Foliations on Open Manifolds, I.

Foliations on Open Manifolds, I

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1. Introduction

Let M be a smooth n-dimensional manifold, with tangent bundle TM. A smooth section in the bundle of p-planes of TM is called a *p-plane field* (also, "p-dimensional distribution") on M. A p-plane field σ gives a p-dimensional subbundle of TM, with fibre over $x \in M$ equal to $\sigma(x)$. *This bundle will also be denoted by* σ. Picking a Riemannian metric for M associates to σ a complementary $(n-p)$-plane field $\sigma^{\perp}$: $\sigma^{\perp}(x)$ is the tangent subspace orthogonal to the p-plane $\sigma(x)$.

The p-plane field σ is called *integrable* if M has a smooth foliation $\mathscr{F}$ (see § 2 for this definition) such that at each $x \in M$ the p-plane $\sigma(x)$ is tangent to $\mathscr{F}$. This is equivalent to saying that each $x \in M$ has a neighborhood U with coordinates $x_1, \ldots, x_n$ such that the tangent vectors $\partial/\partial x_1|_y, \ldots, \partial/\partial x_p|_y$ span $\sigma(y)$ at each $y \in U$. There is a classical criterion for integrability of a p-plane σ, namely that σ be *involutive*. This means that if v and w are vectorfields contained in σ, i.e. such that $v(x) \in \sigma(x)$, $w(x) \in \sigma(x)$ at each point x, then their Poisson bracket $[v, w]$ is also contained in σ. It is easy to see that integrable implies involutive. The converse is FROBENIUS' Theorem [4, Theorem 5.1].

From the point of view of differential topology it is natural to ask which p-plane fields are *homotopic* to integrable fields (see [1], p. 373). This article presents a partial answer to that question.

THEOREM 1.1. *Suppose M is open (i.e. has no compact components). A p-plane field σ on M, whose complementary bundle $\sigma^{\perp}$ is trivial, is homotopic to an integrable field.*

THEOREM 1.2. *Suppose M is open, and n-dimensional. Every $(n-1)$-plane field σ on M is homotopic to an integrable field.*

Remark. The hypothesis, that M be open, seems quite restrictive. For instance, in the case $n=3$ Theorem 1.2 for *compact* M and orientable σ has been proved by JOHN WOOD, a graduate student at Berkeley. On the other hand, it is easy to check that all the foliations constructed in this article are *analytic*, in the sense of [1], p. 368. In this respect, Theorem 1.2 should be compared with the theorem on p. 392 of [1]: if $\pi_1 M$ contains only elements of finite order, then M can carry an analytic foliation of co-dimension 1 only if M is open.

Proof of theorem 1.1. By assumption, the bundle $\sigma^{\perp}$ contains a field ξ of $(n-p)$-frames. The theorem is an immediate consequence of Theorem B of [3] which implies that, since M is open, ξ is homotopic to the gradient $(n-p)$-frame ections

$\nabla F = (\nabla f_1, \ldots, \nabla f_{n-p})$ of a submersion $F = (f_1, \ldots, f_{n-p})$ of M in Euclidean space R^{n-p}. (A *submersion* $M^n \to W^k$ is a smooth map of rank k.) Taking orthogonal complements at each stage of the homotopy deforms σ to a p-plane field orthogonal to ∇F and therefore tangent to the foliation defined by the submanifolds $\{F = \text{constant}\}$.

Example $M = S^2 \times R$. Here every foliation is orientable. The manifold is parallelizable, so homotopy classes of nonzero vectorfields (and of their complementary 2-plane sections) correspond to homotopy classes of maps of M into S^2, i.e. to elements of $\pi_2 S^2 = Z$. A foliation $\mathscr{F}_n$ which corresponds to the map of degree n can be obtained, for $n \geqslant 0$, by stacking the slices of foliations shown below (for $n < 0$, reverse orientation), as follows: $\mathscr{F}_0 = XY$, $\mathscr{F}_1 = XAX$, $\mathscr{F}_2 = XABY$, $\mathscr{F}_3 = XABAX$, etc. It should be clear how to interpolate the missing leaves, and how to fit the slices together to give coherently oriented foliations of $S^2 \times R$. Let us verify that $\mathscr{F}_n$ belongs to the correct homotopy class.

Imagine the stacking to be done vertically in R^3. There is an X-slice on the bottom, then a sequence of A- and B-slices, and on top either a Y-slice or an upside-down X-slice, according as n is even or odd. To calculate the degree of the normal map associated to $\mathscr{F}_n$, it is clearly sufficient to calculate the degree of the map it induces

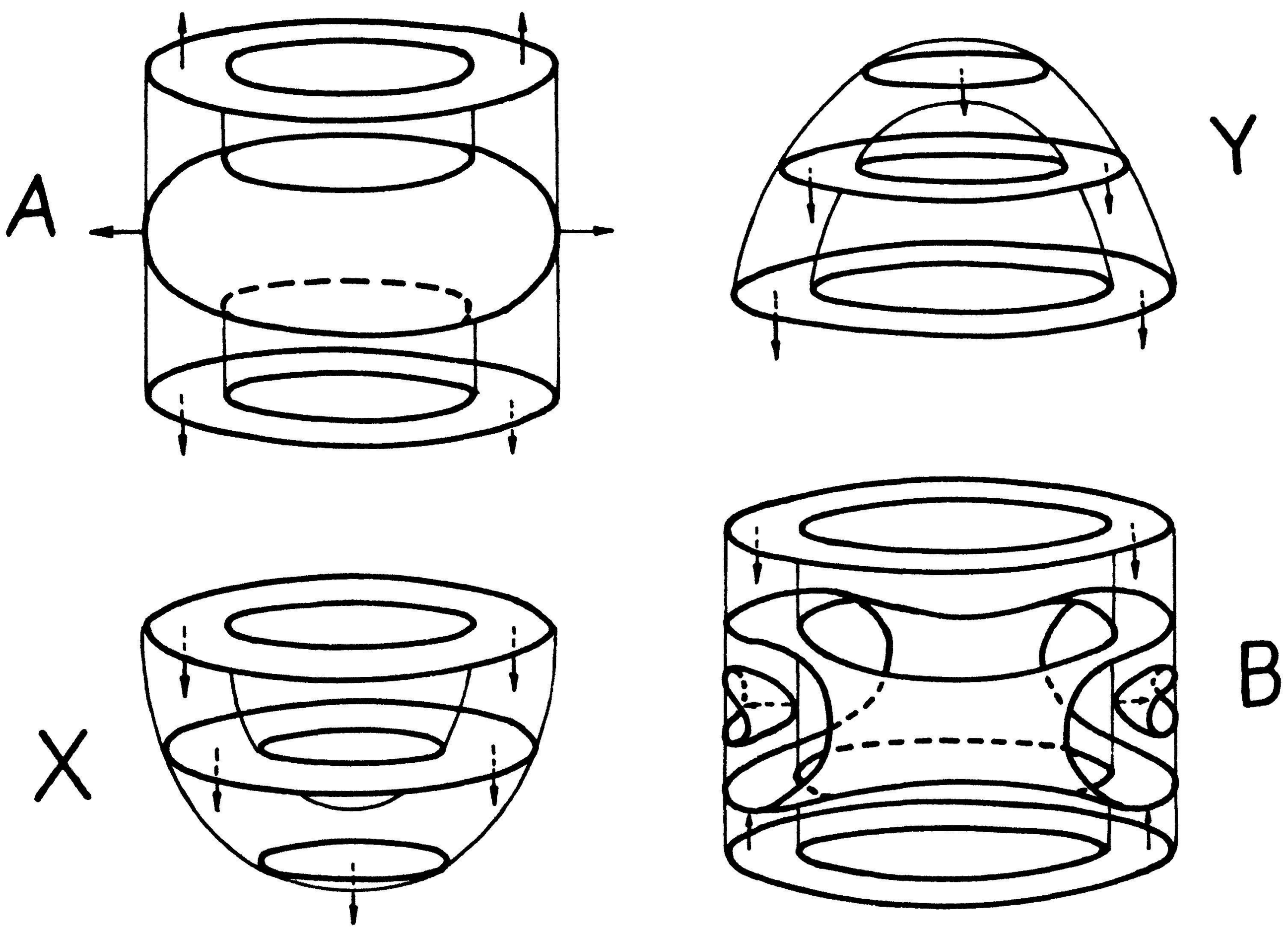

Fig. 1.1

on the S^2 imbedded as $S^2 \times \{0\}$ in $S^2 \times R$. This is well known to be equal to the number of inverse images of a regular value, each one counted plus or minus according as the map preserves or reverses orientation there. Choose as value the point corresponding in Fig. 1.1 to a horizontal arrow pointing to the right. The figure shows that this value is taken precisely once, and with positive orientation, on each A- or B-slice, and not at all on an X- or Y-slice; it follows that $\mathscr{F}_n$ has normal degree n, as claimed.

Outline of proof of theorem 1.2. If the line bundle $\sigma^\perp$ is orientable, this is a special case of the previous theorem. The following sections treat the case where $\sigma^\perp$ is not orientable. Let $f: M \to P^n$ be the classifying map for $\sigma^\perp$, suppose that f intersects $P^{n-1} \subset P^n$ transversally, and let the submanifold N be the inverse image of P^{n-1}. There is a foliation on P^n, studied in § 2, of which P^{n-1} is a leaf. The map f will pull back a foliation $\mathscr{F}$ of an open tubular neighborhood U of N in M. It will be shown in § 3 that $\sigma^\perp$ is homotopic to a line field τ normal to $\mathscr{F}$ near N. Since f sends $M-N$ into the contractible set $P^n - P^{n-1}$ it follows that $\sigma^\perp | M-N$ is trivial, so that the restriction of the homotopic field τ to $M-N$ contains a vectorfield η. The theorem is proved by showing that η is homotopic through non-zero vectorfields to the gradient of a submersion $g: M-N \to R$, by a homotopy leaving η fixed near N. This requires a relative form of the submersion classification theorem (§ 4). The foliation defined on $M-N$ by g matches $\mathscr{F}$ near N; the two fit together to give a foliation of M with tangent field homotopic to σ, as required.

Part II of this article will apply these methods to foliations of co-dimension 2.

I am grateful to MORRIS HIRSCH for bringing this problem to my attention, and for several helpful conversations.

2. Definition of Foliation and an Important Example

Consider a smooth manifold M of dimension n. Let TM_y represent the tangent space to M at $y \in M$.

DEFINITION. (See [1] for a general reference on foliations.) A *smooth foliation $\mathscr{F}$ of dimension p on M* is given by a covering $\{U_\alpha\}$ of M and maps $\varphi_\alpha: U_\alpha \to R^{n-p}$ satisfying 1) and 2).

1) φ_α is a submersion (i.e. has rank $n-p$). Then for each $x \in U$, $\varphi_\alpha^{-1}(\varphi_\alpha(x))$ is a smooth p-dimensional submanifold of U.

2) If $x \in U_\alpha \cap U_\beta$, then $\varphi_\alpha^{-1}(\varphi_\alpha(x)) \cap U_\beta = \varphi_\beta^{-1}(\varphi_\beta(x)) \cap U_\alpha$.

The tangent space $T(\varphi_\alpha^{-1}(\varphi_\alpha(x)))_x$ (the tangent space to the foliation at x) will be denoted by $T\mathscr{F}_x$; $T\mathscr{F}$ will then represent the p-dimensional subbundle of TM whose fibre over $x \in M$ is $T\mathscr{F}_x$. The functions φ_α are called the *distinguished functions* of the foliation.

The *leaf topology* on U_α comes from considering U_α as the disjoint union of the p-dimensional manifolds $\{\varphi_\alpha = \text{constant}\}$. Since these topologies coincide on overlaps they fit together to define the *leaf topology* on M. A connected component of M in this topology is called a *leaf* of the foliation.

Example 1. Let $S^n = \{(x_0, \dots, x_n) \in R^{n+1}, \sum x_i^2 = 1\}$. The function $p_n : S^n \to R$, given by projection on the last coordinate axis, has rank one when restricted to $S^n - (0, \dots, 0, 1) - (0, \dots, 0, -1)$ and defines a foliation of S^n minus the poles by sheets of constant latitude. In this case *one* distinguished function defined the whole foliation. More generally, a submersion $\varphi : M^n \to W^{n-p}$ gives a p-dimensional foliation of M, with leaves the connected components of the submanifolds $\{\varphi = \text{constant}\}$. This is a special case (where $\mathscr{F}$ is the foliation by points) of the next example.

Example 2. Suppose W has a foliation $\mathscr{F}$ of codimension q, with distinguished functions $\{\varphi_\alpha : U_\alpha \to R^q\}$. If M is a smooth manifold and $h : M \to W$ is transversal to the leaves of $\mathscr{F}$, then h *pulls back* $\mathscr{F}$ to give the foliation $h^*\mathscr{F}$ of M with distinguished functions $\{\varphi_\alpha \circ h : h^{-1} U_\alpha \to R^q\}$. In connection with this example there is the following useful result.

LEMMA 2.1. *Let $T\mathscr{F}^\perp$ and $T(h^*\mathscr{F})^\perp$ be the normal q-plane bundles to $\mathscr{F}$ and $h^*\mathscr{F}$ respectively. Then $T(h^*\mathscr{F})^\perp = h^*(T\mathscr{F}^\perp)$, i.e. there is a bundle map*

$$\begin{array}{ccc} T(h^*\mathscr{F})^\perp & \longrightarrow & T\mathscr{F}^\perp \\ \downarrow & & \downarrow \\ M & \xrightarrow{h} & W. \end{array}$$

Proof. Let $p : TW \to T\mathscr{F}^\perp$ be orthogonal projection. Composing p with the differential dh gives a map $p \circ dh$ whose kernel in TM_y is $T(h^*\mathscr{F})_y$, and thereby induces an isomorphism $TM_y / T(h^*\mathscr{F})_y \simeq T(h^*\mathscr{F})_y^\perp \to T\mathscr{F}_{h(y)}^\perp$, for each $y \in M$.

Example 3. This is the example referred to in the section heading. It will play an important role in the proof of Theorem 1.2.

Observe that the foliation of Example 1 is preserved by the antipodal map, and therefore defines a foliation (*the standard foliation*) of the punctured projective space $P^n - x$, where P^n is taken as S^n with antipodal points identified, and $x \in P^n$ corresponds to the poles. Let $\pi : S^n \to P^n$ be the projection. Since π is a local diffeomorphism, it follows that maps of the form $p_n \circ \pi^{-1} | U$, for appropriate U, give a family of distinguished functions for the standard foliation. In particular, notice that π maps the open upper hemisphere diffeomorphically onto $P^n - P^{n-1}$ (here take $P^{n-1} \subset P^n$ as the image of the equatorial S^{n-1}); thus the submersion $\varphi_n = p_n \circ \pi^{-1} : P^n — P^{n-1} — x \to R$ determines the standard foliation on the complement of the leaf P^{n-1}.

LEMMA 2.2. *Let $\alpha \to P^n - x$ be the tangent line bundle normal to the standard foliation. Then α is equivalent to $\gamma_n^1 | P^n - x$, where $\gamma_n^1 \to P^n$ is the canonical line bundle.*

Proof. The two bundles are equivalent over P^{n-1}, a deformation retract of $P^n - x$. In fact, $\alpha | P^{n-1}$ is the normal bundle of P^{n-1} in P^n, which is easily seen to be equivalent to $\gamma_{n-1}^1 = \gamma_n^1 | P^{n-1}$.

3. Proof of Theorem 1.2

The complementary line bundle $\sigma^\perp$ is equivalent to a bundle over a complex of dimension $\leqslant n-1$, since M is open (cf. Proposition 4.1), so there exists a bundle map

$$\begin{array}{ccc} \sigma^\perp & \rightarrow & \gamma_n^1 \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & P^n . \end{array}$$

In fact, one may assume that f misses a point in P^n and, using Lemma 2.2, that there is a map

$$\begin{array}{ccc} \sigma^\perp & \longrightarrow & \alpha \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & P^n - x . \end{array}$$

Finally, it may be assumed that f intersects $P^{n-1} \subset P^n$ transversally and, by Lemma 4.2, proved in § 4, that $N = f^{-1} P^{n-1}$ is an embedded manifold (of dimension $n-1$) with no compact components.

The manifold $P^n - x$ carries the "standard foliation" described in Example 3 of § 2. The intersection of f with a leaf sufficiently near P^{n-1} will also be transversal, so f pulls back (see Example 2 of § 2) a foliation $\mathscr{F}$ of an open tubular neighborhood U of N. Let $\tau \to U$ be the field transverse to $\mathscr{F}$.

LEMMA 3.1. *The line field $\sigma^\perp | U$ is homotopic to τ as sections in the bundle of lines of TU, a bundle with fibre P^{n-1}.*

Proof. The two sections determine isomorphic bundles, since they are both mapped to α by bundle maps covering $f | U$. This is true for $\sigma^\perp$ by definition of f, and follows from Lemma 2.1 for τ.

The obstructions to a homotopy between them lie in $H^i(U; \pi_i P^{n-1})$. Since U is chosen to have N as deformation retract, and N has no compact components, it follows that U has no cohomology in dimensions n or $n-1$; so the only possible obstruction is in $H^1(U; \pi_1 P^{n-1}) = H^1(U; Z_2)$. It is sufficient to show that the obstruction cocycle gives zero when evaluated on any 1-cycle A of U. Suppose that the sections have been deformed to match on the 0-skeleton; then the value of the obstruction cocycle on a 1-simplex Δ^1 of A is 1 or 0 according as the bundle over S^1 formed by $\sigma^\perp | \Delta^1$ on the upper semicircle and $\tau | \Delta^1$ on the lower is orientable or not; and the value of the obstruction cocycle on A will be 1 only if $\sigma^\perp | A$ is orientable and $\tau | A$ is not, or vice-versa, impossible if $\sigma^\perp | A$ and $\tau | A$ are isomorphic bundles.

Let U' be an open neighborhood of N, with closure contained in U. Then the restriction to U' of the homotopy between $\sigma^{\perp}|U$ and τ may be extended to a homotopy deforming all of $\sigma^{\perp}$ to a new line field $\tilde{\tau}$ equal to τ on U'. The orthogonal $(n-1)$-plane field $\tilde{\tau}^{\perp}$ is clearly homotopic to σ.

The next lemma allows one to consider, instead of $M-N$, a manifold $\hat{M}$ which is more convenient for submersion theory.

LEMMA 3.2. *There is an open manifold-with-boundary $\hat{M}$ and a smooth map ψ: $\hat{M}\to M$ which maps* $\operatorname{Int}\hat{M}=\hat{M}-\partial\hat{M}$ *diffeomorphically onto $M-N$, and $\partial\hat{M}$ onto N as a double covering.*

Proof. $\hat{M}$ is constructed by cutting along N, as follows.

The construction may be repeated for each component of N, so suppose that N is connected. Let $\nu\to N$ be the normal bundle of the embedding, assume M to carry a Riemannian metric, and let W be an open neighborhood of N in the total space of ν small enough to be mapped diffeomorphically into M by the exponential map exp.

a) If ν is trivial, orient ν; then let $W^{+}=\{v\in W,\ v\geqslant 0\}$, $W^{-}=\{v\in W,\ v\leqslant 0\}$, and define $\hat{M}$ to be $M-N\cup\cup W^{+}\cup\cup W^{-}$ ($\cup\cup$=disjoint union) with the identification $v\equiv\exp(v)$ for $v\in W^{+}\cup W^{-}$, $v\neq 0$.

b) If ν is non-orientable, let $\tilde{W}\to\tilde{N}$ be the orientable double cover, and $p:\tilde{W}\to W$ the projection. Then define $\hat{M}$ to be $M-N\cup\tilde{W}^{+}$ with the identification $v\equiv\exp(p(v))$ for $v\in W^{+}$, $v>0$.

The natural map $\psi:\hat{M}\to M$ clearly has the required properties. Since N had no compact components, neither does $\partial\hat{M}$; since $\operatorname{Int}\hat{M}$ is also an open manifold, it follows that $\hat{M}$ is an open manifold with boundary. This completes the proof of Lemma 3.2.

Now let $\hat{U}=\psi^{-1}U'\subset\hat{M}$, so $\hat{U}$ is an open neighborhood of $\partial\hat{M}$ in $\hat{M}$. The line field $\tilde{\tau}$ lifts up to a line field $\hat{\tau}$ on $\hat{M}$, which is orientable by construction of $\hat{M}$ (shrink $\hat{M}$ into $\operatorname{Int}\hat{M}$; then $\hat{\tau}$ maps to the trivial bundle $\alpha|P^{n}-P^{n-1}-x$). Let η be a non-zero vectorfield contained in $\hat{\tau}$. The restriction of $\hat{\tau}$ to $\hat{U}$ also contains the non-zero gradient $\nabla(\varphi_n\circ f\circ\psi)$, but the two orientations may or may not coincide. To remedy this, define a new submersion $F:\hat{U}\to R$ by $F(x)=\pm\varphi_n\circ f\circ\psi(x)$, plus or minus according as the two orientations do or do not agree on the connected component of $\hat{U}$ containing x.

Corollary 4.4 now applies. It follows that η is homotopic through non-zero vectorfields to the gradient of a submersion $g:\hat{M}\to R$ such that $g|V=F|V$, for some open neighborhood V of $\partial\hat{M}$. Moving back down to M, the submersion $g\circ\psi^{-1}:M-N\to R$ defines a foliation which clearly agrees with $\mathscr{F}$ on the overlap $\psi(V)\cap M-N$. The proof of Theorem 1.2 is completed by the easy observation that the tangent field of this foliation is homotopic to $\tilde{\tau}^{\perp}$ and therefore to σ.

4. Two Lemmas on Open Manifolds

These lemmas both depend on the following result.

PROPOSITION 4.1. *Let M be an open (no compact components) manifold with (possibly empty) boundary ∂M. Give the pair $(M, \partial M)$ a smooth triangulation. Then M has an $(n-1)$-dimensional subcomplex K containing ∂M, with the following property. Given an open tubular neighborhood M' of K, there is a homotopy of embeddings $\varphi_t: M \to M$ such that φ_0 is the identity, $\varphi_1(M) = M'$, and $\varphi_t(x) = x$ for x belonging to some neighborhood V of K and for all $t \in [0, 1]$.*

Proof. A combinatorial form of this statement is essentially contained in the proof of Theorem 3.2 of [5]. The differentiable form can then be derived by the methods used in [2], Theorem 3.7.

LEMMA 4.2. *Let M be an open manifold, and let $f: M \to W$ be a continuous map. Let $N \subset W$ be a submanifold of codimension p. Then f is homotopic to a smooth map $h: M \to W$ transversal to N and such that the submanifold $h^{-1} N$ (which has codimension p) has a complex of codimension $\geqslant p+1$ (in M) as deformation retract.*

Proof. Let K be the subcomplex of Proposition 4.1. The map f is homotopic to g where g is smooth and transversal to N and such that $g|K$ is transversal to N. The inverse image $g^{-1}N$ is a smooth submanifold of codimension p which intersects K along a subcomplex of codimension p in K. Pick an open tubular neighborhood M' of K small enough so that $g^{-1}N \cap M'$ has $g^{-1}N \cap K$ as deformation retract. Let $\varphi_1: M \to M'$ be the diffeomorphism described above. Then $h = g \circ \varphi_1$ is homotopic to g, and $h^{-1}N = \varphi_1^{-1}(g^{-1}N \cap M')$ has a complex of codimension $\geqslant p+1$ as deformation retract.

LEMMA 4.3. *Let M be an open manifold with boundary ∂M, and let $f: U \to W$ be a submersion defined on a neighborhood U of ∂M. Suppose that the differential $df: TU \to TW$ extends to a tangent bundle map $H: TM \to TW$ of maximal rank. Then H is homotopic through tangent bundle maps of maximal rank to the differential dg of a submersion $g: M \to W$ which is equal to f on some neighborhood of ∂M. The homotopy leaves H fixed near ∂M.*

Proof. This is a relative form of part of [3], Theorem A. The proof is a straightforward application of Proposition 4.1 and the techniques of [3].

In [3], Theorem A has the corollary Theorem B treating the case where $W = R^p$. In precisely the same manner, the following is a consequence of Lemma 4.3.

COROLLARY 4.4. *Let M be an open manifold with boundary ∂M, and let $f: U \to R^p$, $f = (f_1, \ldots, f_p)$, be a submersion defined on a neighborhood U of ∂M. Suppose that the gradient p-frame field $(\nabla f_1, \ldots, \nabla f_p)$ extends to a p-frame field η defined on all of M. Then η is homotopic (as a section in the bundle of p-frames of TM) to the gradient* *p-frame field of a submersion* $g: M \to R^p$ *which is equal to* f *on some neighborhood of* ∂M. *The homotopy leaves* η *fixed near* ∂M.

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