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Remarks on the Koebe Kreisnormierungsproblem[1])

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Two multiply connected domains D and D' are said to be *conformally equivalent* if there exists a schlicht map of D onto D'. KOEBE has conjectured [5] that *every multiply connected domain is conformally equivalent to a circle domain* (a domain bounded by circles and points), and showed that this was true for domains of finite connectivity [6]. Results for infinitely connected domains have been obtained by KOEBE [7], DENNEBERG [3], GRÖTZSCH [4], SARIO [8] and for a large class of domains, including those considered by DENNEBERG and GRÖTZSCH, by STREBEL [10, 11].

We will show that the Koebe conjecture may be reduced to the following deformation question: *Can every plane domain be deformed quasiconformally onto a circle domain?*

Using this result we obtain, in § 3, two theorems which may be used to construct many new examples of domains for which the Koebe conjecture is true. In § 4 we show how a slight modification of the method used in § 1 may be used to demonstrate the conformal equivalence of the domains considered by DENNEBERG and GRÖTZSCH and some of those considered by STREBEL.

§ 1. Main Theorem

1.1. We recall [1] that if $\mu(z)$ is measurable and ess. $\sup|\mu(z)|\leqslant k<1$ in a domain Δ, then a homeomorphic solution of the Beltrami equation $f_{\bar z}=\mu(z)f_z$ (generalized derivatives are understood) is said to be quasiconformal or μ-conformal in D. If f^μ and g^μ are two solutions, then $f^\mu\circ(g^\mu)^{-1}$ is conformal. If μ is given in the entire plane, then the Beltrami equation admits a solution w^μ which is unique up to a Möbius transformation. For completeness we include the proof of the following lemma which is contained in [9]:

LEMMA 1. *Let C be the circle defined by $|z-a|=\varrho$. Suppose that $\mu(z)$, defined in the entire plane, is compatible with reflection in the circle C, $\mu(R(z))=(R_{\bar z}/\overline{R_{\bar z}})\,\overline{\mu(z)}$ where $R(z)=a+\varrho^2/\overline{(z-a)}$. Then any μ-conformal map of the plane maps C into a circle.*

Proof. Denote by $u(z)$ a μ-conformal map of the plane satisfying $u(a)=0$ and $u(\infty)=\infty$. Then $v(z)=1/\overline{u(R(z))}$ satisfies the same Beltrami equation and the same conditions at a and ∞. It follows that $u=\lambda v$ for some constant λ. Then for points ζ on C, $|u(\zeta)|^2=\lambda$ so that λ is real and positive and u maps C into a circle. The same is then true for any μ-conformal map of the plane.

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1.2. Suppose now that D is *quasiconformally equivalent* to a circle domain K_0 so that there exists a quasiconformal map f^μ of K_0 onto D. Then the function $\mu(z)=f_{\bar z}^\mu/f_z^\mu$ is defined in K_0. We extend the definition of μ to all points of the plane which can be obtained from points of K_0 by finite reflections in the *circular* boundary components $\{C_j\}$ of K_0 (of which there are at most a countable number).

More precisely, we define μ in the reflection of K_0 in C_1 by requiring that it be compatible with reflection, $R_1(z)$, in C_1. The function μ, now defined in the domain $K_1=K_0\cup R_1(K_0)$, can be defined in the reflection of K_1 in C_2 by requiring that it be compatible with reflection in C_2. Next we extend the definition of μ by consecutive reflections in C_1, C_2, and C_3. Continuing the process (at the n^{th} stage we reflect consecutively in $C_1,\ldots,C_{n+1}$) we obtain a set K_0^* on which the function $\mu(z)$ is compatible with reflection in each circle C_j.

Since the complement of K_0^* is closed, it is measurable.

Let $\mu=0$ on this set. Then μ satisfies the conditions of the lemma for *each* boundary component C_j. Letting w^μ denote a μ-conformal map of the entire plane, we observe that the map $g=f^\mu\circ(w^\mu)^{-1}$ is a conformal map of the *circle domain* $K=w^\mu(K_0)$ onto D.

We summarize these remarks in

THEOREM 1. *Let D be a planar domain. Then D is conformally equivalent to a circle domain if and only if D is quasiconformally equivalent to a circle domain.*

§ 2. Remarks

2.1. Boundary correspondence. Let $g(z)$ be the conformal map obtained in § 1 of the circle domain K onto D. The point correspondence, induced by g, of corresponding boundary components is clearly one-to-one in the case that the boundary component of D is an isolated Jordan curve. In general we make the following

Remark. Suppose that the quasiconformal map f^μ of § 1 induces a one-to-one point correspondence between the boundary component k_0 of ∂K_0 (the boundary K_0) and the boundary component d of ∂D. Since w^μ is a homeomorphic self map of the plane, the conformal map $g=f^\mu\circ(w^\mu)^{-1}$ of $K=w^\mu(K_0)$ onto D induces a one-to-one point correspondence between the boundary component $k=w^\mu(k_0)$ and d.

2.2. Uniqueness. We say that a map f is *admissible* if it maps some circle domain quasiconformally onto D.

Two conformal admissible maps are usually thought of as "distinct" if they do not differ by a Möbius transformation. In general, one is interested in the relation between the number of distinct conformal equivalence classes and (geometric) characteristics of the domain D. (For example, it is well known that if D is of finite connectivity, then the number of distinct conformal admissible maps is one.)

It might appear, at first, that by including quasiconformal maps in our class of admissible maps, we have lost sight of this problem. The following observations are made in an attempt to clarify this point.

Two admissible maps f_1 and f_2 are said to be *equivalent* if they differ by a quasiconformal homeomorphism w of the entire plane (i.e., $f_1 = f_2 \circ w$ or, more familiarly, $f_2^{-1} = w(f_1^{-1})$). If $f = f^\mu$ is admissible, then, by the procedure of § 1, we are led to a conformal admissible map $g = f^\mu \circ (w^\mu)^{-1}$. We say that f^μ *induces* g. Clearly f^μ and g are equivalent. Hence

Remark. Each equivalence class contains a conformal admissible map.

We may say more if we assume that D has countably many boundary components. We recall [10] that under this assumption a schlicht map of one circle domain onto another is actually a Möbius transformation if the induced point correspondence of corresponding boundary components is one-to-one. We observe, however, that if two conformal admissible maps g_1 and g_2 are equivalent, then $g_1^{-2} \circ g_1$, which is a conformal map of one circle domain onto another, is the restriction to the first domain of a (quasiconformal) homeomorphic self map of the plane, and hence induces a one-to-one point correspondence of the boundary components. Then $g_2^{-1} \circ g_1$ is a Möbius transformation and we may make the following

Remark. If D has countably many boundary components and g_1 and g_2 are conformal admissible maps, then g_1 is equivalent to g_2 if and only if $g_1 = g_2 \circ A$ where A is a Möbius transformation.

We may restate the last two remarks (under the assumption that D has countably many boundary components) in the following way: *Each equivalence class contains a conformal admissible map which is unique up to a Möbius transformation.*

Thus the number of distinct conformal admissible maps is equal to the number of equivalence classes of admissible maps.

§ 3. Examples

3.1. We denote by $R(\alpha, \beta)$ the ring domain bounded by the continua α and β and by $A(r_1, r_2)$ the ring domain $R(c_1, c_2)$ where c_j is the circle $|z| = r_j$; $0 < r_2 < r_1 < \infty$.

LEMMA 2. *Let $s_j(z)$ $(j = 1, 2)$ be a conformal map defined in a neighborhood of c_j and mapping c_j onto itself. Then there exists a quasiconformal map $f(z)$ of $A(r_1, r_2)$ onto itself such that, for $\zeta \in c_j$, $f(\zeta) = s_j(\zeta)$.*

Proof. Let $\varphi(\varrho) = (r_1 - \varrho)/(r_1 - r_2)$. Then $f(\varrho e^{i\theta}) = \varphi(\varrho) s_2(r_2 e^{i\theta}) + (1 - \varphi(\varrho)) s_1(r_1 e^{i\theta})$ is such a map since the hypothesis on $s_j(z)$ implies the existence of constants M_j such that $0 < M_j^{-1} < (d/d\theta)\, s_j(r_j e^{i\theta}) < M_j$.

For any bounded Jordan curve γ we denote by *int* γ (*ext* γ) the bounded (unbounded) region determined by γ.

LEMMA 3. *Let γ_1 and γ_2 be non-degenerate continua contained in* $\mathrm{int}\,\gamma$ *where γ is an analytic Jordan curve. Then there exists a quasiconformal map $f(z)$ of $R(\gamma, \gamma_1)$ onto $R(\gamma, \gamma_2)$ such that for $\zeta \in \gamma, f(\zeta)=\zeta$.*

Proof. Let $g(z)$ and $h(z)$ be conformal maps of $R(\gamma, \gamma_1)$ and $R(\gamma, \gamma_2)$ onto $A(1, r_1)$ and $A(1, r_2)$, respectively. Since γ is an analytic Jordan curve, g and h may be extended to be conformal in a neighborhood of γ, so that $h \circ g^{-1}$ is conformal in a neighborhood of the unit circle and maps it onto itself. Then $f_0(\varrho e^{i\theta})=\varrho(h \circ g^{-1}(e^{i\theta}))$ is a quasiconformal self map of $A(1, r_1)$ with $f_0(\zeta)=h \circ g^{-1}(\zeta)$ for $|\zeta|=1$. Following f_0 by a linear "stretching" in ϱ results in a quasiconformal map f of $A(1, r_1)$ onto $A(1, r_2)$ which also agrees with $h \circ g^{-1}$ on $|\zeta|=1$. Then $h^{-1} \circ f \circ g$ is a quasiconformal map of $R(\gamma, \gamma_1)$ onto $R(\gamma, \gamma_2)$ and is the identity on γ.

PROPOSITION 1. *Let Δ be a simply connected domain whose closure $\bar{\Delta}$ is contained in the domain D. If D is conformally equivalent to a circle domain, then the same is true of $D-\bar{\Delta}$.*

Proof. Let f be a conformal map of D onto a circle domain K. Then $\overline{f(\Delta)} \subset K$ and we may assume that $\overline{f(\Delta)}$ is bounded. Let γ be a bounded analytic Jordan curve separating $\partial f(\Delta)$ from ∂K. If c is a circle contained in $\mathrm{int}\,\gamma$, then, by Lemma 3, there exists a quasiconformal map g of $R(\gamma, c)$ onto $R(\gamma, \partial f(\Delta))$ such that $g(\zeta)=\zeta$ for $\zeta \in \gamma$. Then the function

$$h(\zeta)=\begin{cases}\zeta & \text{for} \quad \zeta \in K-\mathrm{int}\,\gamma \\ g(\zeta) & \text{for} \quad \zeta \in R(\gamma, c)\end{cases}$$

is a quasiconformal map of the *circle domain* $K-\overline{\mathrm{int}\,c}$ onto $K-\overline{f(\Delta)}$. But then $D-\bar{\Delta}$ is quasiconformally and hence, by Theorem 1, conformally equivalent to a circle domain.

THEOREM 2. *Suppose that the domains $D_1, \ldots, D_n$ have disjoint complements D_j' and each is conformally equivalent to a circle domain. Then the intersection $\cap D_j$ is also conformally equivalent to a circle domain.*

Proof. It suffices to obtain the result for two domains D_1 and D_2 which contain the point at infinity. We may draw two (disjoint) analytic Jordan curves α_1 and α_2 such that $D_j' \subset \mathrm{int}\,\alpha_j$ (Fig. 1). The ring domain $R(\alpha_1, \alpha_2)$ is conformally equivalent to an annulus $K_0=A(r_1, r_2)=R(c_1, c_2)$. By Proposition 1 there exist conformal maps f_j of $\Omega_j=D_j \cap \mathrm{int}\,\alpha_j$ onto circle domains K_1 and K_2, respectively, where $K_1 \subset \mathrm{ext}\,c_1$, $K_2 \subset \mathrm{int}\,c_2$, and $f_j(\alpha_j)=c_j$. Using Lemma 2, it is easily seen that there exists a quasiconformal map f_0 of $\Omega_0=\overline{R(\alpha_1, \alpha_2)}$ onto $\overline{K_0}$ which agrees with f_j on α_j. The map g defined by setting it equal to f_j on Ω_j $(j=0, 1, 2)$ is a quasiconformal map of $D_1 \cap D_2$ onto the circle domain $\cup K_j$ and hence (Theorem 1) $D_1 \cap D_2$ is conformally equivalent to a circle domain.

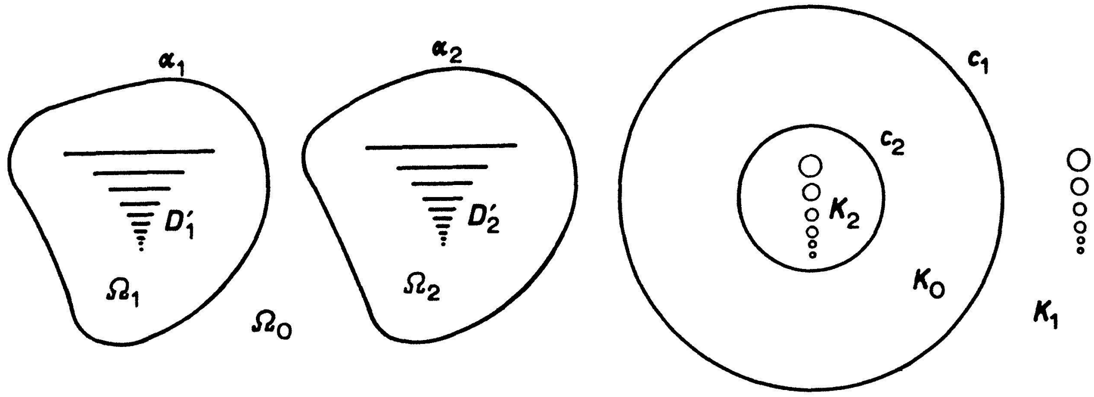

Figure 1

3.2. Suppose that a domain D is symmetric with respect to reflection in the real axis. KOEBE has shown [7] that if the real axis intersects each boundary component of D (or, equivalently, divides D into two simply connected components), then D is conformally equivalent to a circle domain. (It is actually necessary to prove this only in the case that D is the entire plane with slits along the real axis.)

Using Theorem 1 we obtain, easily, the following extension of this result.

THEOREM 3. *(Quasisymmetry). Let $q(z)$ be a quasiconformal reflection of the entire plane in the Jordan curve L. Suppose that L divides a domain D into two simply connected components D_1 and D_2 such that $q(D_1)=D_2$. Then D is conformally equivalent to a circle domain.*

Proof. Let f be a conformal map of D_1 onto the upper half plane. Then $\overline{f\circ q(z)}$ is a quasiconformal map of D_2 onto the lower half plane. Moreover, for $\zeta\in L\cap D$, we have $\overline{f\circ q(\zeta)}=f(\zeta)$. Hence there exists a quasiconformal map of D onto the plane with slits along the real axis. Following this map by the conformal map of the slit plane onto a circle domain (see the above remarks) results in a quasiconformal map of D onto a circle domain and the conclusion follows.

§ 4. Domains With Weak Limit Boundary Components

Let Δ be a bounded open set containing the closed point set κ. It is well known that if the extremal length of the family $\{\gamma\}$ of rectifiable curves contained in Δ and surrounding κ is zero, then κ is necessarily a point. Moreover, since extremal length is a conformal invariant, the point set corresponding to κ under any conformal map of Δ is, in this case, again a point. On the other hand, if a (finite) point is a limit boundary component of an infinitely connected domain D then the extremal length of the family of surrounding curves (in a bounded neighborhood of the point) need not necessarily be zero. If it is, for some such neighborhood, the point is said to be a *weak boundary component*. This definition is modified in the obvious way for the point at infinity.

4.1. DENNEBERG [3] has shown that domains whose boundary components satisfy certain geometric condition are conformally equivalent to circle domains. These conditions, in effect, ensure that the point at infinity is the only limit boundary component and that it is weak, in the sense above. GRÖTZSCH [4] has considered domains with a *finite* number of weak limit boundary components and STREBEL [10] the case of *countably* many (as well having allowed points on non point Jordan boundary components to be limit points of other boundary components, under assumptions analogous to that of weakness [10, 11]).

4.2. We now show how the first result of Strebel stated above can easily be obtained by the methods of § 1. Some of the other, more general results can be obtained in the same way but, for clarity, we restrict ourselves to the simpler situation of only point limit boundary components.

THEOREM 4. (STREBEL). *Let D be a domain with point limit boundary components $\mathscr{L}$ and suppose that each such boundary component is weak. Then D is conformally equivalent to a circle domain.*

Proof. The collection of isolated boundary components is countable. By Lemma 3 we may replace each such (non point) boundary component by a circle, thus obtaining a circle domain K_0 and a homeomorphism f of K_0 onto D which is quasiconformal in the complement (with respect to K_0) of any neighborhood of $\mathscr{L}$. For $\zeta\in\mathscr{L}$ let $\Delta_j(\zeta)$ be a neighborhood of ζ of diameter less than ε_j and $N_j(\zeta)=K_0\cap\Delta_j(\zeta)$. Set $N_j=\cup N_j(\zeta)$ over all $\zeta\in\mathscr{L}$. Let $\mu(z)=f_{\bar{z}}/f_z$ for $z\in K_0$ and define $\mu_j(z)=\mu(z)$ for $z\in K_0-N_j$ and zero for $z\in N_j$. Then $|\mu_j(z)|\leqslant k_j<1$ in K_0 and as in § 1, we obtain a quasiconformal homeomorphism $w_j=w^{\mu_j}$ of the plane onto itself which is symmetric with respect to each (circular) boundary component of K_0-N_j. Letting $\varepsilon_j\to 0$ and following an argument of BERS ([2], Theorem 5) one may show that a subsequence (which we again denote by w_j) converges pointwise to a μ-conformal homeomorphism w of K_0.

We consider the boundary correspondence induced by w and claim that $K=w(K_0)$ is a circle domain. We first observe that if we denote by $\hat{K}_0$ the domain obtained from K_0 by adjoining its reflections in the circular boundary components, together with the circles themselves, then (since for j sufficiently large w_j is symmetric with respect to each circular boundary component) the convergence $w_j\to w$ may be extended to $\hat{K}_0$ and hence, in particular to $\overline{K}_0-\mathscr{L}$. Suppose that γ is a circular boundary component of K_0. For j sufficiently large, $w_j(\gamma)$ is a circle and since $w_j(z)\to w(z)$ for $z\in\gamma$, γ corresponds, under w, to a circle. On the other hand since the point limit boundary components are weak, they correspond to points under the conformal map $w\circ f$ and hence under the map w.

Hence K is a circle domain and $f\circ w^{-1}$ is a conformal map of K onto D.

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