Topological Pontrjagin Classes.

Topological Pontrjagin Classes[1])

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Introduction

We use a variant of the Thom definition of Pontrjagin classes for triangulated manifolds [10] and the transversality theorem of Kirby and Siebenmann [5] to obtain a definition of rational Pontrjagin classes for oriented euclidean bundles over a class of spaces including topological manifolds. These classes possess all the usual properties of characteristic classes associated to a bundle, namely, naturality and multiplicativity. Moreover, if ξ is a vector bundle over a space X, then the classes defined here agree with the differentiable Hirzebruch classes of the inverse bundle to ξ. We also obtain the signature formula, i.e. one defines $l(M^d)$ to be the class associated to any stable normal bundle for M^d, a topological manifold; then one has that $\langle l(M^d), [M^d]\rangle =$ signature of M^d.

This generalization of the Hirzebruch classes to euclidean bundles is then used to give a proof of the topological invariance of rational Pontrjagin classes, first done by Novikov [9] and to show that the natural homomorphism from differentiable cobordism to topological cobordism is a monomorphism. Finally it is shown that if $\tilde{M}^d$ is a finite regular covering of a closed topological manifold M^d, then the signature multiplies, i.e. the signature of $\tilde{M}^d$ equals the order of the cover times the signature of M^d. This last result is false in its most general form, namely if M^d and $\tilde{M}^d$ are Poincaré spaces [12]. The use of some form of transverse regularity seems to be necessary for a positive result to this theorem in an arbitrary subcategory of Poincaré spaces.

The paper is divided into two parts. In the first we set up two homotopy functors w_* and h_*, one related to differentiable bordism $\Omega_*(X, A)$, and one to ordinary singular homology. We define a natural transformation λ between them and show that if (X, A) has finitely generated rational homology, then $\lambda\otimes 1: w_*\to h_*$ is a natural equivalence. In the second part of the paper we restrict our attention to the Thom space $T\xi$ of a euclidean bundle ξ over a suitably nice space and define a homomorphism $\alpha_\xi: \tilde{w}_*(T\xi)\to Z$, where $\tilde{w}_*$ is the reduced group associated to w_*. This homomorphism together with $\lambda_{T\xi}$ gives rise in a natural way to cohomology classes associated to ξ.

We then proceed to prove the results announced in the beginning of this introduction.

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Part I

Let Ω_* denote the oriented bordism theory for topological pairs (X, A). In a standard manner $\Omega_*(X, A)$ is a module over the cobordism ring Ω_* [see [1] for details]. Let Ω_* act on Z via the signature homomorphism $\sigma: \Omega_* \to Z$ and define

$$w_*(X, A) = \Omega_*(X, A) \otimes_{\Omega_*} Z.$$

Introduce a Z_4-grading into $w_*(Z, A)$ as follows,

$$w_j(X, A) = \text{image}\{\sum_k \Omega_{4k+j}(X, A) \to w_*(X, A)\}.$$

Since Ω_* is a homology theory and the induced maps on spaces are Ω_*-module maps, w_* is a functor from the category of topological pairs to Z_4-graded groups. Also since the boundary map in cobordism is an Ω_*-homomorphism, there exists a boundary map $\eth: w_j(X, A) \to w_{j-1}(A)$ commuting with maps induced from space homomorphisms. In fact, it is clear that $\{w_*, \eth\}$ satisfies all the axioms for a Z_4-homology theory with the possible exception of exactness. We will let $\tilde{w}_*(X)$ denote the kernel of $w_*(X) \to w_*(pt)$ which, since $\Omega_*(pt)$ is free as Ω_*-module, is immediately seen to be the same as $\tilde{\Omega}_*(X) \otimes_{\Omega_*} Z$ as Z_4-graded groups.

If (X, A) is a topological pair, let $h_*(X, A)$ denote the Z_4-graded group,

$$h_j(X, A) = \prod_k \text{Hom}(H^{4k+j}(X, A), Z).$$

Define a natural transformation of Z_4-graded functors $\lambda: w_* \to h_*$ as follows. Let $[m^{4s+j}, f]$ be a singular manifold in $\Omega_{4s+j}(X, A)$ (m^{4s+j} may or may not possess boundary), then

$$\lambda([m^{4s+j}, f] \otimes 1)_k (\tau^{4k+j}) = \langle L_{s-k}(M^{4s+j}) \cup f^*(\tau^{4k+j}), [m^{4s+j}]\rangle$$

where $\tau^{4k+j} \in H^{4k+j}(X, A: Z)$, $L_{s-k}(M^{4s+j})$ is the $(s-k)^{\text{th}}$ Hirzebruch class of M^{4s+j} and $[m^{4s+j}]$ denotes the fundamental class in $H_{4s+j}(M^{4s+j}, Z)$ if m^{4s+j} is closed or in $H_{4s+j}(M^{4s+j}, \dot{M}^{4s+j}, Z)$ if m^{4s+j} has boundary.

THEOREM 1. a) *λ is well defined, i.e.*

$$\lambda([m^{4s+j}, f]\,[N^n] \otimes 1)_k = \lambda([m^{4s+j}, f] \otimes \sigma(N^n))_k$$

b) *λ is natural with respect to maps of pairs*

c) *λ commutes with the boundary homomorphisms, where $\delta: h_j(X, A) \to h_{j-1}(A)$ is given by $\prod_k$* Hom $(\delta, 1)$.

Proof. b) is immediate if λ is well defined and a) and c) are similar computations, and therefore we will only do a).

First, λ_k as a map from $\Omega_{4s+j}(X, A)$ into $\text{Hom}(H^{4k+j}(X, A), Z)$ is well defined, since if $[m^{4s+j}, f] \sim 0$ then both the Hirzebruch class of m^{4k+j} and $f^*(\tau)$,

$\tau \in H^{4k+j} \times (X, A)$ are restrictions of classes in $H^*(W{:}Z)$ where W "bounds" m^{4k+j}. Since the fundamental class of m^{4k+j} is ∂_* of the fundamental class of W, the result follows. To show $\lambda([m^{4s+j}, f]\,[N^n] \otimes 1)_k = \lambda([m^{4s+j}, f] \otimes \sigma(N^n))_k$ we notice that if $n \equiv 0 \pmod 4$ both are zero. Suppose $n = 4p$ and let $\tau \in H^{4k+j}(X, A)$. Since the tangent bundle of $M \times N$ is the Whitney sum of the pulbacks, via the projections π_i, of the tangent bundles of M and N, we have that $L_m(M \times N) = \sum_{i+j=m} \pi_1^* L_i(M) \cup \pi_2^* L_j(N) +$ elements of order 2. Therefore

$$\begin{aligned}
\lambda([m^{4s+j}, f]\,[N^n] \otimes 1)_k(\tau) &= \langle L_{s+p-k}(M \times N) \cup \pi_1^* f^*(\tau), [m^{4s+j} \times N]\rangle \\
&= \sum_{i+j=s+p-k} \langle (-1)^j \pi_1^*(L(M) \cup f^*(\tau)) \cup \pi_2^* L_j(N), [m^{4s+j} \times N^n]\rangle \\
&= \sum_{i+j=s+p-k} \langle L_i(M) \cup f^*(\tau), [m^{4s+j}]\rangle \cdot \langle L_j(N), [N^n]\rangle .
\end{aligned}$$

It is immediately seen that the only contribution occurs when $i = s-k$ and $j = p$. However, it now follows from the Hirzebruch Signature Formula [4] that this equals $\lambda([m^{4s+j}, f] \otimes \sigma(N^n))_k\,(\tau)$.

Since λ is natural, we obtain a natural transformation $\tilde{\lambda}$: $\tilde{w}_*(X) \to \tilde{h}_*(X)$. Moreover, an easy calculation shows that $\lambda(pt)$: $w_0(pt) = Z \to h_0(pt) = Z$ is induced from the signature homomorphism and therefore an isomorphism since it is onto. Since both $w_j(pt)$ and $h_j(pt)$ are zero if $j \neq 0$, we have that $\lambda(pt)$: $w_*(pt) \to \tilde{h}_*(pt)$ is an isomorphism. Hence from the 5 Lemma we see that $\tilde{\lambda}$ is an isomorphism if and only if λ is. Since **Q** is a torsion free abelian group, the same is true if everything is tensored with the rationals over Z.

COROLLARY. The following diagram commutes,

$$\begin{array}{ccc}
\tilde{w}_j(X) & \xrightarrow{\lambda(X)} & \tilde{h}_j(X) \\
\approx\downarrow S_w & & \approx\downarrow S_h \\
\tilde{w}_{j+1}(SX) & \xrightarrow{\lambda(SX)} & \tilde{h}_{j+1}(SX)
\end{array}$$

where SX is the (unreduced) suspension of X and S_w, S_h are the suspension isomorphisms.

Proof. This follows immediately from the preceding theorem since both S_w^{-1} and S_h^{-1} are given by the same composition of maps of pairs and boundary homomorphisms and λ commutes with each.

THEOREM 2. If $H_*(X, A{:}Q)$ is *finitely generated, then* $\lambda \otimes 1 : w_*(X, A) \otimes_Z Q \to h_*(X, A) \otimes_Z Q$ *is an isomorphism.*

Proof. From the rational collapsing of the bordism spectral sequence [1], it follows that $w_*(X, A) \otimes Q = \Omega_*(X, A) \otimes_{\Omega_*} Q$ and $h_*(X, A) \otimes Q = H_*(X, A{:}Q)$ are isomorphic finite dimensional rational vector spaces. (Here we are identifying $h_*(X, A)\otimes Q$ and $H_*(X, A:Q)$ by means of a natural equivalence.) Since $\lambda\otimes 1$ is linear, the theorem will follow if we show that $\lambda\otimes 1$ is onto. We first make some preliminary observations. Since $H_*(X, A:Q)$ is finitely generated, there exists a natural isomorphism of vector spaces

$$H_{4k+j}(X, A:Q)\to \mathrm{Hom}_Q(\mathrm{Hom}_Q(H_{4k+j}(X, A:Q), Q), Q)$$

From the universal coefficient theorem we obtain a natural isomorphism

$$\mu: H^{4k+j}(X, A:Q)\to \mathrm{Hom}_Q(H_{4k+j}(X, A, Q), Q).$$

The composition $\mathrm{Hom}(\mu, 1)\cdot \nu$ yields a natural isomorphism

$$\varrho: H_{4k+j}(X, A: Q)\to \mathrm{Hom}_Q(H^{4k+j}(X, A: Q)\, Q).$$

One calculates that if $c\in H_{4k+j}(X, A:Q)$ and $\xi\in H^{4k+j}(X, A; Q)$, then

$$(\varrho c)(\xi)=\langle \xi, c\rangle \in H_0(X:Q)=Q.$$

Now, if $d=\sum c\otimes 1/q\in H_{4k+j}(X, A)\otimes Q=H_{4k+j}(X, A; Q)$ is an arbitrary element, then by results of Thom [11] and Conner-Floyd [1], there exists an odd multiple of c which is Steenrod representable, say $\mu[M^{4k+j}, f]=(2s+1)c$. We are now in a position to see that for any t

$$\lambda_t: w_j(X, A)\otimes Q\xrightarrow{\lambda\otimes 1} h_j(X, A)\otimes Q\xrightarrow{\pi_*}\mathrm{Hom}(H^{4t+j}(X, A: Q), Q)$$

is onto. Let $\varphi\in\mathrm{Hom}_Q\,(H^{4t+j}(X, A; Q), Q)$. Choose $d\in H_{4t+j}(X, A; Q)$ such that $\varrho(d)=\varphi$. And let $d=c\otimes 1/q$. Choose s so that $(2s+1)c$ is Steenrod representable, say by $[M^{4t+j}, \dot M^{4t+j}, f]$.

Claim:

$$\lambda_t\left([M^{4t+j}, \dot M^{4t+j}, f]\otimes\frac{1}{(2s+1)\,q}\right)=\varphi.$$

For let $\tau\in H^{4t+j}(X, A, Q)$, then

$$\begin{aligned}&\lambda_t\left([M^{4t+j}, \dot M^{4t+j}, f]\otimes\frac{1}{(2s+1)\,q}\right)(\tau)\\ &=\frac{1}{(2s+1)\,q}\langle L_0(M^{4t+j})\cup f^*(\tau), [M^{4t+j}, \dot M^{4t+j}]\rangle\\ &=\frac{1}{(2s+1)\,q}\langle \tau, f_*[M^{4t+j}, \dot M^{4t+j}]\rangle\\ &=\frac{1}{(2s+1)\,q}\langle \tau, (2s+1)\,c\rangle=\langle \tau, c\otimes 1/q\rangle\\ &=\langle \tau, d\rangle=\varphi(\tau).\end{aligned}$$

Now λ_r: $w_j(X, A)\otimes Q\to \mathrm{Hom}_Q(H^{4r+j}(X, A; Q), Q)$ is obviously zero if $r<0$, and if $r>t$ it follows that $\lambda_r([M^{4t+j}, \dot{M}^{4t+j}, f]\otimes 1)=0$.

We will now show that

$$\lambda\otimes 1: w_j(X, A)\otimes Q\to h_j(X, A)\otimes Q=\prod_k \mathrm{Hom}(H^{4k+j}(X, A, Q), Q)$$

is onto by induction on k. If $k=0$, the previous remarks give the result. Suppose that if $\varphi\in\prod_{k<p}\mathrm{Hom}_Q(H^{4k+j}(X, A; Q), Q)$ there exists $\alpha\in w_j(X, A)\otimes Q$ such that $\lambda\otimes 1(\alpha)=\varphi$ and let $\psi=(\varphi, \varphi_p)\in\prod_{k\leqslant p}\mathrm{Hom}_Q(H^{4k+j}(X, A; Q), Q)$. By the previous result, there exists $([M^{4p+j}, f]\oplus 1/q)$ such that

$$\lambda_p([M^{4p+j}, \dot{M}^{4p+j}, f]\otimes 1/q)=\varphi_p.$$

Let $\mu_S=\lambda_s(M^{4p+j}, M^{4p+j}, f]\otimes 1/q)\in\mathrm{Hom}_Q(H^{4s+j}(X, A; Q), Q)$. By the induction hypothesis, there exists $\alpha\in w_j(X, A)\otimes Q$ such that $(\lambda\otimes 1)\alpha=\varphi-(\mu_0, \mu_1, \ldots, \mu_{p-1})\in\prod_{k<p}\mathrm{Hom}_Q(H^{4k+j}(X, A; Q), Q)$. Since λ is additive, it follows that $(\lambda\otimes 1)(\alpha+([M^{4p+j}, \dot{M}^{4p+j}, f]\otimes 1/q))=(\varphi, \varphi_p)=\psi$. Since $H_*(X, A: Q)$ is finitely generated, we are done.

Part II

Let ξ^k denote an oriented k-dimensional euclidean bundle (e.b.) (structure group $=H_0^+(R^k)$, the orientation preserving homeomorphisms of euclidean k-space fixing the origin) with compact base space $B(\xi)$ (the restriction that $B(\xi)$ be compact does not seem to be necessary but makes the arguments easier) and total space $E(\xi)$. By a bundle map $f:\xi^k\to\eta^k$ will be meant a fiber preserving map which is an onto homeomorphism when restricted to fibers. The Thom space, $T\xi$, of $E(\xi)$ will be the one point compactification of $E(\xi)$. Since any bundle map f (over a compact base) is a proper map, there exists an extension of f, Tf, to $T\xi$.

PROPOSITION 1. *Let $f:\xi\to\eta$ be an e.b. map, then there exists a continuous map*

$Tf:(T\xi, \infty)\to(T\eta, \infty)$ such that $Tf\,|\,E(\xi)=f$.

PROPOSITION 2. *$H^*(T\xi, \infty)\approx H^*(E\xi, E\varrho_0)$, where $E\xi_0=E\xi-B\xi$.*

Proof. Define a deformation retraction of $E\xi_0\cup\infty$ onto $\{\infty\}$ by using a linear map in each fiber over an open set U where $\xi\,|\,U$ is trivial and piece together using a partition unity. Now use the cohomology sequence of the triple $(T\xi, E\xi\cup\infty, \infty)$ to obtain $H^*(E(\xi)\cup\infty, E\xi_0\cup\infty)\approx H^*(T\xi, \infty)$ and then excise the point at infinity.

Since ξ^k is oriented, we obtain from the Leray-Hirsch theorem, the Thom isomorphisms

$$\phi_\xi : H_n(E(\xi), E(\xi)_0) \xrightarrow{\approx} H_{n-q}(B(\xi)) \qquad \phi_\xi(z) = p_*(U_\xi \cap z)$$

$$\phi^*_\xi : H^r(B(\xi)) \xrightarrow{\approx} H^{r+q}(E(\xi), E(\xi)_0) \qquad \phi^*_\xi(v) = p^* v \cup U_\xi .$$

Moreover, if $f: \xi \to \eta$ is a bundle map, $f^* U_\eta = U_\xi$ and the isomorphisms in Proposition 2 are such that the following diagram commutes

$$\begin{array}{ccc} H^*(T\eta, \infty) & \xrightarrow{(Tf)^*} & H^*(T\xi, \infty) \\ \wr\wr & & \wr\wr \\ H^*(E(\eta), E(\eta)_0) & \xrightarrow{f^*} & H^*(E\xi, E(\xi)_0). \end{array}$$

That is, $(Tf)^*$ commutes with the Thom isomorphisms considered as maps $H^*(B\xi) \to H^*(T\xi, \infty)$.

We may enlarge the collection of fibre preserving maps giving rise to maps on the Thom spaces and commuting with the Thom isomorphisms as follows.

DEFINITION. *A bundle morphisms $h: \xi^k \to \eta^k$ is a fiber preserving map which when restricted to fibers is an imbedding.*

PROPOSITION 3. *Let $h: \xi^k \to \eta^k$ be a bundle morphism with $B(\xi)$ compact, then there exists $\xi_1 \subseteq \xi$ and $\eta_1 \subseteq \eta$, such that $h \mid E(\xi_1): \xi_1 \to \eta_1$ is a bundle map.*

Proof. Since $B(\xi)$ is compact, there exists a neighborhood V of $B(\eta)$ contained in the image of h. V contains a microbundle, which in turn contains euclidean bundle η_1 by the Kister-Mazur theorem [6]. Let $E(\xi_1) = h^{-1} E(\eta_1)$. This is a locally trivial euclidean bundle $\subseteq E(\xi)$ and $h_* \mid E(\xi_1)_*$ is a homeomorphism onto.

Consider the following commutative diagram of bundle morphisms

$$\begin{array}{ccc} \xi_1 & \xrightarrow{h \mid E\xi_1} & \eta_1 \\ \downarrow{\iota_\xi} & & \downarrow{\iota_\eta} \\ \xi & \xrightarrow[h]{} & \eta \end{array}$$

By the Kister-Mazur theorem again, ι_ξ and ι_η are fiber homotopic to bundle maps, g_ξ and g_η covering the identity, and hence homeomorphisms of $E\xi_1$ onto $E\xi$ and $E\eta_1$ onto $E\eta$ respectively. Define $Th = Tg \cdot Th \mid E\xi_1 \cdot Tg^{-1}$. It is immediately seen that Th determines a well defined homotopy class of maps of $(T\xi, \infty)$ into $(T\eta, \infty)$ and since g_ξ, $h \mid E\xi_1$ and g_η are all bundle maps, they all commute with the Thom isomorphisms and hence $(Th)^*$ commutes with the Thom isomorphisms.

We record for later use the following proposition.

PROPOSITION 4. $(T(\xi\oplus\varepsilon'),\infty)\underset{\cong}{\xrightarrow{h}}(\bar{S}T\xi,\infty)$ *where* $\bar{S}$ *denotes the reduced suspension and*

$$h\mid E(\xi\oplus\varepsilon'):E(\xi\oplus\varepsilon')\to \text{image of}\quad E\xi\times(-1,1)$$

is a bundle map covering the natural map $B\xi\to B\xi\times 0$.

Proof. Let γ be any continuous homeomorphism of the real line R onto $(-1, 1)$ which fixes 0. Define $h:T(\xi\oplus\epsilon')\to\bar{S}T\xi$ by $h(\alpha, t)=(\alpha,\gamma(t))$, $h(\infty)=\infty$.

Suppose M^d is a topological manifold and ξ^k is a euclidean bundle, and let $f:M^d\to T\xi^k$.

DEFINITION. *f is transversal to $E(\xi)\subseteq T(\xi^k)$ with normal bundle η* (f is t.r. to ξ) *if there exists an open set $U\subseteq M^d$ such that*

i) *U is the total space of a euclidean bundle $\eta^k\subseteq M^d$.*
ii) *$B(\eta)=f^{-1}(B(\xi))$ is a topological submanifold of M^d.*
iii) *$f\mid U:E(\eta)\to E(\xi)$ is a bundle morphism.*

THEOREM. (Kirby-Siebenmann [5]) *Suppose U is an open neighborhood of a closed set $C\subseteq M^d$. Let $f:M^d\to T\xi^k$ be such that $f\mid U:U\to T\xi^k$ is t.r. to ξ with normal bundle η, then if $\dim M-\dim\xi\geqq 5$ and $B(\xi)$ is a local euclidean retract, f is homotopic to a map $g:M^d\to T\xi^k$ such that*

i) *g is t.r. to ξ with normal bundle $\bar{\eta}$.*
ii) *$g=f$ in some neighborhood of C.*
iii) *$\bar{\eta}\mid V=\eta$ where V is some neighborhood $C\subseteq V\subseteq U$.*

Remarks. 1) Since $E(\eta^k)$ is an open subset in M^d and $E(\eta^k)$ is locally a product, it follows that $B(\eta)$ is a manifold of dimension $d-k$. Moreover, if M^d has a boundary, then $B(\eta)$ has boundary $B(\eta)\cap\dot{M}^d$.

2) If M^d and ξ^k are oriented, then $B(\eta)$ is oriented.

3) If $g_1\cong g_2:M^d\to T\xi^k$, M^d is closed and g_1 and g_2 are both t.r. to ξ, then $g_1^{-1}(B\xi)$ and $g_2^{-1}(B\xi)$ are cobordant in M^d.

4) Suppose $h:E(\xi)\to E(\eta)$ is a bundle map (onto homeomorphism of fibers) and $f:M^d\to E(\xi)$ is t.r. to ξ, then $Th\cdot f$ is t.r. to η.

Proof. 1) and 2) are immediate, while 3) follows from the relative version of the transversality theorem. 4) is true since if $E(\tau)\subseteq M^d$ with $B(\tau)=f^{-1}(B(\xi))$ a manifold and $f\mid E(\tau)$ a bundle morphism, then $B(\tau)=(Th\cdot f)^{-1}(B(\eta))$ and $Th\cdot f\mid E(\tau)$ is a bundle morphism.

Let ξ^k be a k-dimensional oriented Euclidean bundle over a compact local Euclidean neighborhood retract. Choose $j \equiv h \pmod 4$ and define a homomorphism

$$\alpha_\xi : \tilde{w}_j(T\xi^k) \to Z$$

as follows: Recall $\tilde{w}_j(T\xi^k) \approx (\tilde{\Omega}_*(T\xi^k) \otimes_{\Omega_*} Z)_j$ and so $\tilde{w}_j(T\xi^k)$ is generated by elements $[M^{4s+j}, f] \otimes 1$, where M^{4s+j} is a closed manifold whose cobordism class in Ω_* is zero. Let $[M^{4s+j}, f] \otimes 1$ represent a generator of $\tilde{w}_j(T\xi^k)$. Choose $[M^{4s+j}, f] \otimes 1$ such that $4s+j-k \geqq 5$. This is always possible since

$$[M^{4s+j}, f] \otimes 1 = [M^{4s+j} \times CP^{2t}, f\pi_1] \otimes 1 \quad \text{in} \quad \tilde{w}_j(T\xi^k).$$

By the transverse regularity theorem, we may assume f is t.r. to ξ. Let $\alpha_\xi([M^{4s+j}, f] \otimes 1) = \sigma(f^{-1}(B\xi))$. Note that since $j \equiv k \pmod 4$, dimension $f^{-1}(B\xi) = 4s+j-k = 4\varrho$ and since ξ^k and M^{4s+j} are oriented, so is $f^{-1}(B\xi)$.

PROPOSITION 5. *α_ξ is well defined.*

Proof. If $g \simeq f$ is also t.r. to $E(\xi)$ then by Remark 3, $g^{-1}(B\xi)$ and $f^{-1}(B\xi)$ are cobordant and therefore have the same signature. We are therefore left with showing that α_ξ respects the relations in $\tilde{w}_j(T\xi^k)$, i.e.

$$\alpha_\xi([M^d, f]\,[N^t] \otimes 1) = \alpha_\xi([M^d, f] \otimes \sigma[N^t]).$$

This follows for if f is t.r. to ξ with normal bundle η, $f \circ \pi_1$ is t.r. to ξ with normal bundle $\eta \times \varepsilon^0$, where ε^0 is the 0-dimensional bundle over N^t, so that

$$\begin{aligned}\alpha_\eta([M^d, f]\,[N^t] \otimes 1) &= \sigma(B(\eta) \times N^t) = \sigma(B\eta) \cdot \sigma(N^t) \\ &= \sigma([M^d, f] \otimes \sigma(N^t)).\end{aligned}$$

PROPOSITION 6. *If $h: \xi^k \to \eta^k$ is a bundle morphism then $\alpha_\eta (Th)_* = \alpha_\xi$. That is the following diagram is commutative*

$$\begin{tikzcd} \tilde{w}_j(T\xi^k) \arrow[rr, "(Th)_*"] \arrow[dr, "\alpha_\xi"'] & & \tilde{w}_j(T\eta^k) \arrow[dl, "\alpha_\eta"] \\ & Z. & \end{tikzcd}$$

Note: $(Th)_*: \tilde{w}_j(T\xi) \to \tilde{w}_j(T\eta)$ is well defined since $\tilde{w}_j$ is a homotopy functor and if h is a bundle morphism Th is a well determined homotopy class.

Proof. Since Th is the composition of three bundle maps, it is sufficient to prove the proposition if h is a bundle map. However, this is an immediate consequence of Remark 4.

The homomorphism α_ξ and the natural equivalence $\lambda_{T\xi^k}$ allow one to define rational cohomology classes associated to any oriented Euclidian bundle ξ^k whose base space $B(\xi)$ is a compact, local Euclidian neighborhood retract such that $H_*(B(\xi):Q)$ is finitely generated. We proceed as follows:

Consider the homomorphism

$$\alpha_\xi \otimes 1 : \tilde{w}_j(T\xi^k) \otimes Q \to Q$$

Composing with the natural equivalence

$$\tilde{\lambda}_{T\xi}^{-1} \otimes 1 : \tilde{h}_j(T\xi^k) \otimes Q \to \tilde{w}_j(T\xi^k) \otimes Q$$

we obtain a homomorphism

$$\varrho_\xi : \tilde{h}_j(T\xi) \otimes Q \to Q.$$

As we have previously noted, there is a natural identification of $\tilde{h}_j(T\xi)\otimes Q$ with $\prod_t \mathrm{Hom}_Q(\tilde{H}^{4t+j}(T\xi, Q), Q)$.

We will make this identification and consider ϱ_ξ as a map from

$$\prod_t \mathrm{Hom}_Q(\tilde{H}^{4t+j}(T\xi, Q), Q) \to Q,$$

i.e.

$$\varrho_\xi \in \mathrm{Hom}_Q[\prod_t \mathrm{Hom}_Q(\tilde{H}^{4t+j}(T\xi, Q), Q), Q]$$

which is naturally isomorphic to

$$\prod_t \tilde{H}^{4t+j}(T\xi^k, Q),$$

since $\tilde{H}^*(T\xi^k, Q)$ is finitely generated.

Under these natural identifications ϱ_ξ corresponds to an element $s(\xi)\in\prod_t H^{4t+j}\times(T\xi^k, Q)$.

Suppose $f: E(\xi^k)\to E(\eta^k)$ is a bundle morphism. Since $\lambda_{T\xi^k}$ is natural and since $\alpha_\xi = \alpha_\eta (Tf)_*$ we have that

$$\varrho_\xi = (\alpha_\xi \otimes 1)(\tilde{\lambda}_{T\xi}^{-1} \otimes 1) = \varrho_\eta \circ (Tf)_*$$

where $(Tf)_* : \tilde{h}_j(T\xi^k)\otimes Q \to \tilde{h}_j(T\eta^k)\otimes Q$, i.e. Hom $(Tf_*, 1)(\varrho_\eta) = \varrho_\xi$.

Since the identification of $\mathrm{Hom}(\tilde{h}_j(T\xi^k), Q) \to \prod_t \tilde{H}^{4t+j}(T\xi^k, Q)$ is natural we obtain

$$Tf^*(s(\eta)) = s(\xi),\ Tf^* : \prod_t \tilde{H}^{4t+j}(T\eta^k, Q) \to \prod_t \tilde{H}^{4t+j}(T\xi^k, Q).$$

Let $\phi^* : H^*(B(\xi), Q) \to \tilde{H}^{*+k}(T\xi^k, Q)$ be the Thom isomorphism (in the ordinary integral Thom isomorphism tensored with the identity of Q).

DEFINITION. $\bar{l}(\xi) = (\phi^*)^{-1} s(\xi) H^{4*}(B(\xi):Q)$.

THEOREM 3. *If $\bar{f}:\xi\to\eta$ is a bundle morphism then $f^*(\bar{l}(\eta))=\bar{l}(\xi)$ where $f:B(\xi)\to B(\eta)$ is the map induced by $\bar{f}$.*

Proof. Since $T\bar{f}^*(s(\eta))=s(\xi)$ it is sufficient to show the diagram

$$\begin{tikzcd}
\tilde{H}^*(T\xi) & \tilde{H}^*(T\eta) \arrow[l, "Tf_*"'] \\
H^*(B\xi) \arrow[u, "\phi_\xi"] & H_*(B\eta) \arrow[l, "f_*"] \arrow[u, "\phi_\eta"']
\end{tikzcd}$$

commutes. However, this is just the naturality of the Thom iso. with respect to bundle morphisms.

COROLLARY. If ε^n is a trivial bundle over Y then $\bar{l}_t(\varepsilon^n)=0$ for $t>0$.

THEOREM 4. *Let ξ^k be an oriented Euclidian bundle over Y and ε^n a trivial bundle over Y, then $\bar{l}(\xi^k\oplus\varepsilon^n)=\bar{l}(\xi^k)$ (the $\bar{l}$-classes are invariants of the stable class of the bundle).*

Proof. By induction it is clearly sufficient to do the case $n=1$.

Now as we have seen $T(\xi\oplus\varepsilon')$ is homeomorphic to $\bar{S}T(\xi)$ by a map h^{-1} which is a bundle map when restricted to $E(\xi\oplus\varepsilon')$.

LEMMA 1.

$$\begin{tikzcd}
\tilde{w}_j(T\xi^k) \arrow[r, "S_*"] \arrow[drr, "\alpha_\xi"'] & \tilde{w}_{j-1}(ST\xi^k) \arrow[r, "\mu_*"] & \tilde{w}_{j+1}(\bar{S}T\xi^k) \arrow[r, "h_*"] & \tilde{w}_{j+1}T(\xi\oplus\varepsilon') \arrow[dl, "\alpha_\xi\oplus\varepsilon"] \\
 & & Z &
\end{tikzcd}$$

is a commutative diagram where S_* is the suspension homomorphism, μ_2 is the natural map $SX\to\bar{S}X$. (Suspension to reduced suspension) and h is the homeo described above.

Proof. Let $[M^d,f]\in\tilde{w}_j(T\xi^k)$, where $d-k\geqq 5$ and f is t.r. to $E(\xi^k)$, say with normal bundle $E(\eta)\subseteq M^d\cdot\alpha_\xi([M^d,f]\otimes 1)=\sigma(f^{-1}B(\xi))$.

The suspension map S is given by

$$S([M^d,f]\otimes 1)=[(I\times B^{d+1})^0,F]$$

where

$$M^d = \dot{B}^{d+1} \quad \text{and} \quad \begin{aligned} F \mid I \times M^d &= \mu_1(1d \times f) \\ F \mid -1 \times B^d &= \text{N.P.} \\ f \mid 1 \times B^d &= \text{S.P.} \end{aligned}$$

Therefore $h_*\mu_{2*}S\,([M^d, f]\otimes 1) = [(I\times B^{d+1})^0, h\mu_2 F]$ where

$$\begin{aligned} h\mu_2 F \mid I \times M^d &= h\mu(1d \times f) \\ h\mu_2 F \mid -1 \times B^d &= \infty \\ h\mu_2 F \mid 1 \times B^d &= \infty\,. \end{aligned}$$

Claim $h\mu_2 F$ is t.r. to $E(\xi\oplus\varepsilon')$ in $T(\xi\oplus\varepsilon')$. For let $U=(-1,1)\times E(\eta)\subseteq I\times M^d$. U is open and is the total space of an m.b. $\subseteq I\times M\cdot(h\mu_2 F)^{-1}B(\xi\oplus\varepsilon')=F^{-1}\mu_2^{-1}h^{-1}B(\xi\oplus\varepsilon')$. But h^{-1} maps $B(\xi\oplus\varepsilon')$ to $0\times B(\xi)$ in $\bar{S}T\xi$, and since μ_2 is a relative homeomorphism, $\mu_2^{-1}(0\times B(\xi))=0\times B(\xi)$ and

$$F^{-1}(0\times B(\xi)) = 0 \times f^{-1}B(\xi) = 0 \times B(\eta)$$

is a topological submanifold of $I\times M^d$. Now $h\mu_2 F\,|\,(-1,1)\times E(\eta)$ is a bundle morphism since f is and since μ_2 is a relative homeo and h is a bundle map.

Therefore $\alpha_{\xi\oplus\varepsilon'}(h_*\mu_{2*}S\,([M^d,f]\otimes 1))=\sigma(h\mu_2 F^{-1}(B\xi))=\alpha_\xi([M^d,f]\otimes 1)$.

Since each of the maps in Lemma 1 commute with λ (the first from a previous proposition and the last two because they are space maps we have

$$\begin{tikzcd}[column sep=small] \tilde h_j(T\xi^k)\otimes Q \arrow[r, "S_h"] \arrow[drr, "\rho_\xi"'] & \tilde h_{j+1}(ST\xi^k)\otimes Q \arrow[r, "\mu_{2*}"] & \tilde h_{j+1}(\bar{S}T\xi^k)\otimes \arrow[r, "h_*"] & \tilde h_{j+1}(T(\xi\oplus\varepsilon')) \arrow[dl, "\rho_{\xi\oplus\varepsilon'}"] \\ & & Q & \end{tikzcd}$$

Because $s(\xi)$ is the class corresponding to ϱ_ξ in $\prod_t \tilde H^{4t+j}(T\xi^k, Q)$ it follows that the composite

$$\prod_t \tilde H^{4t+j+1}(T(\xi\oplus\varepsilon'), Q) \xrightarrow{h^*} \prod_t \tilde H^{4t+j+1}(\bar{S}T\xi^k, Q) \xrightarrow{\mu^*_2} \prod_t \tilde H^{4t+j+1}(ST\xi^k, Q)$$
$$\xrightarrow{S^*_h} \prod_t \tilde H^{4t+j}(T\xi^k, Q)$$

maps $s(\xi\oplus\varepsilon')$ to $s(\xi)$.

Now let $\sigma: H_n(X, *) \xleftarrow{\cong} H_{n+1}(\bar{C}X, X) \xrightarrow{\cong} H_{n+1}(\bar{S}X, *)$ be the reduced suspension isomorphism. Then it is clear that if S_h denotes the ordinary suspension homomorphism that $\sigma = \mu_{2*} \cdot S_h$, where $\mu_2 \colon SX \to \tilde{S}X$ is the collapsing map. Therefore to show that $\bar{l}(\xi) = \bar{l}(\xi \oplus \varepsilon')$ in $H^*(Y, Q)$ we need the following lemma.

LEMMA 2. *The following diagram commutes up to* $(-1)^s$.

$$\begin{tikzcd}
\tilde{H}^{k+1+s}(T(\xi \oplus \varepsilon')) \arrow[r, "h^*"] & \tilde{H}^{k+1+s}(\tilde{S}T\xi) \arrow[r, "\sigma^*"] & H^{k+s}(T\xi) \\
& H^s(Y) \arrow[ul, "\phi^*_{\xi \oplus \varepsilon'}"] \arrow[ur, "\phi^*_{\xi}"'] &
\end{tikzcd}$$

Proof. We will orient $\xi \oplus \varepsilon'$ by choosing $U_{\xi \oplus \varepsilon'}$ to be the image of $h^{*-1}\sigma^{*-1}$ of U_ξ.

Now if γ is the generator (suspension of $1 \in H^0(S^0, pt)$) in $H^1(S^1, pt)$ then it is shown in (3, Prop. 1.C) that for any $(X, *)$

$$\begin{tikzcd}
H^k(X, *) \arrow[r, "\gamma \times"] \arrow[dr, "\sigma^*"'] & H^{k+1}(S^1 \times X, * \times X \cup S^1 \times *) \arrow[d, "\text{natural iso.}"] \\
& H^{k+1}(\tilde{S}X, *)
\end{tikzcd}$$

commutes.

It follows that for $u \in H^*(X, A)$, $v \in H^*(X)$ $\sigma^*(u \cup v) = (-1)^{\deg u} u \cup \sigma^* v$. In particular $\sigma^* \phi^*_\xi(u) = (-1)^{\deg u} p^* u \cup \sigma^* U$. Since $h \mid E(\xi \oplus \varepsilon')$ is a bundle map, we have that

$$\begin{aligned}
h^* \sigma^* \phi^*_\xi(u) &= (-1)^{\deg u} h^* p^* u \cup h^* \sigma^* U \\
&= (-1)^{\deg u} p'^* u \cup U_{\xi \oplus \varepsilon'} \\
&= (-1)^{\deg u} \phi^*_{\xi \oplus \varepsilon'}(u)
\end{aligned}$$

where p' is the projection $E(\xi \oplus \varepsilon') \to Y$.

The proof of the theorem now follows since

$$\phi^*_\xi(\bar{l}(\xi)) = s(\xi), \; \phi^*_{\xi \oplus \varepsilon'}(\bar{l}(\xi \oplus \varepsilon')) = s(\xi \oplus \varepsilon')$$

and the $\bar{l}$-classes are even dimensional classes.

The following sequence of more or less obvious remarks constitutes a proof for the multiplicity of the $\bar{l}$-classes. Since most have been done in detail for the case η is trivial, we only indicate the main steps.

1) If ξ and η are e.b. over compact bases then $T(\xi \times \eta) \simeq T\xi \circledast T_\eta$ (smash product) by a homeomorphism h which is a bundle map when restricted to $E(\xi \times \eta) = E\xi \times E\eta$.

2) If K denotes the Künneth map in cohomology then the following diagram

commutes

$$\begin{tikzcd}
H^*(B\xi \times B\eta) \arrow[r, "\phi^*_{\xi\times\eta}"] & \tilde{H}^*(T(\xi \times \eta)) \\
 & \tilde{H}^*(T\xi \divideontimes T\eta) \arrow[u, "\approx"', "h^*"] \\
H^*(B\xi)\otimes H^*(B\eta) \arrow[uu, "\approx"', "\kappa"] \arrow[r, "\phi^*_\xi\otimes\phi^*_\eta"'] & \tilde{H}^*(T\xi)\otimes \tilde{H}(T\eta) \arrow[u, "\approx"', "\kappa"]
\end{tikzcd}$$

3) Consider the Künneth map in bordism $\Omega_*(X)\times\Omega_*(Y)\to\Omega_*(X\times Y)$ given by $[M^n, f]\times[N^n, g]\to[M^n\times N^n, f\times g]$. This map gives rise to a homomorphism $\Omega_*(X)\otimes_{\Omega_*}\Omega_*(Y)\xrightarrow{\kappa}\Omega_*(X\times Y)$. Consider

$$w_*(X)\otimes_z w_*(Y)=(\Omega_*(X)\otimes_{\Omega_*}Z)\otimes_z(\Omega_*(Y)\otimes_{\Omega_*}Z)\simeq\Omega_*(X)\otimes_{\Omega_*}\Omega_*(Y)\otimes_{\Omega_*}Z$$

where the identification is

$$\begin{aligned}[M^n,f]\cdot[P^p]\otimes[N^n,g]\otimes n&=[M^n,f]\otimes\sigma(P^p)\,[N^n,g]\otimes n\\&=[M^n,f]\otimes[N^n,g]\otimes\sigma(P^p)\cdot n.\end{aligned}$$

From the Künneth map κ we therefore obtain a map

$$w_*(X)\otimes w_*(Y)\xrightarrow{\kappa\otimes 1}w_*(X\times Y)$$

It is not difficult to see that the following diagram commutes

$$\begin{tikzcd}
w_*(T\xi)\otimes w_*(T\eta) \arrow[r, "\lambda_{T\xi}\otimes\lambda_{T\eta}"] \arrow[d, "\kappa\otimes 1"] & h_*(T\xi)\otimes h_*(T\eta) \arrow[d, "\kappa"] \\
w_*(T\xi\times T\eta) \arrow[r, "\lambda_{T\xi\times T\eta}"] & h_*(T\xi\times T\eta).
\end{tikzcd}$$

4) The following diagram commutes

$$\begin{tikzcd}
\tilde{w}_*(T\xi)\otimes\tilde{w}_*(T\eta) \arrow[r, "\alpha_\xi\otimes\alpha_\eta"] \arrow[d, "\kappa\otimes 1"] & Z\otimes Z \arrow[ddd, "\text{multiplication}"] \\
\tilde{w}_*(T\xi\times T\eta) \arrow[d, "\text{collapsing map } c"] & \\
\tilde{w}_*(T\xi\divideontimes T\eta) \arrow[d, "h_*"] & \\
\tilde{w}_*(T(\xi\times\eta)) \arrow[r, "\alpha_{\xi\times\eta}"] & Z
\end{tikzcd}$$

since $hc(f\times g)\colon M^m\times N^n\to T(\xi\times\eta)$ is t.r. to $B(\xi\times\eta)=B\xi\times B\eta$ and $\sigma(f^{-1}B\xi\times g^{-1}B\eta)$ $=\sigma(f^{-1}B\xi)\cdot\sigma(g^{-1}B\eta)$.

Putting these four fact together and the facts that $\phi_\xi^*\bar{l}(\xi)$ is the class corresponding to the homomorphism $(\alpha_\xi\otimes 1)(\lambda_{T\xi}^{-1}\otimes 1)$ we see that $\bar{l}(\xi\times\eta)=\bar{l}(\xi)\otimes\bar{l}(\eta)\in H^{4*}(B\xi\times B\eta)$.

Since the Whitney sum of ξ and η is $\Delta^*(\xi \times \eta)$ where $\Delta: B\xi \to B\xi \times B\xi$ is the diagonal map, we obtain.

THEOREM 5. *If ξ and η are e.b. over X, then $\bar{l}(\xi \oplus \eta) = \bar{l}(\xi) \cdot \bar{l}(\eta)$.*

Suppose M^d is a closed oriented topological manifold. In [8] it is shown that M^d has a stable normal microbundle, that is, there exists an embedding of M^d into S^{d+N} such that M^d has a normal bundle, ν_M, in S^{d+N}. By the Kister-Mazur theorem, there exists a Euclidian bundle, also called ν_M, contained in the microbundle and unique up to a Euclidian bundle equivalence.

DEFINITION. *The l-class of M^d, $l(M^d)$, is the $\bar{l}$-class of any stable normal bundle for M^d.*

Remark: By the last theorem, this is well defined since any two stable normal bundles are stably equivalent.

In [7], Milnor characterizes the combinatorial Pontrjagin-Hirzebruch classes [10], $l'(K^n)$, where K^n is a simplical, rational homology manifold as follows.

If $n \geqq 8i+2$, then $l'_i(K^n)$ is the unique $4i$-dimensional rational cohomology class satisfying

$$\langle l'_i(K) \cup f^*(\mu), [K] \rangle = \sigma(f^{-1}(y))$$

where f is any simplical map $K^n \to S^{n-4i}$, and μ is the standard generator of $H^{n-4i}(S^{n-4i}, Z)$.

We will show the classes $\bar{l}(M^d)$, M^d a closed topological manifold agrees with the Combinatorial Pontrjagin-Hirzebruch class, if M^d is a PL manifold by showing $l(M^d)$ satisfies the characterization of Milnor. Before we obtain this result we need some preliminary facts.

Fact 1

Suppose $j: N^n \to M^m$ is an embedding of closed topological manifolds and that N^n has a normal bundle ν in M. Let k denote the map of M^m to the Thom space of ν, obtained by collapsing the complement of $E(\nu)$ to a point. If $\phi_{\nu *}$ is the Thom isomorphism then the map $k^* \phi_{\nu *}: H^t(N^n) \to H^{t+m-n}(M^m)$ is the Gysin homomorphism $j_!$. That is $k^* \cdot \phi_\nu{}^*$ is the map making the following diagram commutative

$$\begin{array}{ccc} H^t(N^n) & \xrightarrow{j_!} & H^{m-n+t}(M^m) \\ {\scriptstyle [N^n]\cap}\downarrow & & {\scriptstyle [M^m]\cap}\downarrow{\scriptstyle \simeq} \\ H_{n-t}(N^n) & \xrightarrow[j_*]{} & H_{n-t}(M^m) \end{array}$$

This follows from a careful analysis of the Thom isomorphism as pointed out to me by F. Raymond.

Fact 2

If $f: M^d \to T\xi$ is transversal to $E\xi$ with normal bundle ν, then the following diagram is homotopy-commutative

$$\begin{tikzcd} M^d \arrow[r, "f"] \arrow[dr, "k"'] & T\xi \\ & T_\nu \arrow[u, "T(f \mid E(\nu))"'] \end{tikzcd}$$

Proof. We first note that f is homotopic to a map $\tilde{f}: M^d \to T\xi$ such that $\tilde{f} \mid E(\nu) = f$ and $\tilde{f}$ maps the complement of $E(\nu)$ to the base point of $T\xi$. We define a homotopy as follows. Let H denote the strong deformation retraction of $T\xi - B\xi$ onto the base point. Then $F: M^d \times I \to T\xi$ is defined by

$$F \mid \overline{E(\nu)} \times I = f \mid \overline{E(\nu)}$$
$$F(x, t) = H(f(x), t) \text{ for } x \in \text{complement of } E(\nu).$$

This map will be continuous if $f(\overline{E(\nu)} - E(\nu)) = \text{base point}$. However, this follows since if $x \in \overline{E(\nu)} - E(\nu)$ and $f(x) \neq$ base point, then $f(x) \in E(\xi)$. Let U be a compact neighborhood of $f(x)$ not containing the base point. Since $f \mid E(\nu)$ is proper $(f \mid E(\nu))^{-1}(U)$ is compact. Therefore any sequence in $E(\nu)$ converging to x, must be in $(f \mid E(\nu))^{-1}(U)$ and converge to some point of $E(\nu)$ which is impossible since $x \notin E(\nu)$.

Now if $f \mid E(\nu)$ is a bundle map, then $T(f \mid E(\nu))$ is just the extension of $f \mid E(\nu)$ to the one point compactification of $E(\nu)$. Since $f \mid E(\nu) = \tilde{f} \mid E(\nu)$, $T(f \mid E(\nu)) = T(\tilde{f} \mid E(\nu))$ and clearly $T(\tilde{f} \mid E(\nu)) \circ k = \tilde{f}$. Therefore $T(f \mid E(\nu)) \circ k = \tilde{f} \simeq f$.

If $f \mid E(\nu)$ is only a bundle morphism, then $T(f \mid E(\nu))$ is obtained by choosing subbundles $\nu_1 \subseteq \nu$, and $\xi_1 \subseteq \xi$ and *letting* $T(f \mid E(\nu)) = T(h\xi_1) \circ T(f \mid E(\nu_1)) \circ (Th\nu_1)^{-1}$ where $h\xi_1$ and $h\nu_1$ are bundle maps fiber homotopic to the respective inclusion maps.

Now if $k_1: M^d \to T\nu_1$ is the collapsing map for $E(\nu_1)$ then $Th_{\nu_1} \circ k_1 \simeq k$. This follows for if h_t is the fiber preserving homotopy of $i: E(\nu_1) \to E(\nu)$ to h_{ν_1}, then we may for each t define a continuous fiber preserving map from $h_t(E(\nu_1))$ onto $E(\nu)$ by $h_{\nu_1} \circ h_t^{-1}$ Let $H: M^d \times I - T\nu$ be defined by $H_t \mid$ complement of $h_t(E(\nu_1)) = \text{base point}$. $H_t \mid h_t(E(\nu_1)) = h_{\nu_1} \circ h^{-1}$. Then H is a homotopy from $Th_{\nu_1} \circ k_1$ to k.

Consider the map $Th_{\xi_1}^{-1} \circ f: M^d \to T\xi_1$, this is t.r. to $E(\xi_1)$ with normal bundle ν_1 and $Th_{\xi_1}^{-1} \circ f \mid E(\nu_1)$ is a bundle map, hence from the preceding $Th_{\xi_1}^{-1} \circ f \simeq T(f \mid E(\nu_1)) \circ k_1$. That is

$$\begin{aligned} f &\simeq Th_{\xi_1} \circ T(f \mid E(\nu_1))\, k_1 \\ &= Th_{\xi_1} \circ T(f \mid E(\nu_1))\, Th_{\nu_1}^{-1} \circ Th_{*_1} \circ k_1 \\ &\simeq Th_{\xi_1} \circ T(f \mid E(\nu_1))\, Th_{\nu_1}^{-1} \circ k \\ &= T(f \mid E(\nu)) \cdot k. \end{aligned}$$

THEOREM 6. *Suppose* $f: M^d \to T\xi^k$ *is transversal to* $E(\xi)$ *with normal bundle* ν, *where* M^d *is a closed differentiable manifold. Then if* $d \equiv k$ (mod 4), $\sigma(B(\nu)) = \langle j^* L(M^d) \cup \bar{l}_*(\nu), [B\nu]\rangle$ *where* $j: B(\nu) \to M^d$ *is the inclusion map and* $L(M^d)$ *is the differentiable Hirzebruch class of* M^d.

Proof. From the definition of the $\bar{l}$-classes we have that

$$\sigma(B(\nu)) = \sigma(f^{-1}B(\xi)) = \langle L(M^d) \cup f^* \phi_{\xi *} \bar{l}(\xi), [M^d]\rangle$$

which by the previous two facts equals

$$\langle L(M^d) \cup k_\nu^* \phi_\nu^* f^* \bar{l}(\xi), [M^d]\rangle .$$

By the naturality of $\bar{l}$-classes with respect to bundle *morphisms* this equals $\langle L(M^d) \cup k_\nu^* \phi_\nu^* \bar{l}(\nu), [M^d]\rangle$. But $k^* \phi_\nu^*$ is the Gysin homomorphism $j_!$ and so this equals

$$\langle j_* L(M^d) \cup \bar{l}(\nu), [B\nu]\rangle .$$

COROLLARY. If P^{4p} is a closed topological manifold, then $\sigma(P^{4p}) = \langle l_p(P^{4p}), [P^{4p}]\rangle$.

Proof. Embed P^{4p} in S^d with normal bundle ν_p and let $k: S^d \to T\nu_p$ be the collapsing map. k is obviously transversal to $E(\nu_p)$ with normal bundle ν_p. By Theorem 5

$$\sigma(P^{4p}) = \langle j_* L(S^d) \cup \bar{l}(\nu_p), [P^{4p}]\rangle .$$

But $L(S^d) = 1$ so this is just $\langle \bar{l}_p(\nu_p), [P^{4p}]\rangle$.

COMPATIBILITY THEOREM. *If* M^d *is a closed PL-manifold, then* $l(M^d) = l'(M^d)$.

Proof. From Milnor's characterization of the classes $l'(M^d)$ we only need to show that if $g: M^d \to S^{d-4i}$ is a simplicial map, then

$$\sigma(g^{-1}(y)) = \langle l_i(M^d) \cup g^*(\mu), [M^d]\rangle .$$

Now $N^{4i} = g^{-1}(y)$ has a neighborhood in M^d homeomorphic to $N^{4i} \times \mathbf{R}^{d-4i}$, i.e. has a trivial normal bundle in M^d, [8]. Moreover N^{4i} is a PL-submanifold of M^d. Embed M^d is a large sphere S^k with normal bundle ν_p. Since N^{4i} has a normal bundle in M^d and M^d has a normal bundle in S^k it follows that N^{4i} has a normal bundle ν_N in S^k. Moreover we have

$$\nu_N \oplus \tau_N \simeq i^* \nu_M \oplus \nu_N^M \oplus \tau_N$$

and therefore ν_N is stably equivalent to $i^* \nu_M \oplus \nu_N^M$. But ν_N^M is trivial and so ν_N is stably equivalent to $i^*\nu_M$, and therefore $\bar{l}(\nu_N)=i^*\bar{l}(\nu_M)$. From the Corollary to Theorem 5 we have

$$\begin{aligned}\sigma(N^{4i}) &= \langle l_i(N^{4i}), [N^{4i}]\rangle \\ &= \langle i^*\bar{l}_i(\nu_M), [N^{4i}]\rangle \\ &= \langle \bar{l}_i(\nu_M), i_*[N^{4i}]\rangle .\end{aligned}$$

But using the fact that the Gysin homomorphism

$$i_!:H^*(N^{4i})\to H^*(M^d)$$

is $k^*\phi_\nu^*$ it is easy to see that $i_*[N^{4i}]\in H_{4i}(M^d)$ is the Poincaré dual of $g^*(\mu)\in H^{d-4i}(M^d)$. Therefore

$$\sigma(N^{4i}) = \langle \bar{l}_i(\nu_M)\cup g^*(\mu), [M^d]\rangle .$$

Once we are in possession of the compatibility theorem (that is the differentiably defined classes agree with the topological classes) we can easily obtain the following results.

COROLLARY 1. (Topological invariance of rational Pontrjagin classes, [9]). Suppose M_1^d and M_2^d are closed differentiable manifolds and $h:M_1^d\to M_2^d$ is a homeomorphism, then

$$h^*(i_*L(M_2^d)) = i_*L(M_1^d)$$

where i: $Z\to Q$ is the coefficient homomorphism.

Proof. From the compatibility theorem we only need to show $h^*(l(M_2^d))=l(M_1^d)$. But h being a topological homeomorphism induces an e.b. map of the topological tangent bundle of M_1^d to that of M_2^d. Since the stable normal bundles are (stable) inverses to the topological tangent bundles, the result follows.

COROLLARY 2. The natural map from differentiable cobordism, Ω_*^{DIFF}, to topological cobordism, Ω_*^{TOP}, is a monomorphism.

Proof. Since the differentiable cobordism classes are completely determined by the Pontrjagin and Whitney numbers and since we have a definition of Pontrjagin clases in the topological category, the standard proof will show that if M^d bounds topologically, all Pontrjagin and Whitney numbers are zero and hence if M^d is differentiable it represents the zero class in Ω_*^{DIFF}.

Let X be a sufficiently nice space (so that if ξ is an e.b. over X the classes $\bar{l}(\xi)$ are defined.

DEFINITION. *If ξ is an e.b. over X, let $l(\xi)=\bar{l}(\xi^{-1})$.*

THEOREM 1. *If ξ is a vector bundle over X, a C.W. complex, the $l(\xi)=i_*L(\xi)$ where $L(\xi)$ is the differentiable Hirzebruch class of ξ.*

Proof. Embed X in a high dimensional space and take a regular nbd U of X. Let $\bar{\xi}$ be the vector bundle over U corresponding to ξ. The tangent bundle of the total space of $\bar{\xi}$ decomposes as the Whitney sum $\pi^*\tau_U \oplus \pi^*\bar{\xi}$, where π is the projection of $E(\bar{\xi})$ onto U. Hence if s denotes the inclusion of the zero section into $E(\bar{\xi})$ we have that stably

$$[\bar{\xi}] = s^*[\tau_M] - [\tau_U].$$

The compatibility theorem says that the topological and differentiable classes agree for tangent bundles (after applying the coefficient homomorphism) and so the same is true for ξ, since the l-classes are multiplicative.

We finish this paper with a proof of a theorem which started the whole investigation.

THEOREM 8. If M^d is a closed topological manifold and $\tilde{M}^d$ is a finite regular covering of M^d, then $\sigma(M^d)=(\text{order of covering})\cdot\sigma(M^d)$.

Proof. The projection $\pi\colon \tilde{M}^d \to M^d$ is a local homeomorphism and so induces a bundle map of the topological tangent bundles, that is $\pi^*l(M^d)=l(\tilde{M}^d)$. Since $\pi_*[\tilde{M}^d]=n\cdot[M^d]$, where n is the order of the covering, the result follows from the corollary to Theorem 6.

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