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Quadratic Spaces of Countable Dimension over Algebraic Number Fields

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Introduction

Because of work done by I. KAPLANSKY (see, for instance, [6]) and more recent work done by H. R. FISCHER and H. GROSS (see, for instance, [1], [2], [3], and [4]) it now seems natural to study vector spaces of countable dimension over algebraic number fields.

One of the major accomplishments in the theory of quadratic forms is the characterization of quadratic spaces of finite dimension over arithmetic fields by a set of invariants which include the discriminant of the space, the dimension of the space, the Hasse symbols at all discrete spots, and the positive indices at all real spots (see [7], pp. 154–189).

In this paper we shall show that a denumerably infinite-dimensional vector space supplied with a symmetric non-degenerate bilinear form over an algebraic number field K is characterized by a set of $2n$ invariants. Here n is the number of distinct orderings of K (or equivalent n is the number of distinct real archimedean spots on K). Hence, if K is non-formally real, there is (up to orthogonal isomorphism) only one denumerably infinite-dimensional vector space with a non-degenerate bilinear form over K (cf. theorem 3 in [6]).

Sylvester's theorem holds for countably infinite-dimensional quadratic spaces over ordered fields (see [8], p. 522). Thus, if $n>0$, we shall obtain our $2n$ invariants by the use of Sylvester's theorem.

Many of the ideas in this paper were inspired by GROSS' work in [4].

1. Notation and basic concepts

Let K be a field. By a quadratic space (E, Φ) over K we mean a vector space E over K with a symmetric bilinear form $\Phi: E\times E\to K$. By a quadratic subspace (F, Φ) of (E, Φ) we shall mean that F is a linear subspace of E together with the form Φ restricted to $F\times F$. If there is no risk of confusion we shall write F instead of (F, Φ). A quadratic space (E, Φ) is semisimple (regular) if and only if Φ is non-degenerate.

If (E, Φ) is a quadratic space over K and A and B are linear subspaces of E such that $A\cap B=(0)$ then we shall denote the direct sum of A and B by $A\oplus B$. If we also have that $\Phi(A, B)=\{0\}$ then we shall write $A\overset{\perp}{\oplus}B$. If F is a linear subspace of E then we denote by $F^{\perp}$ the linear subspace of E comprising all those vectors

x of E such that $\Phi(x, y)=0$ for every y in F. If (E, Φ) is semisimple, and F is a finite-dimensional linear subspace of E such that (F, Φ) is semisimple then $E=F\overset{\perp}{\oplus}F^{\perp}$.

Suppose (E, Φ) is a quadratic space over K and $\{e_t\}_{t\in\Lambda}$ is a basis of (E, Φ) where Λ is some indexing set. $\{e_t\}_{t\in\Lambda}$ is an orthogonal basis if and only if $\Phi(e_t, e_{t'})=0$ for $t\neq t'$. $\{e_t\}_{t\in\Lambda}$ is an orthonormal basis if and only if $\{e_t\}_{t\in\Lambda}$ is an orthogonal basis and $\Phi(e_t, e_t)=1$ for every $t\in\Lambda$.

If char $K\neq 2$ then any quadratic space of countable dimension over K has an orthogonal basis. This is not true in general for quadratic spaces of higher dimension (see, for instance, GROSS' "Vandermonde" example [1], pp. 321–323).

We shall say that two quadratic spaces (E, Φ) and (G, Ψ) over K are orthogonally isomorphic if and only if there is a vector space isomorphism α from E onto G such that $\Phi(x, y)=\Psi(\alpha x, \alpha y)$ for every pair (x, y) in $E\times E$. If (E, Φ) and (G, Ψ) are orthogonally isomorphic we denote it by $(E, \Phi)\simeq(G, \Psi)$. If the dimension of E is countably infinite then we shall denote it by $\dim E=\aleph_0$ or $\dim(E, \Phi)=\aleph_0$.

II. Definition of the invariants and more terminology

From now on we assume that K is an algebraic number field.

Suppose K has precisely n orderings $(n>0)$. We shall denote these orderings by $\prec_1,\cdots,\prec_n$. Let (E, Φ) be a semisimple $\aleph_0$-dimensional quadratic space over K. Since (E, Φ) has an orthogonal basis, for each ordering $\prec_i$ we may decompose E such that $E=E_i\overset{\perp}{\oplus}E_i^-$ where $0\prec_i\Phi(x, x)$ for each $x\in E_i$ with $x\neq 0$ and $\Phi(x, x)\prec_i 0$ for each $x\in E_i^-$ with $x\neq 0$. Furthermore, since Sylvester's theorem holds in the countable case (see [8], p. 522), the dimensions of E_i and E_i^- are unique (E_i and E_i^- are not unique).

Define $\dim E_i=n_i$ and $\dim E_i^-=n_i^-$ for each i $(1\leqslant i\leqslant n)$. It is obvious that $\{n_1,\ldots, n_n, n_1^-,\ldots, n_i^-\}$ is a set of invariants for quadratic spaces orthogonally isomorphic to (E, Φ). We shall show that this is a complete set of (cardinal) invariants.

Let us consider for a moment the orderings on K. Each of these orderings is an archimedean ordering and each defines an archimedean valuation (I should say spot) on K say $|\ |_i$ from $\prec_i$. Furthermore, these valuations are all inequivalent and are all of the real archimedean valuations on K (see [7], pp. 30–35 and [5], pp. 287–289).

Now if $\sum_1^5\alpha_i X_i^2$ is a quadratic form with coefficients in K such that $\sum_1^5\alpha_i X_i^2$ is indefinite under each ordering of K then from the Hasse–Minkowski theory and what is stated above the form has a non-trivial zero in K. For an exposition of the Hasse–Minkowski theory see [7], pp. 154–189.

Again consider the orderings $\prec_1,\cdots,\prec_n$ and the corresponding valuations $|\ |_1,\ldots, |\ |_n$ defined by them. First of all, we may take $|\ |_i$ such that $|N|_i=N$ for each natural number N and all i $(1\leqslant i\leqslant n)$. Since $|\ |_1\ldots, |\ |_n$ is a set of inequivalent valuations, given $\alpha_1,\ldots, \alpha_n$ in K and a positive integer M there is a number α in K such that

$|\alpha_i - \alpha|_i \prec M^{-1}$ for all i between 1 and n. This is simply the weak approximation theorem and will be used frequently in the following exposition.

We now introduce signature functions which will reduce the number of words we must use later. For each i $(1 \leqslant i \leqslant n)$ we introduce the map $\psi_i : K^* \to \{-1, 1\}$ such that for $\alpha \in K^*$, $\Psi_i(\alpha) = 1$ if $0 \prec_i \alpha_i$ or $\psi_i(\alpha) = -1$ if $\alpha \prec_i 0$.

Here 1 and -1 are in K. (This is not important.)

We now define the map $\psi : K \to \{-1, 1\}^n$ such that for $\alpha \in K^*$,

$$\psi(\alpha) = (\psi_1(\alpha), \dots, \psi_n(\alpha)).$$

We are now in a position to prove a very useful lemma.

LEMMA 1. *Let (E, Φ) be a semisimple quadratic space over K. Suppose there exist vectors $x_1, \dots, x_n$ in E (not necessarily distinct) such that $\Phi(x_i, x_i) \neq 0$ and $\psi_i(\Phi(x_i, x_i)) = s_i$ for each i. Then there is a vector $x \in E$ such that $\psi(\Phi(x, x)) = (s_1, \dots, s_n)$.*

Proof. By Sylvester's theorem in the finite case E must contain a semisimple subspace F with $\dim F = q \leqslant n$ and an orthogonal basis $\{e_1, \dots, e_q\}$ such that for each i $(1 \leqslant i \leqslant n)$ there exists e_j with $\psi_i(\Phi(e_j, e_j)) = s_i$.

Let N be a positive integer (we shall assume that N is in K or in the reals whenever convenient). For each i $(1 \leqslant i \leqslant n)$ define

$$\alpha_{li} = 0 \quad \text{if} \quad \psi_i(\Phi(e_l, e_l)) = -s_i \quad \text{or} \quad \alpha_{li} = N \quad \text{if} \quad \psi_i(\Phi(e_l, e_l) = s_i.$$

Now from the weak approximation theorem we can find α_l in K such that $|\alpha_l - \alpha_{li}|_i < N^{-1}$ for each i.

Do this for each l between 1 and q. Then by taking N large enough we have, by the continuity of multiplication and addition over valuated fields, that $x = \sum_1^q \alpha_l e_l$ is the vector desired.

We must note, however, that several rather delicate points in the above proof must be cleared up. First of all, we have taken a fixed field of real numbers R with an ordering $\prec$. The completion of K under each valuation $|\ |_i$ yields a field K_i which is order isomorphic to R. The ordering of K_i when restricted to K is $\prec_i$. Without confusion we may say that the ordering on K_i is $\prec_i$. Secondly, if μ is the order isomorphism between K_i and R and $]\ [_i$ is the ordinary absolute value in K_i then we have that

(1) $\alpha <_i \beta$ if and only if $\mu(\alpha) \prec \mu(\beta)$ $\quad (\alpha, \beta \in K_i)$,

(2) $\mu([\alpha]_i) = |\mu(\alpha)| = |\alpha|_i$ $\quad (\alpha \in K)$,

(3) $]\alpha[_i \prec_i N^{-1}$ if and only is $|\alpha|_i < N^{-1}$ $\quad (\alpha \in K)$.

Thus, we now note that $|\alpha_l - \alpha_{li}|_i < N^{-1}$ for each i implies that $-N^{-1} \prec_i \alpha_l - \alpha_{li} \prec_i N^{-1}$ for each i.

and the "limit part" of the proof the Lemma 1 becomes clear.

III. The characterization theorem

We are now in a position to state the main assertion of the paper.

CHARACTERIZATION THEOREM. *Let* (E, Φ) *be a semisimple* $\aleph_0$*-dimensional quadratic space over an algebraic number field K with n orderings. Then (a) if K is non-formally real,* (E, Φ) *has an orthonormal basis; (b) if* $n>0$, (E, Φ) *is completely characterized, up to orthogonal isomorphism, by the invariants* $\{n_1,\dots, n_n, n_1^-,\dots, n_n^-\}$ *defined in Section II.*

The proof of (a) follows from a known result of GROSS' (see Corollary 4 [2], p. 290) which states that if a semisimple $\aleph_0$-dimensional quadratic space (G, Φ) over a non-formally real field has an $\aleph_0$-dimensional subspace F such that $\Phi(F, F)=\{0\}$ then (G, Φ) has an orthonormal basis. Since (E, Φ) has an orthogonal basis and every five-dimensional quadratic space over a non-formally real algebraic number field contains a non-zero isotropic vector (i.e. a vector $x\neq 0$ such that $\Phi(x, x)=0$) it follows that (E, Φ) contains such a subspace.

A major part of the proof of part (b) will be contained in the following three lemmas. The proof of part (b) of the characterization theorem will be completed in Section 4. From now on, assume K is an algebraic number field with n orderings ($n>0$).

LEMMA 2. *Let* (E, Φ) *and* (F, Ψ) *be two semisimple quadratic spaces over K* $\dim E=\dim F=\aleph_0$).

Suppose (1) $\dim E_i=\dim F_i$ *each i* $(1\leqslant i\leqslant n)$,
(2) $\dim E_i^-=\dim F_i^-$ *each i* $(1\leqslant i\leqslant n)$,
(3) $\dim E_i<\aleph_0$ or $\dim E_i^-<\aleph_0$ *each i*,

then $(E, \Phi)\simeq(F, \Psi)$.

Proof. We first consider the case where $\dim E_i=0$ or $\dim E_i^-=0$ for each i $(1\leqslant i\leqslant n)$. Let $\{x_i\}_{i\geqslant 1}$ be a fixed basis of E and let $\{y_i\}_i\geqslant 1$ be a fixed basis of F. We note that in this particular case $\psi(\Phi(x, x))=\psi(\Psi(y, y)$ for every $x\neq 0$, $x\in E$ and for every $y\neq 0$, $y\in F$.

To prove that E and F are orthogonally isomorphic we shall use the standard inductive procedure.

Suppose we have been able to find finite dimensional semisimple spaces ${}_pE\subset E$ and ${}_pF\subset F$ such that ${}_pE\simeq{}_pF$ with $x_i\in{}_pE$ for $i\leqslant p$, and $y_i\in{}_pF$ for $i\leqslant p$. Let x_l be the first vector of the fixed basis not in ${}_pE$. Since ${}_pE$ is semisimple we may assume that $x_l\in{}_pE^\perp$. Let y_m be the first vector of the fixed basis not in ${}_pF$. Since ${}_pF$ is semisimple we may assume that $y_m\in{}_pF^\perp$.

If we can find a semisimple finite dimensional space ${}_p\bar{E}$ in ${}_pE^\perp$ and a finite dimensional semisimple space ${}_p\bar{F}$ in ${}_pF^\perp$ such that $x_l\in{}_p\bar{E}$, $y_m\in{}_p\bar{F}$ and ${}_p\bar{E}\simeq{}_p\bar{F}$ we shall be done. Take four linearly independent mutually orthogonal vectors $\{y_m, f_1, f_2, f_3\}$

in ${}_pF^{\perp}$. Then the quadratic form

$$-\Phi(x_l, x_l)\, z_5^2 + \Psi(y_m, y_m)\, z_4^2 + \textstyle\sum_1^3 \Psi(f_i, f_i)\, z_i^2$$

has a non-trivial zero since the form is indefinite with respect to each ordering on K by the particular assumption of this case. Thus, the space generated by $\{y_m, f_1, f_2, f_3\}$ has an orthogonal basis $\{u_0, u_1, u_2, u_3\}$ with $\Phi(x_l, x_l) = \Psi(u_0, u_0)$. Call this space ${}_pF$. Using the same sort of argument above we may find vectors $\bar{u}_1, \bar{u}_2, \bar{u}_3$ in ${}_pE^{\perp}$ such that $\{x_l, \bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is a set of linearly independent mutually orthogonal vectors such that $\Phi(\bar{u}_i, \bar{u}_i) = \Psi(u_i, u_i)$; $i = 1, 2, 3$. Denote by ${}_p\bar{E}$ the space generated by $\{x_l, \bar{u}_1, \bar{u}_2, \bar{u}_3\}$. Then ${}_p\bar{E} \simeq {}_pF$, each are semisimple with $x_l \in {}_p\bar{E}$ and $y_m \in {}_pF$ and we are done with this case.

We now finish the proof of Lemma 2 by the use of induction. Suppose for some integer $l > 0$ we have proved: if $n_i < l$ or $n_i^- < l$ for each i $(1 \leqslant i \leqslant n)$ then $E \simeq F$. Thus suppose for each i that $n_i \leqslant l$ or $n_i^- \leqslant l$. Furthermore, assume that for some i $\dim E_i = l$ or $\dim E_i^- = l$. For otherwise we are done by induction hypothesis. We pick $(s_1, \ldots, s_n) \in \{-1, 1\}^n$ in the following manner: for each i

$$s_i = \begin{cases} 1 & \text{if } 1 \leqslant \dim E_i < \aleph_0, \\ 1 & \text{if } \dim E_i^- = 0, \\ -1 & \text{if } 1 \leqslant \dim E_i^- < \aleph_0, \\ -1 & \text{if } \dim E_i = 0. \end{cases}$$

By Lemma 1 there are $x \in E$ and $y \in F$ such that

$$\psi(\Phi(x, x)) = \psi(\Psi(y, y)) = (s_1, \ldots, s_n).$$

Write $E = K(x) \overset{\perp}{\oplus} \bar{E}$ and $F = K(y) \overset{\perp}{\oplus} \bar{F}$. Here $K(x)$ and $K(y)$ are the vector spaces over K generated by x and y respectively. Now take $(t_1, \ldots, t_n) \in \{-1, 1\}^n$ such that for each i

$$t_i = \begin{cases} 1 & \text{if } \dim E_i = \aleph_0, \\ -1 & \text{if } \dim E_i^- = \aleph_0. \end{cases}$$

Again by Lemma 1 (used three times together with Sylvester's theorem) we choose three linearly independent mutually orthogonal vectors $\{y_1, y_2, y_3\}$ in $\bar{F}$ such that

$$\psi(\Psi(y_i, y_i)) = (t_1, \ldots, t_n) \quad \text{and}$$

$$F = K(y) \overset{\perp}{\oplus} K(y_1) \overset{\perp}{\oplus} K(y_2) \overset{\perp}{\oplus} K(y_3) \overset{\perp}{\oplus} H.$$

Now it is easy to see that the quadratic form

$$-\Phi(x, x)\, Z_5^2 + \Psi(y, y)\, Z_4^2 + \Psi(y_1, y_1)\, Z_1^2 + \Psi(y_2, y_2)\, Z_2^2 + \Psi(y_3, y_3)\, Z_3^2$$

is indefinite under each ordering $\prec_i$. Hence, the form has a non-trivial zero in K and, thus, the space

$$K(y)\overset{\perp}{\oplus} K(y_1)\overset{\perp}{\oplus} K(y_2)\overset{\perp}{\oplus} K(y_3)$$

contains a vector $\bar{x}$ such that $\Phi(x,x)=\Psi(\bar{x},\bar{x})$. Therefore $E=K(x)\overset{\perp}{\oplus}K(x)^{\perp}$, $F=K(\bar{x})\overset{\perp}{\oplus}K(\bar{x})^{\perp}$ and $K(x)\simeq K(\bar{x})$. The invariants of $K(x)^{\perp}$ and $K(\bar{x})^{\perp}$ are $\{m_1,\dots,m_n,m_1^-,\dots,m_n^-\}$ where $m_i=n_i$ if $n_i=\aleph_0$; $m_i=n_i-1$ if $n_i<\aleph_0$; $m_i^-=n_i^-$ if $n_i^-=\aleph_0$; and $m_i^-=n_i^--1$ if $n_i^-<\aleph_0$. By induction hypothesis we have that $K(x)^{\perp}\simeq K(\bar{x})^{\perp}$ and hence $E\simeq F$. This completes the proof of Lemma 2.

We continue our extensive task of bookkeeping by stating the following lemma.

LEMMA 3. *Let (E,Φ) be a semisimple $\aleph_0$-dimensional quadratic space over K. Further suppose that the set of integers $\{i_1,\dots,i_k\}\subset\{1,\dots,n\}$ such that $\dim E_{i_j}=\dim E_{i_j}^-=\aleph_0\ (1\leqslant j\leqslant k)$ is not empty and that if $m\in\{1,\dots n\}-\{i_1,\dots,i_k\}$ either $\dim E_m=0$ or $\dim E_m^-=0$. Then there is a semisimple $\aleph_0$-dimensional quadratic space (G,Ω) with $G=G^+\overset{\perp}{\oplus}G^-$ and such that*

(1) *if $x\in G^+$ $(x\neq 0)$ then*

$$\psi_{i_j}(\Omega(x,x))=1,\quad \forall i_j\in\{i_1,\dots,i_k\},$$

and if $m\in(1,\dots,n)-(i_1,\dots,i_k)$ then

$$\psi_m(\Omega(x,x))=\begin{cases}1 & \text{if } \dim E_m=\aleph_0,\\ -1 & \text{if } \dim E_m^-=\aleph_0;\end{cases}$$

(2) *if $x\in G^-$ $(x\neq 0)$ then*

$$\psi_{i_j}(\Omega(x,x))=-1\quad \forall i_j\in\{i_1,\dots,i_k\},$$

and if $m\in\{1,\dots,m\}-\{i_1,\cdots,i_k\}$ then

$$\psi_m(\Omega(x,x))=\begin{cases}1 & \text{if } \dim E_m=\aleph_0,\\ -1 & \text{if } \dim E_m^-=\aleph_0;\end{cases}$$

(3) $(G,\Omega)\simeq(E,\Phi)$.

Note. It follows that $\dim G^+=\dim G^-$.

Proof. We first show the existence of such a quadratic space (G,Ω). Let $G=K\{e_i\}_{i\geqslant 1}\oplus K\{f_i\}_{i\geqslant 1}$ be an arbitrary $\aleph_0$-dimensional vector space over K. We have only to define the bilinear form Ω.

Define $(s_1,\dots,s_n)\in\{-1,1\}^n$ such that

$$\begin{aligned} s_{i_j}&=1 && \text{if } i_j\in\{i_1,\dots,i_k\},\\ s_m&=1 && \text{if } \dim E_m=\aleph_0,\\ &\ \ -1 && \text{if } \dim E_m^-=\aleph_0, \end{aligned}$$

for $m\in\{1,\dots,n\}-\{i_1,\dots,i_k\}$.

Now by the weak approximation theorem, given a positive integer N there is $\alpha \in K$ such that

$$|\alpha - s_j|_j < N^{-1} \quad \text{for each } j, \quad 1 \leqslant j \leqslant n.$$

Thus, if we take N large enough we have

$$\psi(\alpha) = (s_1, \dots, s_n).$$

Now define $(t_1, \dots, t_n) \in \{-1, 1\}^n$ such that

$$\begin{aligned} t_{i_j} &= -1 \quad \text{if } i_j \in \{i_1, \dots, i_k\}, \\ t_m &= 1 \quad \text{if } \dim E_m = \aleph_0, \\ &\;\; -1 \quad \text{if } \dim E_m^- = \aleph_0, \end{aligned}$$

for $m \in \{1, \dots, n\} - \{i_1, \dots, i_k\}$.

Again by the weak approximation theorem we can find $\beta \in K$ such that

$$\psi(\beta) = (t_1, \dots, t_n).$$

Thus we define $G^+ = K\{e_i\}_{i \geqslant 1}$, $G^- = K\{f_i\}_{i \geqslant 1}$, and Ω by

$$\begin{aligned} &\Omega(e_i, f_j) = 0 \quad \text{all } i, j \geqslant 1, \\ &\Omega(e_i, e_j) = \alpha\delta_{ij}, \\ &\Omega(f_i, f_j) = \beta\delta_{ij} \end{aligned}$$

where δ_{ij} is the Kronecker delta. Hence we have proved (1) and (2) of the lemma. It remains to prove (3).

Let $\{x_i\}_{i \geqslant 1}$ be a fixed orthogonal basis for (E, Φ). The space $K\{e_1, e_2, f_1, f_2\}$ represents $\Phi(x_1, x_1)$; for the form

$$-\Phi(x_1, x_1) Z_1^2 + \alpha Z_2^2 + \alpha Z_3^2 + \beta Z_4^2 + \beta Z_5^2$$

is indefinite under every ordering of K. Thus,

$$K\{e_1, e_2, f_1, f_2\} = K\{u_1, u_2, u_3, u_4\}$$

where $\{u_i\}_{1 \leqslant i \leqslant 4}$ is an orthogonal basis with $\Omega(u_1, u_1) = \Phi(x_1, x_1)$.

On the other hand, by Sylvester's theorem and three successive uses of Lemma 1 we can find three linearly independent mutually orthogonal vectors in $K(x_1)^\perp$ $(E = K(x_1) \overset{\perp}{\oplus} K(x_1)^\perp)$ say $\bar{u}_2, \bar{u}_3, \bar{u}_4$ such that $\Phi(\bar{u}_i, \bar{u}_i) = \Omega(u_i, u_i)$ for $2 \leqslant i \leqslant 4$.

Set $K\{u_1, \dots, u_4\} = {}_1\bar{G}$ and $K\{x_1, \bar{u}_2, \bar{u}_3, \bar{u}_4\} = {}_1\bar{E}$. Now ${}_1\bar{E} \simeq {}_1\bar{G}$ and we can continue this procedure in ${}_1\bar{E}^\perp$ and ${}_1\bar{G}^\perp$ to "finally" show that $E = \overset{\perp}{\oplus}{}_1^\infty {}_i\bar{E}$; $G = \overset{\perp}{\oplus}{}_1^\infty {}_i\bar{G}$ with ${}_i\bar{E}_i \simeq \bar{G}$ all $i \geqslant 1$. Thus, $(E, \Phi) \simeq (G, \Omega)$ and this completes the proof of Lemma 3.

LEMMA 4. *Let (E, Φ) have the same properties as in Lemma 3. Then if (F, Ψ) is another quadratic space with*

$$\dim E_i = \dim F_i,$$
$$\dim E_i^- = \dim F_i^-$$

for all i $(1 \leqslant i \leqslant n)$ *we have that* $(E, \Phi) \simeq (F, \Psi)$.

Proof. The proof is obvious from the existence of (G, Ω) in Lemma 3.

IV. Completion of the proof of the characterization theorem

We are now in a position to complete the proof of (b) in the characterization theorem. We have in this case that the number n of orderings of K is positive. Let (E, Φ) and (F, Ψ) be two semisimple quadratic spaces over K such that

$$\dim E_i = \dim F_i = n_i \quad \text{each } i, \quad 1 \leqslant i \leqslant n;$$
$$\dim E_i^- = \dim F_i^- = n_i^- \quad \text{each } i, \quad 1 \leqslant i \leqslant n.$$

Let $A = \{1, \ldots, n\}$

$B = \{i \in A \mid \ \dim E_i = \dim E_i^- = \aleph_0\}$

$C = \{i \in A \mid \ \text{either} \ \dim E_i < \aleph_0 \ \text{or} \ \dim E_i^- < \aleph_0\}$

Then $A = B \cup C$ and $B \cap C$ is null. We now divide the proof of part (b) into several cases.

Case 1. If B is null then Lemma 2 is the proof of part (b).

Case 2. If C is null then Lemma 4 is the proof of part (b).

Let

$$\bar{C} = \{i \in C \mid \ \dim E_i = 0 \quad \text{or} \quad \dim E_i^- = 0\}$$
$$\bar{\bar{C}} = \{i \in C \mid \ 1 \leqslant \dim E_i < \aleph_0 \quad \text{or} \quad 1 \leqslant \dim E_i^- < \aleph_0\}.$$

Case 3. If C is not null and B is not null we shall show that we may reduce the proof to the proof of Case 2.

Case 3a. If $\bar{\bar{C}}$ is null then again we may apply Lemma 4.

Case 3b. We are now left with the case where B is not null and $\bar{\bar{C}}$ is not null.

Define $(t_1, \ldots, t_n) \in \{-1, 1\}^n$ such that for each i

$$t_i = \begin{cases} 1 & \text{if } \dim E_i = \dim E_i^- = \aleph_0, \\ -1 & \text{if } \dim E_i = 0, \\ 1 & \text{if } 1 \leqslant \dim E_i < \aleph_0, \\ 1 & \text{if } \dim E_i^- = 0, \\ -1 & \text{if } 1 \leqslant \dim E_i^- < \aleph_0. \end{cases}$$

Define $(s_1, \ldots, s_n) \in \{-1, 1\}^n$ such that for each i

$$s_i = \begin{cases} 1 & \text{if } \dim E_i = \aleph_0, \\ -1 & \text{if } 0 \leqslant \dim E_i < \aleph_0 \end{cases}$$

By Lemma 1 there are $x \in E$ and $y \in F$ such that

$$\psi(\Phi(x, x)) = \psi(\Psi(y, y)) = (t_1, \dots, t_n).$$

By successive uses of Lemma 1 (and Sylvester's theorem) there are three mutually orthogonal linearly independent vectors in $K(y)^{\perp}$ say y_1, y_2, y_3 such that

$$F = K(y) \overset{\perp}{\oplus} K(y_1) \overset{\perp}{\oplus} K(y_2) \overset{\perp}{\oplus} K(y_3) \overset{\perp}{\oplus} \bar{F}$$

and

$$\psi(\Psi(y_i, y_i)) = (s_1, \dots, s_n), \quad 1 \leqslant i \leqslant 3.$$

By the choice of $(t_1, \dots, t_n)$ and $(s_1, \dots, s_n)$ the form

$$-\Phi(x, x)\, z_5^2 + \Psi(y, y)\, z_4^2 + \Psi(y_1, y_1)\, z_1^2 + \Psi(y_2, y_2)\, z_2^2 + \Psi(y_3, y_3)\, z_3^2$$

is indefinite under every ordering on K. Thus, the form has a non-trivial zero in K. Hence, $K(y) \overset{\perp}{\oplus} K(y_1) \overset{\perp}{\oplus} K(y_2) \overset{\perp}{\oplus} K(y_3)$ has a vector $\bar{x} \neq 0$ such that $\Phi(x, x) = \Psi(\bar{x}, \bar{x})$. Hence $E = K(x) \overset{\perp}{\oplus} K(x)^{\perp}$, $F = K(y) \overset{\perp}{\oplus} K(y)^{\perp}$ and $K(x) \simeq K(\bar{x})$. Continue to reapply this procedure (the second time on $K(x)^{\perp}$ and $K(\bar{x})^{\perp}$. After a finite number of steps we obtain $\bar{E} \subset E$ and $\bar{F} \subset F$ which are finite dimensional semisimple quadratic spaces such that $\bar{E} \simeq \bar{F}$ and if $E = \bar{E} \overset{\perp}{\oplus} \bar{E}^{\perp}$ and $F = \bar{F} \overset{\perp}{\oplus} \bar{F}^{\perp}$ then (again using Sylvester's theorem) for each i,

$$\dim \bar{E}_i^{\perp} = \dim \bar{F}_i^{\perp},$$
$$\dim \bar{E}_i^{\perp -} = \dim \bar{F}_i^{\perp}$$

and either

$$(1) \quad \dim \bar{E}_i^{\perp} = \dim \bar{E}_i^{\perp -} = \aleph_0$$

or

$$(2) \quad \dim \bar{E}_i^{\perp} = 0 \quad \text{or} \quad \dim \bar{E}_i^{\perp -} = 0.$$

We can thus apply Lemma 4 to show that $\bar{E}^{\perp} \simeq \bar{F}^{\perp}$. Hence $(E, \Phi) \simeq (F, \Psi)$ in the Case 3a and the characterization theorem is proved.

Remark. In retrospect, since the discriminant and the Hasse symbols in the sense of finite-dimensional quadratic spaces fail to make sense in the denumerably infinite case, one would expect that a denumerably infinite-dimensional quadratic space over an algebraic field would have to be described by the positive and negative indices at all real archimedean spots. For quadratic spaces of higher dimension, the field seems to lose much of its importance as far as describing the quadratic spaces. In this case, inductive proofs such as the ones used in the proofs of Lemma 2 and Lemma 3 are, of course, not applicable. Again, as pointed out in the introduction, there are examples of semisimple quadratic spaces of dimension greater than countable which do not possess an orthogonal basis and, furthermore, cannot be imbedded in a quadratic space which has an orthogonal basis.

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