk}(\varepsilon)$ *in Theorem* 5.1 *have the asymptotic form*

$$\gamma_{kk}(\varepsilon) \sim \varepsilon^{nb_k(\varepsilon)/(n+1)} \sum_{r=0}^{\infty} \check{\gamma}_{kk,\, r}\varepsilon^{r/(n+1)}, \quad \text{as} \quad \varepsilon \to 0+,$$

where $b_k(\varepsilon)$ *is the* kth *diagonal element of the matrix* $B(\varepsilon)$ *in Theorem* 3.1, *and*

$$\check{\gamma}_{kk,\, r} = \sum_{\nu=0}^{r} c^k_{r-\nu,\, r+(n+1)\nu,\, \nu}, \qquad k=1, 2, \ldots, n$$

The coefficients $c^k_{p_1 p_2 p_3}=c_p$ *were defined in* (7.11) *and* (7.3).

Proof of Lemma 7.1. Assume the lemma is false. If both (a) and (b) are false there exists a first coefficient f_ϱ in the series $\sum_{r=0}^{\infty} f_r\ \varepsilon^{r/(n+1)}$ and a first positive integer, say μ, such that for some multiindex $p=\hat{p}$ one has $|\hat{p}|=\mu$, $B_{\hat{p}}\neq 0$ and $c_{\hat{p}}\neq 0$. There cannot be a second multiindex p with the same three properties for which also

$$A_p + \frac{\alpha}{n} B_p = A_{\hat{p}} + \frac{\alpha}{n} B_{\hat{p}},$$

identically in α. For then the three equations

$$\frac{1}{n+1} p_2 - p_3 = A_{\hat{p}}, \qquad p_1 - p_2 + (n+2)p_3 = B_{\hat{p}}, \quad p_1 + p_2 + p_3 = \mu$$

for p_1, p_2, p_3

would have to have more than one solution, which is not the case. Hence, the assumption that

$$A_{\hat{p}} + \frac{\alpha}{n} B_{\hat{p}} < A_p + \frac{\alpha}{n} B_p, \quad \text{for all} \quad p$$

with $|p| = \mu$, $p \neq \hat{p}$, in some open subinterval of (7.13) entails no loss of generality. It then follows from (7.14) that, in this α-interval,

$$\left.\begin{aligned} &\gamma(\varepsilon)\,\varepsilon^{-nd(\varepsilon)/(n+1)} = [1 + \alpha f_\varrho \varepsilon^{\varrho/(n+1)} \log \varepsilon + o(\varepsilon^{\varrho/(n+1)} \log \varepsilon)] \\ &\quad \times [\phi_\mu(\varepsilon) + c_{\hat{p}} \varepsilon^{A\hat{p} + \alpha B\hat{p}/n} + o(\varepsilon^{A\hat{p} + \alpha B\hat{p}/n})] = \phi_\mu(\varepsilon) + \alpha \phi_\mu(\varepsilon) \\ &\quad \times f_\varrho \varepsilon^{\varrho/(n+1)} \log \varepsilon + c_{\hat{p}} \varepsilon^{A\hat{p} + \alpha B\hat{p}/n} + o(\varepsilon^{\varrho/(n+1)} |\log \varepsilon| + \varepsilon^{A\hat{p} + \alpha B\hat{p}/n}\,. \end{aligned}\right\} \tag{7.17}$$

Here $\phi_\nu(\varepsilon)$ is independent of α and $\phi(0) = 1$, because of (3.7) and (4.14). The formula remains valid if only (b) is false, if one then sets $f_\varrho = 0$ and permits ϱ to be taken as large as one pleases. Similarly, if only (a) is false one may set $c_{\hat{p}} = 0$ and permit $A_{\hat{p}} + \alpha B_{\hat{p}}/n$ to be arbitrarily large. Let α_1, α_2 be two values of α for which (7.17) is true and $(\alpha_2 - \alpha_1)\, B_{\hat{p}} > 0$. Then subtraction of (7.17) for the two values of α leads to

$$\begin{aligned} 0 &= (\alpha_2 - \alpha_1)\, \phi_\mu(\varepsilon) f_\varrho \varepsilon^{\varrho/(n+1)} \log \varepsilon \\ &\quad + c_{\hat{p}} \varepsilon^{A\hat{p} + \alpha_1 B\hat{p}/n} (\varepsilon^{\alpha_2 - \alpha_1()B\hat{p}/n} - 1) + o(\varepsilon^{\varrho/(n+1)} |\log \varepsilon| + \varepsilon^{A\hat{p} + \alpha_1 B\hat{p}/n}). \end{aligned}$$

This is possible, if and only if all three terms vanish identically, contrary to our assumption.

8. Remarks on Theorem 3.1.

Theorem 3.1 differs from Theorem 1.1 of [10] in four respects. First, formula (3.3) contains the constant scalar factor $\varepsilon^{-(1-n)/2(n+1)}$, which does not appear in [10]. It has been introduced to simplify the matching formulas. Second, Theorem 3.1 deals only with the special case of Theorem 1.1 of [10] obtained by setting $h = 1$ in the latter theorem. The remaining two points require a more detailed analysis, which will now be given.

Formula (3.3) contains a factor $x^{(1-n)/2n}$, which appears in [9], [10] only in the less precise form x^C, where C is some unspecified diagonal matrix (see [9], p. 666, formula (5.11); the notation there is different). To calculate C one has to carry out the diagonalization process described in [9] §§ 3, 4 through the first two terms and to compute the contribution of the form $Cx^{-1}\varepsilon$ that appears in the second term. One finds – we omit some details – that C is the diagonal matrix formed with the diagonal entries of

$$-\lim_{x \to 0} \left\{ x M^{-1}(x) \frac{dM(x)}{dx} \right\}, \tag{8.1}$$

where

$$M(x)=\begin{bmatrix}1 & 1 & \dots 1\\ \lambda_1(x) & \lambda_2(x) & \dots \lambda_n(x)\\ \cdot & \cdot & \dots \cdot\\ \lambda_1^{n-1}(x) & \lambda_2^{n-1}(x) \dots & \lambda_n^{n-1}(x)\end{bmatrix}. \tag{8.2}$$

(The matrix M, here differs from the matrix with the same name in [9], (3.7) in its higher order terms, but his does not affect the limit in (8.1).)

A short calculation shows that

$$\lim_{x\to 0}\left\{xM^{-1}(x)\frac{dM(x)}{dx}\right\}=T^{-1}ST,$$

where T is the Vandermonde matric called $\tilde{U}_0(0)$ in (3.7), and

$$S=\frac{1}{n}\begin{pmatrix}0 & & & & 0\\ & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ 0 & & & & n-1\end{pmatrix}. \tag{8.3}$$

Hence

$$C=-\operatorname{diag}(T^{-1}ST). \tag{8.4}$$

The symbol "diag" here means that one forms the diagonal matrix with the same diagonal entries as $T^{-1}ST$. To calculate C we observe that

$$TP=\Omega(\omega)\,T,$$

if P is the permutation matrix

$$P=\begin{pmatrix}0 & 0 & 0\dots 0 & 1\\ 1 & 0 & 0\dots 0 & 0\\ 0 & 1 & 0\dots 0 & 0\\ \cdot & \cdot & \cdot\dots\cdot & \cdot\\ 0 & 0 & 0\dots 1 & 0\end{pmatrix}.$$

Therefore,

$$\begin{aligned}-P^{-1}CP=P^{-1}\operatorname{diag}(T^{-1}ST)\,P&=\operatorname{diag}(P^{-1}T^{-1}STP)\\ &=\operatorname{diag}(T^{-1}\Omega(\omega)\,S\Omega(\omega)\,T)=\operatorname{diag}(T^{-1}ST)=-C.\end{aligned}$$

This means that the diagonal of C is unchanged under a cyclic permutation of its entries. These entries must therefore, be all equal. Their sum, the trace of C is the same as the trace of $-S$, by (8.4). The trace of S, according to (8.3), is $(n-1)/2$. It follows, that, indeed,

$$C = -\frac{n-1}{2n} I.$$

After division by ε and integration of the diagonalized differential equation, this term $Cx^{-1}\varepsilon$ gives rise to the factor x^C.

The fourth new feature of Theorem 3.1 is the factoring of $x^{B(\varepsilon)}$ in the series representation for $U(x, \varepsilon)$. The series appearing in Theorem 1.1 of [10] can be obtained from (3.4) by writing

$$x^{B(\varepsilon)} = \exp\{B(\varepsilon)\log x\} = \sum_{r=0}^{\infty} \frac{1}{r!}(B(\varepsilon)\log x)^r$$

and expanding everything in powers of ε.

To derive the factored form of $U(x, \varepsilon)$ in (3.4), (3.5), (3.6) one follows the presentation of [9] as far as Theorem 4.1 of [9]. The formally transformed differential equation presents itself then as

$$\frac{dZ^*}{dt} = t^n \sum_{r=0}^{\infty} C_r(t)(t^{-n-1}\varepsilon)^r Z^*, \qquad t = x^{1/n} \tag{8.5}$$

with diagonal matrices $C_r(t)$ holomorphic in $|t| < x_0^{1/n}$. (Again the notation differs from that in [9].) A formal expression for a solution Z^* is

$$Z^*(t, \varepsilon) = \exp\left\{\sum_{r=0}^{\infty} \varepsilon^{r-1} \int C_r(t)\, t^{-n(r-1)-r}\, dt\right\} \tag{8.6}$$

where the choice of the indefinite integrals still has to be defined. If also each $C_r(t)$ is expanded in powers of t, there will arise in the integrand terms of the form $F_k\varepsilon^k t^{-1}$, $k = 1, 2, \ldots$. We deviate from the procedure in [9] by collecting and integrating these terms separately. This gives us, at least formally, a factor of the form

$$\exp\left\{\left(\sum_{k=1}^{\infty} F_k\varepsilon^k\right)\log t\right\} = x^{\sum_{k=1}^{\infty} B_k\varepsilon^k}, \qquad B_k = \frac{1}{n} F_k.$$

The remaining terms in the integrands of (8.6) are then integrated as in [9], p. 665. This leads to formulas (3.4), (3.5), (3.6) as formal results. The proof that the formal expansion represents asymptotically a true solution differs only insignificantly from the one in [10].

9. Remarks on Theorem 4.1.

We outline a method of proof which leads to formulas (4.10) through (4.14) instead of the somewhat less precise Theorem 1.2 of [10].

Let the matrix form of equation (4.3), i.e.,

$$\frac{dV}{ds} = H(s, \varepsilon)\, V \tag{9.1}$$

be transformed by setting

$$V = W s^{D(\varepsilon)}, \tag{9.2}$$

where $D(\varepsilon)$ is a diagonal matric with an asymptotic expansion

$$D(\varepsilon) \sim \sum_{r=1}^{\infty} D_r \varepsilon^{r/(n+1)}. \tag{9.3}$$

Suitable choices for the matrices D_r will be described below. Equation (9.1) becomes

$$\frac{dW}{ds} = H(s, \varepsilon)\, W - W D(\varepsilon)\, s^{-1}. \tag{9.4}$$

It possesses formal solutions

$$W = \sum_{r=0}^{\infty}{}' \, W_r(s)\, \varepsilon^{r/(n+1)} \tag{9.5}$$

with

$$\frac{dW_0}{ds} = H_0(s)\, W_0, \tag{9.6}$$

$$\frac{dW_r}{ds} = H_0(s)\, W_r + G_r(s) - W_0(s)\, D_r s^{-1}, \tag{9.7}$$

$$G_r(s) = \sum_{j=1}^{r-1} \left(H_j(s)\, W_{r-j}(s) - W_{r-j}(s)\, D_j s^{-1}\right) + H_r(s)\, W_0(s). \tag{9.8}$$

In [9], § 7 there is described a paricular $V_0(s)$ of the differential equation (9.6), whose asymptotic expansion in the whole s-plane are known thanks to the work of Turrittin [8]. In particular,

$$V_0(s) = s^{(1-n)/2n}\, \Omega(s^{1/n})\, \hat{W}_0(s)\, e^{Q(s)}, \tag{9.9}$$

where

$$\hat{W}_0(s) \sim \sum_{r=0}^{\infty} \hat{W}_{0r} s^{-(n+1)r/n} \quad \text{as} \quad s \to \infty \text{ in } \textstyle\sum. \tag{9.10}$$

(The notation differs slightly from that in [9].) We take $W_0 = V_0(s)$. The integral

$$W_r(s) = V_0(s) \int_{\Gamma(s)} V_0^{-1}(\sigma)\, [G_r(\sigma) - V_0(\sigma)\, D_r \sigma^{-1}]\, d\sigma \tag{9.11}$$

represents a solution of (9.7) if $\Gamma(s)$ is any path in the σ-plane ending at $\sigma = s$, or even a set of n^2 such paths, one for each entry of the matrix in the integrand (see [9], p. 670). With this integral one can repeat the arguments of [9], § 8:

We define $\hat{W}_r(s)$, $\hat{G}(s)$, $r = 1, 2, \ldots$, by

$$W_r(s) = \Omega(s^{1/n})\, \hat{W}_r(s)\, e^{Q(s)}$$
$$G_r(s) = \Omega(s^{1/n})\, \hat{G}_r(s)\, e^{Q(s)} \tag{9.12}$$

and recall from [9], formula (6.8), that

$$H_r(s) = s^{(r+1)/n}\, \Omega(s^{1/n})\, H_r^*(s)\, \Omega(s^{-1/n}) \tag{9.13}$$

with

$$H_r^*(s) = \sum_{\nu=0}^{\infty} H_{r\nu} s^{-\nu/n}. \tag{9.14}$$

The last series has only finitely many terms. Formula (9.8) shows that

$$\hat{G}_r(s) = \sum_{j=1}^{n-1} (s^{(j+1)/n} H_j^*(s)\, \hat{W}_{r-j}(s) - \hat{W}_{r-j}(s)\, D_j s^{-1}) + s^{(r+1)/n} H_j^*(s)\, \hat{V}_0(s). \tag{9.15}$$

The integral representation (9.10) becomes

$$\hat{W}_r(s) = \hat{V}_0(s) \int_{\Gamma(s)} e^{Q(s)-Q(\sigma)} [\hat{V}_0^{-1}(\sigma)\, \hat{G}_r(\sigma) - D_r \sigma^{-1}]\, e^{Q(\sigma)-Q(s)}\, d\sigma. \tag{9.16}$$

The $\hat{W}_r(s)$ can now be successively calculated.

For the same choice of paths as in [9] Theorems 8.1, 8.2, and by the same method, one proves successively that

$$\hat{W}_r(s) = \tilde{Y}_r(s)\, s^{(n+2)r/n-(1-n)/2n}$$

with $\tilde{Y}_r(s)$ bounded, as $s \to \infty$ in Σ. Beyond that, a simple inductive argument shows that the $\tilde{Y}_r(s)$ have an asymptotic representation involving asymptotic series in powers of $s^{-1/n}$ and polynomials in $\log s$. However, the appearance of logarithmic terms can be prevented by choosing the D_r successively in a suitable way. In fact, logarithmic terms are introduced by the integrations whenever the series for a diagonal entry in the matrix $\hat{V}_0^{-1}(\sigma)\, \hat{G}_r(\sigma) - D_r \sigma^{-1}$, appearing in (9.16) contains a term in σ^{-1}, and in no other way. Hence, the proper choice of D_r is the one which makes these terms zero.

With the D_r determined in this way the existence of a formal series solution of (9.4), and hence of equation (2.1), with the properties of Theorem 4.1 is this established. The proof of the analytic validity of the expansions (4.12), (4.13) also resembles closely the reasoning in the proof of Theorem 1.2 of [10]. This completes the proof of Theorem 4.1.

Theorem 4.1, as stated, leaves one important question unanswered, viz., the computation of $Y(x, \varepsilon)$ when $|s| \leqslant s_0$. It is true that, thanks to the knowledge of $V_0(s)$ in the whole plane through the work of Turrittin in [8] the functions $\hat{W}_r(s)$ in (9.16) can be calculated for $0 < |s| \leqslant s_0$, as well as for large s in Σ, but at $s = 0$ these functions have singularities so that the series in (9.5) cannot be expected to represent a solution of (9.4) asymptotically in regions containing the origin. The series (4.5), however, can be used for such a representation in the whole disk $|s| \leqslant s_0$.

THEOREM 9.1. *Let*

$$\sum_{r=0}^{\infty} V_r(s)\, \varepsilon^{r/(n+1)} \tag{9.17}$$

be the formal solution of the differential equation (9.1) *for which*

$$V_r(1) = W_r(1), \qquad r = 1, 2, \dots .$$

Then the solution $Y(x, \varepsilon)$ *of equation* (2.1), *as given in Theorem* 4.1 *has, uniformly for* $|s| \leqslant s_0$, *the asymptotic expansion*

$$Y(x, \varepsilon) \sim \Omega(\varepsilon^{1/(n+1)}) \sum_{r=0}^{\infty} V_r(s)\, \varepsilon^{r/(n+1)}, \quad \text{as} \quad \varepsilon \to 0+ .$$

Proof: An elementary argument shows that the series (9.17) represents asymptotically in $|s| \leqslant s_0$ a solution of (9.1) with initial values at $s = 1$ that are asymptotic to $\sum_{r=0}^{\infty} W_r(1)\varepsilon^{r/(n+1)}$. By (9.2) the same is true for the solution $W(s, \varepsilon)\, s^{D(\varepsilon)}$ of (9.1). Hence those two solutions are asymptotically equal in $|s| \leqslant s_0$. As $Y(x, \varepsilon) = \Omega(\varepsilon^{1/(n+1)})$ $W(s, \varepsilon)\, s^{D(\varepsilon)}$, by definition, the proof of the theorem is at hand.

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