Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 43 (1968)

Artikel: An Approximate Reciprocity Formula for Some Exponential Sums.

Autor: Chandrasekharan, K. / Narasimhan, R.

DOI: https://doi.org/10.5169/seals-32924

An Approximate Reciprocity Formula for Some Exponential Sums

by K. CHANDRASEKHARAN and RAGHAVAN NARASIMHAN

§ 1. The Dedekind zeta-function $\zeta_K(s, \mathfrak{C})$ of an ideal class $\mathfrak{C}$ in a quadratic field $K = Q(\sqrt{d})$ can be represented by the Dirichlet series $\sum a_m m^{-s}$ in the half-plane $\operatorname{Re} s > 1$, where a_m is the number of non-zero integral ideals of class $\mathfrak{C}$ with norm m. In a recent paper [4] we showed that

$$\sum_{m \le T} a_m \exp(2\pi i m x) = o(T), \quad \text{as} \quad T \to \infty, \tag{1.1}$$

for any irrational x, and used this fact to prove that $\zeta_K(\tfrac{1}{2} + it, \mathfrak{C})$ vanishes for an infinity of real values of t. This result goes through also for zeta-functions with grössencharacters associated with quadratic fields.

We shall here prove the following

THEOREM 1. *If $\lambda = |d|^{1/2}$, and $0 < x \le \lambda$, we have*

$$\sum_{m \le X} a_m \exp(2\pi i m x/\lambda) = \frac{c_1}{x} \sum_{m \le X x^2} a_m \exp(-2\pi i m/\lambda x) + O(X^{1/2} \log X), \tag{1.2}$$

provided that $X x^2 \ge 1/\lambda > 0$. Here $c_1 = 1$, or i, according as the field K is real or imaginary.

If $0 < X x^2 < 1/\lambda$, then

$$\sum_{m \le X} a_m (\exp 2\pi i m x/\lambda) = \frac{c_2}{2\pi i x} (\exp(2\pi i x X/\lambda) - 1) + O(X^{1/3}), \tag{1.3}$$

where c_2 is a constant which depends on the field. (If κ denotes the residue of $\zeta_K(s, \mathfrak{C})$ at $s = 1$, then $c_2 = \kappa\lambda$, or $\kappa\lambda/2\pi$, according as K is real or imaginary). The error-terms in (1.2) and (1.3) are uniform with respect to x.

From this we deduce the following

COROLLARY. *If x is rational, and $x = 1/(kd)$, where k is an integer, then*

$$\sum_{m \le X} a_m \exp(2\pi i m x) = c_k X + O(X^{1/2} \log X), \quad \text{as} \quad X \to \infty, \tag{1.4}$$

where c_k is a constant which depends on k and the field.

When $a_m = d(m)$, the number of divisors of m, formulas of the type (1.2) and (1.3) were first given by J. R. WILTON [6], who used them to study the order of magnitude of sums of the type

$$\sum_{m \le X} d(m) \cos(2\pi m x),$$

for x belonging to various classes of numbers. Such a study was originated by HARDY and LITTLEWOOD, and later carried on by a number of authors. References to the literature can be found in WILTON's paper [6].

The proof of Theorem 1 requires, among other things, properties of convergence of a class of infinite series of Bessel functions with coefficients a_m. The study of such series when $a_m = d(m)$, or $r(m)$ (the number of representations of m as a sum of two squares), originated with VORONOI, HARDY, and LANDAU [5]. In two of our earlier papers [1, 2] we gave a general method for attacking the convergence problem for a wide class of such series. While the proof of Theorem 1 can be effected *without* using the sharpest known results on such series, it seems possible, by using them, to obtain a general summation formula for a_m. This is given in Theorem 2, § 5, and includes the VORONOI–HARDY–LANDAU formula when $K = Q(\sqrt{-4})$. Our proof goes through also in the case $d=1$ of equation (2.1), when ζ_K is the square of Riemann's zeta-function, so that Voronoi's summation formula for the divisor function is also included.

We are indebted to Professor C. L. SIEGEL for critically reading a first version of this paper and making several helpful comments. He pointed out to us that in the case of an imaginary quadratic field, the constant $c_1 = i$, and that an additional term $f(0)/w$ appears in (5.6). He has given an alternative method for sharpening the Corollary to Theorem 1, which we quote, with his permission, in § 6.

§ 2. We shall first prove Theorem 1 for a real quadratic field $K = Q(\sqrt{d})$, $d > 0$. In that case, we have

$$\zeta_K(s, \mathfrak{C}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} = \zeta_K(s, \tilde{\mathfrak{C}}), \qquad \operatorname{Re} s > 1,$$

where $\tilde{\mathfrak{C}}$ is the ideal class conjugate to $\mathfrak{C}$, and the following functional equation holds:

$$\left(\frac{\sqrt{d}}{\pi}\right)^s \Gamma^2\left(\frac{s}{2}\right) \zeta_K(s, \mathfrak{C}) = \left(\frac{\sqrt{d}}{\pi}\right)^{1-s} \Gamma^2\left(\frac{1-s}{2}\right) \zeta_K(1-s, \mathfrak{C}).$$

This is of the form

$$\Gamma^2\left(\frac{s}{2}\right) \phi(s) = \Gamma^2\left(\frac{1-s}{2}\right) \phi(1-s), \tag{2.1}$$

where

$$\phi(s) = \sum_{m=1}^{\infty} \frac{a_m}{\mu_m^s}, \qquad \mu_m = \frac{m\pi}{\lambda}, \qquad \lambda = \sqrt{d}. \tag{2.2}$$

Let

$$A_\mu^\varrho(x) = \frac{1}{\Gamma(\varrho+1)} \sum_{\mu_n \le x}' a_n (x - \mu_n)^\varrho,$$

for $x>0$, $\varrho\geq 0$, the dash denoting that the last term has to be multiplied by $\frac{1}{2}$, if $\varrho=0$ and $x=\mu_n$. By the standard convention that an empty sum is zero, $A_\mu^\varrho(x)=0$ for $0\leq x<\mu_1$, $\varrho\geq 0$.

According to a previous result of ours [2, p. 116], equation (2.1) implies the existence of an identity, the precise form of which is given by the following

LEMMA 1. *Equation* (2.1) *implies the identity*

$$A_\mu^\varrho(x)-P_\varrho(x)=-2^{-\varrho}\sum_{m=1}^{\infty}a_m\left(\frac{x}{\mu_m}\right)^{(1+\varrho)/2}F_{1+\varrho}\{4(x\mu_m)^{1/2}\}, \tag{2.3}$$

where $x>0$, $F_\nu(x)=Y_\nu(x)+(-1)^{\nu-1}(2/\pi)K_\nu(x)$, Y_ν *and* K_ν *being the well-known Bessel functions,* ϱ *is an integer, such that* $\varrho\geq 0$, $\varrho\geq 2\beta-\frac{5}{2}$, *where* $\beta=1+\varepsilon>1$, *and*

$$P_r(x)=\frac{1}{2\pi i}\int_{\mathscr{C}_r}\frac{\Gamma(s)\,\varphi(s)\,x^{s+r}\,ds}{\Gamma(s+r+1)},$$

where $\mathscr{C}_r$ *is a curve which encloses all the singularities of the integrand. The series in* (2.3) *converges uniformly in any interval of the positive real axis in which the function on the left is continuous. If* $\varrho=0$, *it converges boundedly in any interval* $0<x_1\leq x\leq x_2<\infty$.

This lemma is a consequence of Theorem 7.1(c) of [2], since the condition given there, namely

$$\sup_{0\leq h\leq 1}\Big|\sum_{n^2<\mu_m<(n+h)^2}a_m\mu_m^{1/2-\beta}\Big|=o(1),$$

as $n\to\infty$, is fulfilled, because of the known estimate [2, p. 128]

$$\sum_{m\leq x}a_m=\kappa x+O(x^{1/3}), \tag{2.4}$$

where κ is the residue of $\zeta_K(s,\mathfrak{C})$ at its only pole $s=1$, and of the fact that $\beta=1+\varepsilon>1$. The method of proof is that of equiconvergent trigonometric integrals used by us in [1]. The assumption that ϱ is an integer is not necessary; but our applications do not require more.

In the case of a real quadratic field, it is known that

$$P_0(x)=\frac{2\log\eta}{\pi}x,$$

$\eta>1$ being the fundamental unit in K, while

$$P_1(x)=-\zeta_K(-1,\mathfrak{C})\frac{\pi}{\sqrt{d}}+\frac{\log\eta}{\pi}x^2.$$

Thus, if we take $\varrho=0$, and write πy for x, we get

$$A(y)=\sum_{\lambda_m\leq y}{}' a_m=P_0(\pi y)-\sum_{m=1}^{\infty}\frac{a_m}{\lambda_m^{1/2}}y^{1/2}F_1[4\pi(\lambda_m y)^{1/2}], \tag{2.5}$$

where now

$$\lambda_m = \frac{m}{\lambda}, \lambda = \sqrt{d}, P_0(\pi y) = 2\log\eta \cdot y = c \cdot y, \quad \text{say.} \tag{2.6}$$

we define for $y>0$, $\varrho \geq 0$,

$$A^\varrho(y) = A^\varrho_\lambda(y) = \frac{1}{\Gamma(\varrho+1)} \sum_{\lambda_m \leq y}{}' a_m (y-\lambda_m)^\varrho .$$

If we set

$$I_\nu(y) = -\, y^{(\nu+1)/2} F_{\nu+1}(4\pi y^{1/2}), \tag{2.7}$$

then we can rewrite (2.5) as

$$A(y) = c\,y + \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m} I_0(\lambda_m y), \quad \lambda_m = \frac{m}{\lambda}. \tag{2.8}$$

Because of Lemma 1, with $\varrho=0$, the infinite series on the right-hand side of (2.8) converges boundedly for $0<\alpha\leq y\leq\alpha'<\infty$. But this result is *not* necessary for the proof of Theorem 1.

The case $\varrho=1$ of Lemma 1 gives

$$A^1(y) = \tfrac{1}{2} c\, y^2 + \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^2} I_1(\lambda_m y), \tag{2.9}$$

since

$$\frac{d}{dx}[x^{\nu/2} F_\nu\{4\pi(\lambda_m x)^{1/2}\}] = 2\pi\lambda_m^{1/2} x^{(\nu-1)/2} F_{\nu-1}[4\pi(\lambda_m x)^{1/2}]$$
$$= -\,2\pi\lambda_m^{1/2} I_{\nu-2}(\lambda_m x),$$

or

$$\frac{d}{dx}[I_{\nu-1}(\lambda_m x)] = 2\pi\lambda_m \cdot I_{\nu-2}(\lambda_m x), \tag{2.10}$$

and the series in (2.9) is uniformly convergent for $y\geq 0$. This result is much easier to establish than the convergence of the series in (2.8).

If f is a function which is twice continuously differentiable in $[0, \infty)$, then by Abel's Lemma on partial summation, we have

$$\sum_{\lambda_m \leq X} a_m f(\lambda_m) = A(X) f(X) - \int_{\lambda_1}^{X} A(t) f'(t)\, dt,$$

where f' is the derivative of f. If we integrate by parts, once more, we have

$$\sum_{\lambda_m \leq X} a_m f(\lambda_m) = A(X) f(X) - A^1(X) f'(X) + \int_0^X A^1(t) f''(t)\, dt.$$

If we choose $f(t)=\exp(2\pi i x t)$, $x>0$, and use (2.9) and (2.4) we get

$$\sum_{\lambda_m \le X} a_m f(\lambda_m) = c X f(X) - \tfrac{1}{2} c X^2 f'(X) + \int_0^X \tfrac{1}{2} c t^2 f''(t)\, dt + O(X^{1/3})$$

$$-\frac{f'(X)}{2\pi} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^2} I_1(\lambda_m X)$$

$$+\int_0^X \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^2} I_1(\lambda_m t) f''(t)\, dt,$$

the series occurring in the last integral being uniformly convergent. The error-term $O(X^{1/3})$ comes from formula (2.4). Hence

$$\sum_{\lambda_m \le X} a_m f(\lambda_m) = c \int_0^X f(t)\, dt + O(X^{1/3})$$

$$+\frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^2} \left[\int_0^X I_1(\lambda_m t) f''(t)\, dt - I_1(\lambda_m X) f'(X) \right],$$

and the last term equals

$$\frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^2} \left\{ [I_1(\lambda_m t) f'(t)]_0^X - \int_0^X (2\pi \lambda_m) I_0(\lambda_m t) f'(t)\, dt - I_1(\lambda_m X) f'(X) \right\}.$$

The first term in the curly brackets vanishes at $t=0$ because of (2.9). Taking $f(t)=\exp(2\pi i x t)$ we get the formula

$$\sum_{\lambda_m \le X} a_m \exp(2\pi i \lambda_m x) = c \int_0^X \exp(2\pi i x t)\, dt + O(X^{1/3}) + \sum_{m=1}^{\infty} a_m r_m, \qquad (2.11)$$

where

$$r_m = -\frac{2\pi i x}{\lambda_m} \int_0^X I_0(\lambda_m t) \exp(2\pi i x t)\, dt. \qquad (2.12)$$

We shall prove Theorem 1 by estimating the sum $\sum_{m=1}^{\infty} a_m r_m$ in (2.11) by means of the following lemma due to J. R. WILTON [6].

LEMMA 2 (WILTON). *Let*

$$J_m = \int_0^X I(\lambda_m t)\exp(2\pi i x t)\,dt, \quad I(t) = \frac{d}{dt}[I_0(t)], \tag{2.13}$$

and

$$N = X x^2.$$

Let

$$R_m = J_m - \frac{c_1}{x}\exp - (2\pi i \lambda_m/x), \quad \text{if} \quad \lambda_1 \le \lambda_m \le N, \tag{2.14}$$

where c_1 is defined as in (1.2), *and*

$$R_m = J_m - \frac{1}{\lambda_m} I_0(\lambda_m X)\exp(2\pi i x X), \quad \text{if} \quad \lambda_m > N \ge c' > 0. \tag{2.14'}$$

Then we have the estimates:

$$R_m = O(x^{-1/2}\lambda_m^{-1/4} N^{-1/4}) + O\left(\frac{x}{\lambda_m^2}\right), \quad \text{if} \quad \lambda_m < (1-\varepsilon) N, \quad 0 < \varepsilon < 1; \tag{2.15}$$

$$R_m = O(x^{-1/2} N^{1/2} |N - \lambda_m|^{-1}), \quad \text{if} \quad c_2 N < \lambda_m < c_3 N, \quad c_2 > 0, \quad \lambda_m \ne N; \tag{2.16}$$

$$R_m = O\left(\frac{1}{x}\right), \quad \text{if} \quad N - c_4\sqrt{N x} < \lambda_m < N + c_5\sqrt{N x}, \quad c_4, c_5 > 0; \tag{2.17}$$

$$R_m = O(x^{-1/2} N^{3/4} \lambda_m^{-5/4}), \quad \text{if} \quad \lambda_m > (1+\varepsilon) N > N. \tag{2.18}$$

The O's are uniform in x, for $0 < x \leqslant A$, for any constant A, ε being fixed.

An integration by parts applied to (2.12) yields the relation

$$r_m = J_m - \frac{1}{\lambda_m} I_0(\lambda_m X)\exp(2\pi i x X), \tag{2.19}$$

where J_m is defined as in (2.13). Therefore, by (2.14'),

$$r_m = R_m, \quad \text{if} \quad \lambda_m > N \ge c' > 0, \tag{2.20}$$

and, by (2.19) and (2.14), we have

$$r_m = \frac{c_1}{x}\exp - (2\pi i \lambda_m/x) + R_m - \frac{1}{\lambda_m} I_0(\lambda_m X)\exp(2\pi i x X), \quad \text{if} \quad \lambda_1 \le \lambda_m \le N. \tag{2.21}$$

Combining (2.20) and (2.21) we get

$$\sum_{m=1}^{\infty} a_m r_m = \frac{c_1}{x}\sum_{\lambda_m \le N} a_m \exp - (2\pi i \lambda_m/x) - \sum_{\lambda_m \le N} \frac{a_m}{\lambda_m} I_0(\lambda_m X)\exp(2\pi i x X) + \sum_{m=1}^{\infty} a_m R_m. \tag{2.22}$$

It is known that $I_0(t)=O(t^{1/4})$, as $t\to\infty$ [2, p. 96], so that

$$\sum_{\lambda_m\leq N} \frac{a_m}{\lambda_m} I_0(\lambda_m X)\exp(2\pi i x X) = O\left(X^{1/4}\sum_{\lambda_m\leq N}\frac{a_m}{\lambda_m^{3/4}}\right) = O(X^{1/2}), \tag{2.23}$$

because of (2.4).

In order to estimate the sum $\sum_{m=1}^{\infty} a_m R_m$, we split it up into three parts. If $0<\varepsilon<1$, we write

$$\sum_{m=1}^{\infty} a_m R_m = \sum_{\lambda_m\leq(1-\varepsilon)N} + \sum_{(1-\varepsilon)N<\lambda_m<(1+\varepsilon)N} + \sum_{\lambda_m\geq(1+\varepsilon)N}$$
$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \quad \text{say.} \tag{2.24}$$

If we use (2.15) in $\sum_1$, we get

$$\left|\sum\nolimits_1\right| \leq c\cdot \sum_{\lambda_m\leq(1-\varepsilon)Xx^2} \frac{a_m}{\lambda_m^{1/4}} X^{-1/4} x^{-1}$$
$$\leq c\cdot (Xx^2)^{3/4} X^{-1/4} x^{-1} \leq c\cdot X^{1/2}. \tag{2.25}$$

(N.B. Here, as elsewhere, c is a constant with possibly different values at different occurrences).

If we use (2.18) in $\sum_3$, we get

$$\left|\sum\nolimits_3\right| \leq c\cdot \sum_{\lambda_m>Xx^2} a_m \lambda_m^{-5/4}(Xx^2)^{3/4} x^{-1/2}$$
$$\leq c\cdot (Xx^2)^{-1/4+3/4} x^{-1/2} \leq c\cdot X^{1/2}. \tag{2.26}$$

Finally we consider $\sum_2$. Here we have, by (2.17),

$$\left|\sum_{N-\sqrt{Nx}\leq\lambda_m\leq N+\sqrt{Nx}} a_m R_m\right| \leq c\cdot\frac{1}{x}\sum_{N-\sqrt{Nx}\leq\lambda_m\leq N+\sqrt{Nx}} a_m = \frac{c}{x}[c\sqrt{Nx}+O(N^{1/3})]$$
$$= O(X^{1/2}) + O(X^{1/3}x^{-1/3}) = O(X^{1/2}), \tag{2.27}$$

if $Nx\geq c>0$, and $0<x\leq\lambda$. If Nx is so small that $\lambda\sqrt{Nx}<\frac{1}{2}$, then the sum on the right-hand side reduces to a single term a_m, and the estimate is

$$O\left(\frac{N^{\varepsilon}}{x}\right) = O(X^{1/2}),$$

since $N\geqslant 1/\lambda>0$. On the other hand, we have, because of (2.16),

$$\left|\sum_{N+\sqrt{Nx}\leq\lambda_m\leq(1+\varepsilon)N} a_m R_m\right| \leq c \sum_{N+\sqrt{Nx}\leq\lambda_m\leq(1+\varepsilon)N} \frac{a_m}{\lambda_m - N} x^{-1/2} N^{1/2}.$$

Now

$$c \sum_{N+\sqrt{N}x \le \lambda_m \le (1+\varepsilon)N} \frac{a_m}{\lambda_m - N} = O\left(\int_{N+\sqrt{N}x}^{(1+\varepsilon)N} \frac{dA(t)}{t-N}\right),$$

and, by (2.4), $A(t)=\kappa t+E(t)$, where $E(t)=O(t^{1/3})$. Thus

$$\Big|\sum_{N+\sqrt{N}x \le \lambda_m \le (1+\varepsilon)N} a_m R_m\Big| = O(X^{1/2}\log X), \tag{2.28}$$

and similarly

$$\Big|\sum_{(1-\varepsilon)N \le \lambda_m \le N-\sqrt{N}x} a_m R_m\Big| = O(X^{1/2}\log X). \tag{2.29}$$

Hence (2.27), (2.28), and (2.29) together give

$$|\textstyle\sum_2| = O(X^{1/2}\log X). \tag{2.30}$$

On combining (2.24), (2.25), (2.26), and (2.30), we get

$$\sum_{m=1}^{\infty} a_m R_m = O(X^{1/2}\log X). \tag{2.31}$$

If we use this, together with (2.23), in (2.22), then we get

$$\sum_{m=1}^{\infty} a_m r_m = \frac{c_1}{x}\sum_{\lambda_m \le N} a_m \exp(-2\pi i \lambda_m/x) + O(X^{1/2}\log X). \tag{2.32}$$

Further

$$\left|\int_0^X \exp(2\pi i x t)\,dt\right| \le \frac{c}{x} = O(X^{1/2}), \tag{2.33}$$

since, by hypothesis, $N=Xx^2\ge 1/\lambda>0$.

Formula (2.11), combined with (2.32) and (2.33), gives (1.2) in the case of a real quadratic field, provided that $N\ge 1/\lambda$.

If $N<1/\lambda$, then (2.20) gives

$$\sum_{m=1}^{\infty} a_m r_m = \sum_{m=1}^{\infty} a_m R_m,$$

in which case we can use (2.18), so that

$$\left|\sum_{m=1}^{\infty} a_m R_m\right| \le \frac{c(Xx^2)^{3/4}}{x^{1/2}} \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^{5/4}} \le cX^{3/4}x \le cX^{1/4},$$

since $Xx^2=N<c$. Hence, by formula (2.11),

$$\sum_{\lambda_m \le X} a_m \exp(2\pi i \lambda_m x) = \frac{c}{2\pi i x}(\exp(2\pi i x X) - 1) + O(X^{1/3}),$$

uniformly for $0<x\leq\lambda$, which proves (1.3) in the case of a real quadratic field, and thereby completes the proof of Theorem 1 in that case. We note that the term $O(X^{1/3})$ comes from (2.4).

§ 3. The proof of Theorem 1 for an imaginary quadratic field $K=Q(\sqrt{-d})$, $d>0$, is similar. Here we have a functional equation with a simple gamma factor, namely

$$\left(\frac{\sqrt{d}}{2\pi}\right)^s \Gamma(s)\,\zeta_K(s,\mathfrak{C})=\left(\frac{\sqrt{d}}{2\pi}\right)^{1-s}\Gamma(1-s)\,\zeta_K(1-s,\mathfrak{C}),$$

which takes the form

$$\Gamma(s)\,\varphi(s)=\Gamma(1-s)\,\varphi(1-s), \tag{3.1}$$

where

$$\varphi(s)=\sum_{m=1}^{\infty}\frac{a_m}{\mu_m^s},\ \mu_m=\frac{2m\pi}{\lambda},\ \lambda=\sqrt{d}.$$

We define $A_\mu^\varrho(x)$ as before. As in Lemma 1, equation (3.1) again implies the existence of an identity with the Bessel function $J_\nu(x)$ – not to be confused with the integral J_m in Lemma 2 – in place of the special function $F_\nu(x)$:

$$A_\lambda^\varrho(x)-Q_\varrho(x)=(2\pi)^{-\varrho}\sum_{m=1}^{\infty}a_m\left(\frac{x}{\lambda_m}\right)^{(1+\varrho)/2}J_{1+\varrho}\{4\pi(x\lambda_m)^{1/2}\}, \tag{3.2}$$

where $\lambda_m=m/\lambda$, $\varrho\geq 2\beta-\frac{5}{2}$, $\beta=1+\varepsilon>1$. The series converges absolutely for $\varrho>\frac{1}{2}$. Further, it converges boundedly in any interval of the real axis when $\varrho=0$, and uniformly whenever the function on the left-hand side is continuous [1, Th. III]. The asymptotic behaviour of $J_\nu(x)$ is similar to that of $Y_\nu(x)$: $J_\nu(x)=O(x^{-1/2})$, as $x\to\infty$. The analogue of (2.4) is true; there is an additive constant which is absorbed by the term $O(x^{1/3})$. The proof of Lemma 2 goes through with these changes; in fact, it becomes simpler than WILTON's original proof. In (2.14) we now have $c_1=i$, instead of $c_1=1$. Estimate (2.15) holds without the error-term $O(x/\lambda_m^2)$. The rest of the proof of (1.2) proceeds on the same lines as before, and we omit details.

If we take $x=1/k\lambda$, where k is an integer, in (1.2), we get

$$\begin{aligned}\sum_{m\leq X}a_m\exp(2\pi i\,m/k\lambda^2)&=\frac{c_1}{x}\sum_{m\leq Xx^2}a_m\exp(-2\pi i\,m\,k)+O(X^{1/2}\log X)\\&=\frac{c_1}{x}\sum_{m\leq Xx^2}a_m+O(X^{1/2}\log X)\\&=c_kX+O(X^{1/2}\log X),\ c_k=\frac{c_1}{k\lambda},\end{aligned}$$

which is the Corollary to Theorem 1.

§ 4. The estimate in (1.1) for irrational x was based on purely arithmetical considerations [4], though the proof that $\zeta_K(\frac{1}{2}+it)$ vanishes for an infinity of values of t made use of the functional equation satisfied by $\zeta_K(s)$. The proof of the reciprocity formula, and of the estimate in (1.4) for *rational* x, was based on the functional equation for $\zeta_K(s)$, and the consequent existence of an arithmetical identity like (2.8). This fact enables us to extend the formula to zeta-functions with Grössencharacters $\zeta(s, \Lambda)$. The functional equation in that case, the analogue of identity (2.8), namely

$$A_\mu^\varrho(x) - P_\varrho(x) = \sum_{m=1}^{\infty} \frac{b_m}{\mu_m^{1+\varrho}} I_\varrho(\mu_m x),$$

and of the estimate (2.4), were considered by us in [2, p. 128]. The properties of $I_\varrho(x)$ are known [3, p. 33], and enable us to uphold the validity of Wilton's lemma. Thus Theorem 1 is valid for zeta-functions with Grössencharacters associated with quadratic fields.

§ 5. We shall now make use of the relatively difficult case $\varrho=0$ of Lemma 1 to obtain an exact summation formula instead of the asymptotic formula in (2.11).

Let a_n, λ_n, and $A_\lambda^\varrho(x)$ be defined as hitherto, and f a function which is twice continuously differentiable in $[0, \infty)$. By Abel's lemma on partial summation, we have, if $0<\alpha<\lambda_1$,

$$\sum_{\lambda_m \leq x} a_m f(\lambda_m) = A(x) f(x) - \int_\alpha^x A(t) f'(t)\, dt. \tag{5.1}$$

If we first confine ourselves to the case of a real quadratic field, then because of the case $\varrho=0$ of Lemma 1, we have

$$\int_\alpha^x A(t) f'(t)\, dt = \int_\alpha^x c\, t f'(t)\, dt + \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m} \int_\alpha^x I_0(\lambda_m t) f'(t)\, dt. \tag{5.2}$$

The first term on the right-hand side is continuous in $0 \leq \alpha \leq 1$; so is the second, for by partial integration, we have

$$\sum_{m=1}^{\infty} \frac{a_m}{\lambda_m} \int_\alpha^x I_0(\lambda_m t) f'(t)\, dt = \left[\sum_{m=1}^{\infty} \frac{a_m}{2\pi \lambda_m^2} I_1(\lambda_m t) f'(t) \right]_\alpha^x$$

$$- \sum_{m=1}^{\infty} \frac{a_m}{2\pi \lambda_m^2} \int_\alpha^x I_1(\lambda_m t) f''(t)\, dt. \tag{5.3}$$

Lemma 1, with $\varrho=1$, shows that this is also continuous in $0\leq\alpha\leq 1$. Hence from (5.1) we obtain

$$\sum_{\lambda_m\leq x} a_m f(\lambda_m) = A(x)f(x) - \int_0^x A(t)f'(t)\,dt$$

$$= cxf(x) - \int_0^x ctf'(t)\,dt + \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m}\left[I_0(\lambda_m x)f(x) - \int_0^x I_0(\lambda_m t)f'(t)\,dt\right] \tag{5.4}$$

An integration by parts gives

$$\sum_{\lambda_m\leq x} a_m f(\lambda_m) = c\int_0^x f(t)\,dt + \sum_{m=1}^{\infty} a_m \int_0^x I(\lambda_m t)f(t)\,dt, \tag{5.5}$$

where

$$I(y) = \frac{d}{dy}[I_0(y)] = -2\pi\left(Y_0(4\pi\sqrt{y}) - \frac{2}{\pi}K_0(4\pi\sqrt{y})\right),$$

and $c=\kappa\sqrt{d}$, where κ is the residue of $\zeta_K(s, \mathfrak{C})$ at $s=1$.

In the case of an imaginary quadratic field $K=Q(\sqrt{-d})$, $d>0$, the term $Q_0(x)$ in (3.2) gives

$$Q_0(x) = \zeta_K(0, \mathfrak{C}) + \frac{x}{w} = -\frac{1}{w} + \frac{x}{w},$$

where w is the number of roots of unity in K, so that (5.5) becomes

$$\sum_{\lambda_m\leq x} a_m f(\lambda_m) + \frac{f(0)}{w} = c'\int_0^x f(t)\,dt + \sum_{m=1}^{\infty} a_m \int_0^x I(\lambda_m t)f(t)\,dt, \tag{5.6}$$

where $c'=(\kappa\sqrt{d}/2w)$, where κ is the residue of $\zeta_K(s, \mathfrak{C})$ at $s=1$.

If we make the convention that $\lambda_0=0$, and $a_0=0$, or $1/w$, according as the field K is real or imaginary, we obtain the following

THEOREM 2. *If f has a continuous second derivative in* $[0, \infty)$, *then for any $X>0$, we have*

$$\sum_{\lambda_m\leq X}^{*} a_m f(\lambda_m) = c\int_0^X f(t)\,dt + \sum_{m=1}^{\infty} a_m \int_0^X I(\lambda_m t)f(t)\,dt, \tag{5.7}$$

where the star $*$ indicates the convention regarding a_0, and $c=\kappa\lambda$, or $\kappa\lambda/2\pi$, according as K is real or imaginary, κ being the residue of $\zeta_K(s, \mathfrak{C})$ at $s=1$, and $\lambda=|d|^{1/2}$.

We can take $f(t)=\exp(2\pi ixt)$, $x>0$, and obtain an exact formula in place of (2.11). The case $d=+1$ goes back to VORONOI; the case $d=-4$ is Satz 559 in [5]. The condition that f be twice continuously differentiable in $[0,\infty)$ is *not* fulfilled if we take $f(t)=\exp(2\pi it^{1/2}x)$. To accommodate this case, which is of interest in several applications, we shall formulate more general assumptions on f near the origin.

We remark, first of all, that if $0<a\leq b$, and f has a continuous derivative in $(0,\infty)$, then

$$\sum_{a<\lambda_m\leq b} a_m f(\lambda_m) = c\int_a^b f(t)\,dt + \sum_{m=1}^{\infty} a_m \int_a^b I(\lambda_m t) f(t)\,dt. \tag{5.8}$$

In order to be able to replace a by 0, we now assume that f is twice continuously differentiable in $(0,\infty)$, and satisfies, near the origin, the condition

$$\int_0^1 t^p |f''(t)|\,dt < \infty, \tag{5.9}$$

for some $p<1$. We then have

$$\begin{aligned} t^p|f'(t)| - c_3 \leq t^p \int_t^1 |f''(u)\,du \leq \int_t^1 u^p|f''(u)|\,du \\ \leq \int_0^1 u^p|f''(u)|\,du < \infty, \end{aligned}$$

where c_3 is a constant, so that $tf'(t)\to 0$, as $t\to 0$, and $\int_0^1 t|f'(t)|td<\infty$. We again have (5.1) and (5.2). The first term on the right-hand side of (5.2) tends to $\int_0^x ctf'(t)\,dt$, as $\alpha\to 0$, since

$$\int_0^x |ctf'(t)|\,dt < \infty,$$

because of assumption (5.9), while the second term, by partial integration, leads to (5.3), namely

$$\left[\sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} I_1(\lambda_m t) f'(t)\right]_\alpha^x - \sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_\alpha^x I_1(\lambda_m t) f''(t)\,dt. \tag{5.10}$$

Because of identity (2.9), we have

$$\sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} I_1(\lambda_m t) = O(t), \tag{5.11}$$

as $t \to 0$. Hence, if we let $\alpha \to 0$, the first term in (5.10) gives just

$$\sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} I_1(\lambda_m x) f'(x),$$

because of assumption (5.9). To consider the second term in (5.10), we take the series

$$\sum \frac{a_m}{2\pi\lambda_m^2} \int_0^1 | I_1(\lambda_m t) \cdot f''(t) | \, dt, \tag{5.12}$$

and define

$$e_m = \sup_{0 \le t \le 1} \frac{| I_1(\lambda_m t) |}{t^p},$$

where $p<1$, given as in (5.9). If $\lambda_m t<1$, then, since $I_1(t)=O(t^{3/2})$, as $t \to 0$, we have

$$t^{-p} | I_1(\lambda_m t) | = O(\lambda_m^{3/2} t^{3/2-p}) = O(\lambda_m^p),$$

while, if $\lambda_m t \ge 1$, then, since $I_1(t)=O(t^{3/4})$, as $t \to \infty$, we have

$$t^{-p} | I_1(\lambda_m t) | = O(\lambda_m^{3/4} t^{3/4-p}) = O(\lambda_m^{3/4} + \lambda_m^p),$$

since $0 \le t \le 1$. Hence

$$e_m = O(\lambda_m^q), \tag{5.13}$$

where $q=\max(\frac{3}{4}, p)<1$. Thus, by assumption (5.9), we have

$$\sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_0^1 | I_1(\lambda_m t) f''(t) | \, dt < c \sum_{m=1}^{\infty} \frac{a_m e_m}{\lambda_m^2} < \infty.$$

Hence the series

$$\sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_\alpha^x I_1(\lambda_m t) f''(t) \, dt$$

is uniformly convergent in $0<\alpha \le 1$, and we have

$$\lim_{\alpha \to 0} \sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_\alpha^x I_1(\lambda_m t) f''(t) \, dt = \sum_{m=1}^{\infty} \frac{a_m}{2\pi\lambda_m^2} \int_0^x I_1(\lambda_m t) f''(t) \, dt \tag{5.14}$$

Thus in (5.10), and so also in (5.2), we can pass to the limit as $\alpha \to 0$. The rest of the argument is the same as in the case where f has a continuous second derivative in $[0, \infty)$. Thus we obtain

THEOREM 3. *If f has a continuous second derivative in* $(0, \infty)$, *and satisfies the condition*

$$\int_0^1 t^p \, |f''(t)| \, dt < \infty ,$$

for some $p<1$, *then formula* (5.7) *is valid.*

§ 6. Professor SIEGEL's treatment of the Corollary to Theorem 1, referred to in § 1, runs as follows.

"I start from the Gaussian sum

$$G = \sum_{p,\, q \,(\mathrm{mod}\, dk)} \exp(2\pi i(a p^2 + b p q + c q)/dk) = \begin{cases} k d\sqrt{d}, \\ i k d \sqrt{d}, \end{cases}$$

where $b^2-4ac=+d>0$ in the first case, and $b^2-4ac=-d<0, a>0$, in the second case. In the second case the expression

$$S = \sum_{ap^2+bpq+cq^2 \le X} \exp 2\pi i(a p^2 + b p q + c q^2)/dk$$

has to be evaluated. The area of the ellipse $ap^2+bpq+cq^2 \le X$ is $2\pi X/\sqrt{d}$, and therefore the number of lattice points with any given residue classes of p and q modulo dk is $(dk)^{-2}\, 2\pi X/\sqrt{d}+O(X^{1/2})$. Hence

$$S = (dk)^{-2} \frac{2\pi X}{\sqrt{d}} G + O(X^{1/2}),$$

$$S = \frac{2\pi i}{k d} X + O(X^{1/2}).$$

In the first case the ellipse is replaced by a sector of a hyperbola. It is clear how to improve the error-term $O(X^{1/2})$. The same idea goes through if the ratio $1/dk$ is replaced by any rational number. The result is a little more general than in your text, since d need not be the discriminant of a quadratic number field; even a square number is allowed."

REFERENCES

[1] K. CHANDRASEKHARAN and RAGHAVAN NARASIMHAN, *Hecke's Functional Equation and Arithmetical Identities*, Annals of Math. *74* (1961), 1–23.

[2] ———, *Functional Equations with multiple gamma factors and the average order of arithmetical functions*, Annals of Math. *76* (1962), 93–136.

[3] ———, *The approximate functional equation for a class of zeta-functions*, Math. Annalen, *152* (1963), 30–64.

[4] ———, *Zeta-functions of ideal classes in quadratic fields and their zeros on the critical line* (to appear in Commentarii, Vol. 43/2).
[5] E. LANDAU, Vorlesungen über Zahlentheorie (1937).
[6] J. R. WILTON, *An approximate functional equation with applications to a problem of Diophantine Approximation*, J. reine angew. Math. *169* (1932), 219–237.

Forschungsinstitut für Mathematik E.T.H., Zürich

and

Université de Genève

Received November 11, 1967