On Characteristic Classes for Spherical Fibre Spaces

by J. MILNOR, Princeton, New Jersey

Let $G(n)$ denote the associative H-space consisting of all homotopy equivalences from the sphere S^{n-1} to itself.[1] According to STASHEFF [21] this H-space $G(n)$ has a "classifying space" $B_{G(n)}$ which serves as universal base space for fibre spaces having a homotopy $(n-1)$-sphere as fibre. (See also DOLD [7, § 16].)

The object of this paper is to make a preliminary study of the singular cohomology groups of $B_{G(n)}$; and particularly of the stable groups

$$H^k(B_{G(n)}; Z_p), \qquad k < n,$$

with mod p coefficients, which will be denoted briefly by $H^k(B_G; Z_p)$. Following THOM and WU one can use the Steenrod operations P^i to define characteristic classes

$$q_i \in H^{2i(p-1)}(B_{G(n)}; Z_p)$$

for any odd prime p. Our main result is the following.

THEOREM 4. *In dimensions less than $2p(p-1)-1$ the cohomology ring $H^*(B_G; Z_p)$ is isomorphic to the tensor product of a polynomial algebra freely generated by the Wu classes*

$$q_1, q_2, q_3, \dots$$

and a Grassmann algebra freely generated by the Bockstein coboundaries

$$\beta q_1, \beta q_2, \beta q_3, \dots$$

of the Wu classes.

Briefly speaking we will say that $H^*(B_G; Z_p)$ is *free commutative*[2]) in this range of dimensions, with the q_i and βq_i as *independent* generators.

This is proved (in § 3 and § 4) by a cumbersome argument which depends on inductively building up a complete description of the Postnikov system of B_G in dimensions less than $2p(p-1)-1$.

GITLER and STASHEFF [10] and STASHEFF [22] have succeeded in computing somewhat further. They show that Theorem 4 breaks down precisely in the dimension $2p(p-1)-1$. More precisely:

$$H^{2p(p-1)-1}(B_G; Z_p) \cong Z_p,$$

---

1) This notation is non-standard. Compare § 2.

2) The word *commutative* means of course that

$$ba = (-1)^{\dim a \dim b} ab.$$

where the generator is a new kind of characteristic class, which cannot be expressed in terms of the q_i and βq_i.

Even in dimensions greater than $2p(p-1)$ it is easy to see that the q_i are algebraically independent. However, I do not know whether all of the βq_i, $1 \leqq i < \infty$, are independent. This question is discussed in Appendix 3.

Just for completeness let me mention what is known about the cohomology of B_G with other coefficients. BROWDER, LIULEVICIUS and PETERSON [4] have shown that the algebra $H^*(B_G; Z_2)$ is isomorphic to a tensor product

$$H^*(B_O; Z_2) \otimes A^*$$

where A^* is a certain 2-connected Hopf algebra over the mod 2 Steenrod algebra. Unfortunately very little is known about A^*.

With rational coefficients, B_G has the cohomology of a point. This follows immediately from the fact that the homotopy groups of B_G are all finite. (We will see in § 3 that $\pi_1 B_G \cong \pi_0 G$ has two elements, and that $\pi_i B_G \cong \pi_{i-1} G$ is isomorphic to the stable homotopy group $\pi_{n+i-1} S^n$ for all $1 < i < n$.) However:

THEOREM 5. *The cohomology algebra*

$$H^*(B_{G(2t)}; Q) \cong H^*(B_{G(2t+1)}; Q)$$

is free commutative on one generator p_t of dimension $4t$, providing that $t \geq 1$.

This will be proved in Appendix 1. The notation p_t is chosen since this generator corresponds to the t-th Pontryagin class[3]) in $H^{4t}(B_{O(2t)}; Q)$.

I want to thank STASHEFF for his interest and help.

§ 1. The Wu characteristic classes

A Hurewicz fibre space $\pi: E \to B$ is called $(n-1)$-*spherical* if each fibre has the homotopy type of an $(n-1)$-sphere.

Let $\bar{B} \supset E$ be the mapping cylinder of π, and let $\bar{\pi}: \bar{B} \to B$ be the natural projection. If π is orientable (that is if the fibres can be coherently oriented) then Thom showed that the cohomology group $H^k B$ is isomorphic to $H^{k+n}(\bar{B}, E)$, using any commutative coefficient ring Λ.

A more precise statement is the following. Note that the cup product

$$H^k(\bar{B}; \Lambda) \otimes H^l(\bar{B}, E; \Lambda) \to H^{k+l}(\bar{B}, E; \Lambda)$$

makes the graded group $H^*(\bar{B}, E; \Lambda)$ into a module over the graded algebra $H^*(\bar{B}; \Lambda)$.

---

3) For general information about classical characteristic classes see for example HUSEMOLLER, "Fibre Bundles", Mc Graw Hill, 1966 (§ 16).

Assertion. *$H^*(\bar{B}, E; \Lambda)$ is a free module over $H^*(\bar{B}; \Lambda) \cong H^*(B; \Lambda)$ on one generator u of dimension n.*

This can be proved, for example, by a spectral sequence argument.

The generator u, also called the *fundamental cohomology class* of the fibration, is uniquely determined up to multiplication by units in $H^0(\bar{B}; \Lambda)$. Any particular choice of generator u will be called a *Λ-orientation*[4]) of the fibration.

Given such a generator u the *Thom isomorphism*

$$\Phi: H^k(B; \Lambda) \to H^{k+n}(\bar{B}, E; \Lambda)$$

is defined by the formula

$$\Phi(a) = (\bar{\pi}^* a) \cup u .$$

Fundamental construction. Any cohomology operation θ which operates on $H^*(\bar{B}, E; \Lambda)$ gives rise to a corresponding characteristic class

$$\Phi^{-1} \theta \Phi(1) \in H^*(B; \Lambda).$$

If θ is linear (that is if $\theta(\lambda a + \lambda' a') = \lambda \theta(a) + \lambda' \theta(a')$), note that this construction does not depend on the choice of orientation. For $(\lambda \Phi)^{-1} \theta (\lambda \Phi)$ is then equal to $\Phi^{-1} \theta \Phi$.

In particular, starting with the Steenrod operation

$$P^i: H^n(\ ; Z_p) \to H^{n+2i(p-1)}(\ ; Z_p)$$

(see [9]) we obtain a corresponding cohomology class

$$\Phi^{-1} P^i \Phi(1) = \Phi^{-1}(P^i u)$$

in $H^{2i(p-1)}(B; Z_p)$; which will be denoted by q_i or $q_i(\pi)$, and called the i-th *Wu characteristic class* of the fibration π. Here i can be any positive integer.

Since P^i is linear, the Wu class q_i does not depend on the choice of orientation.

Similarly the mod 2 Steenrod operation $\mathrm{Sq}^i H^n(\ ; Z_2) \to H^{n+i}(\ ; Z_2)$ gives rise to the *Stiefel-Whitney class* $w_i(\pi) = \Phi^{-1} \mathrm{Sq}^i \Phi(1)$ in $H^i(B; Z_2)$.

Here is an important example of a non-linear cohomology operation. For any ring Λ the cup product squaring operation

$$a \mapsto a \cup a = a^2$$

gives rise to a characteristic class

$$x = \Phi^{-1}(u^2) \in H^n(B; \Lambda)$$

which is called the *Euler class* of the fibration. This class x *does* depend on the particular choice of orientation.

---

[4]) In the special case $\Lambda = Z$ one speaks simply of an *orientation*. Of course any Z-orientation determines a Λ-orientation using the unique homomorphism $Z \to \Lambda$.

If n is odd note that $2x=0$. This follows from the commutativity of the cup product. Note also the identity $\Phi(ax)=\Phi(a)\,u$. From this we see inductively that

$$\Phi(1) = u, \quad \Phi(x) = u^2, \quad \Phi(x^2) = u^3, \dots$$

and in general $\Phi(x^i)=u^{i+1}$.

For any cohomology class $a \in H^*(\ ; Z_p)$ the Steenrod operations are known to satisfy the following two identities:

$$P^i a = 0 \quad \text{if} \quad i > \tfrac{1}{2}\dim(a), \tag{1.1}$$

$$P^i a = a^p \quad \text{if} \quad i = \tfrac{1}{2}\dim(a). \tag{1.2}$$

Applying these to the fundamental cohomology class u in $H^n(\bar{B}, E; Z_p)$ we obtain corresponding relations for the Wu classes q_i of any oriented $(n-1)$-spherical fibration:

$$q_i = 0 \quad \text{if} \quad i > n/2, \tag{1.3}$$

$$q_i = x^{p-1} \quad \text{if} \quad i = n/2, \tag{1.4}$$

where x is the mod$p$ Euler class.

[Similarly the Cartan formula for $P^h(a \cup b)$ gives rise to a formula for the Wu classes of a Whitney join. If π and ϱ are spherical fibre spaces over B then the *Whitney join* $\pi * \varrho$ is the fibre space over B such that each fibre $(\pi * \varrho)^{-1} b$ is equal to the join of $\pi^{-1} b$ and $\varrho^{-1} b$. (See HALL [12] or SPIVAK [20].) The required formula is now

$$q_h(\pi * \varrho) = \textstyle\sum_{i+j=h} q_i(\pi) \cup q_j(\varrho);$$

where $q_0=1$ by definition.]

The formulas (1.3) and (1.4) are polynomial relations which must be satisfied by the mod$p$ characteristic classes $q_1, q_2, q_3, \dots$ and x of *every* $(n-1)$-spherical fibration. For n odd one has the additional relation $x=0$. We will next give an example to show that these are the only such universal polynomial relations.

LEMMA 1. (WU). *There exists a space B and a fibre bundle with fibre S^{2m-1} over B so that the characteristic classes $q_1, \dots, q_{m-1}$ and x in $H^*(B; Z_p)$ do not satisfy any polynomial relations.*

Proof. First consider the case $m=1$. Consider the universal complex line bundle γ over the infinite complex projective space $P(C)$. Then $H^*(P(C); Z_p)$ is known to be a polynomial algebra generated by the Chern class

$$c_1 = c_1(\gamma) \in H^2(P(C); Z_p).$$

But c_1 is equal to the Euler class x of the associated 1-sphere bundle $\dot{\gamma}$. Hence

$$q_1(\dot{\gamma}) = x(\dot{\gamma})^{p-1} = c_1^{p-1}.$$

Now consider the cartesian product ξ of m copies of γ. This is a complex vector bundle over the product $P(C)\times\cdots\times P(C)$. The "total Wu class"

$$1+q_1+q_2+\cdots+q_m\in\oplus_j H^j(P(C)\times\cdots\times P(C);Z_p)$$

of the associated $(2m-1)$-sphere bundle $\hat{\xi}$ is equal to the cross product

$$\begin{aligned}&(1+q_1(\hat{\gamma}))\times\cdots\times(1+q_1(\hat{\gamma}))\\&=(1+c_1^{p-1})\times\cdots\times(1+c_1^{p-1}).\end{aligned}$$

In other words $q_i(\hat{\xi})$ is equal to the i-th elementary symmetric function of the m algebraically independent elements

$$1\times\cdots\times c_1^{p-1}\times\cdots\times 1\in H^{2(p-1)}(P(C)\times\cdots\times P(C)\colon Z_p).$$

This proves that the Wu classes $q_1(\hat{\xi}),\ldots,q_m(\hat{\xi})$ are algebraically independent. Since $q_m(\hat{\xi})$ is equal to $x(\hat{\xi})^{p-1}$, it follows that the classes $q_1(\hat{\xi}),\ldots,q_{m-1}(\hat{\xi})$ and $x(\hat{\xi})$ are also algebraically independent; which completes the proof.

Now consider an arbitrary spherical fibre space, with base space B and with Wu classes q_i. We will study the action of the Steenrod operations P^h on the Wu classes, and also on the Bockstein coboundaries

$$\beta q_i\in H^{2i(p-1)+1}(B;Z_p)$$

of the Wu classes.

[We will see later that βq_i is non-zero in general. Of course if we happen to be working with a sphere bundle (having the orthogonal group as structural group), rather than a general fibration, then q_i comes from an integral cohomology class, and hence βq_i must be zero.]

THEOREM 1. *The cohomology class $P^h q_i$ can be expressed as a polynomial in the Wu classes $q_1,q_2,\ldots,q_{h+i}$; the coefficient of q_{h+i} in this polynomial being equal to the binomial coefficient*

$$(-1)^h\binom{(p-1)i-1}{h}.$$

Similarly each $P^h(\beta q_i)$ can be expressed as a linear combination

$$f_1\beta q_1+f_2\beta q_2+\cdots+f_{h+i}\beta q_{h+i}$$

of the classes $\beta q_1,\ldots,\beta q_{h+i}$; the coefficients in this case being polynomials

$$f_t=f_t^{h,i}(q_1,q_2,\ldots,q_{h+i-t})$$

in the Wu classes.

Proof. Let θ be any element of degree k in the Steenrod algebra $A(p)$. In other words let

$$\theta\colon H^i(\ ;Z_p)\to H^{i+k}(\ ;Z_p)$$

be a universally defined mod$p$ cohomology operation. It is convenient to define a homomorphism

$$[\theta]: H^i(B; Z_p) \to H^{i+k}(B; Z_p)$$

by the formula

$$[\theta] = \Phi^{-1} \theta \Phi,$$

where Φ is the Thom isomorphism. (This homomorphism $[\theta]$ is of course not a cohomology operation in the usual sense.) Note the four identities:

$$[\theta \theta'] = [\theta]\,[\theta'], \tag{1.5}$$

$$[P^h]\,1 = q_h, \quad \text{and more generally} \tag{1.6}$$

$$[P^h]\,\alpha = P^h \alpha + q_1 P^{h-1} \alpha + q_2 P^{h-2} \alpha + \cdots + q_{h-1} P^1 \alpha + q_h \alpha, \tag{1.7}$$

$$[\beta] = \beta; \tag{1.8}$$

where $\beta \in A(p)^1$ is the Bockstein operation.

Proof. The first two are clear from the definitions, and the last is proved as follows:

$$\begin{aligned}[\beta]\,\alpha &= \Phi^{-1} \beta \Phi \bar{\alpha} = \phi^{-1} \beta(\bar{\alpha} u) \\ &= \Phi^{-1}((\beta \bar{\alpha})\, u \pm \bar{\alpha}(\beta u)) = \beta \alpha,\end{aligned}$$

since $\beta u = 0$. Formula (1.7) is proved similarly, using the Cartan formula for $P^h(\bar{\alpha} u)$.

We will show by induction on $h+i$ that $P^h q_i$ can be expressed as a polynomial in the Wu classes. We may assume that $h < pi$ since otherwise $P^h q_i$ would be zero by (1.1).

First consider the cohomology class

$$[P^h]\, q_i = [P^h]\,[P^i]\,1 = [P^h P^i]\,1.$$

According to the Adem relations [9, p. 77] the composition $P^h P^i$ is equal to

$$(-1)^h \binom{(p-1)\,i-1}{h} P^{h+i}$$

plus a sum of terms of the form $\lambda P^r P^s$, with

$$r > ps > 0, \qquad r + s = h + i,$$

and with $\lambda \in Z_p$. Now $[P^{h+i}]\,1$ is equal to q_{h+i}; and each

$$[P^r P^s]\,1 = [P^r]\, q_s$$

is equal to

$$P^r q_s + q_1 P^{r-1} q_s + \cdots + q_r q_s$$

by (1.7). But the inequality $r > ps$ implies that $P^r q_s = 0$ by (1.1). The remaining terms $q_t P^{r-t} q_s$ can all be expressed as polynomials in the q_j, using the induction hypothesis. *Therefore* $[P^h]\, q_i$ *is equal to a polynomial in the* q_j.

But, using (1.7) together with the induction hypothesis again, this implies that $P^h q_i$ can also be expressed as such a polynomial.

The proof for $P^h \beta q_i$ is similar; and is again by induction on $h+i$. Again we may assume that $h<pi$. The Adem relations state that $P^h \beta P^i$ is equal to

$$\lambda_1 \beta P^{h+i} + \lambda_2 P^{h+i} \beta \qquad (\text{with } \lambda_1, \lambda_2 \in Z_p)$$

plus a linear combination of terms

$$\beta P^r P^s \quad \text{and} \quad P^r \beta P^s,$$

with

$$r > ps > 0, \quad r + s = h + i.$$

Hence the class

$$[P^h]\, \beta q_i = [P^h \beta P^i]\, 1$$

is equal to

$$[\lambda_1 \beta P^{h+i} + \lambda_2 P^{h+i} \beta]\, 1 = \lambda_1 \beta q_{h+i}$$

plus a linear combination of terms of the form

$$[\beta P^r P^s]\, 1 = \beta [P^r]\, q_s$$

and

$$[P^r \beta P^s]\, 1 = [P^r]\, \beta q_s.$$

But $[P^r] q_s$ can be expressed as a polynomial $f(q_1, \ldots, q_{r+s})$ by the portion of Theorem 1 which has already been established; hence

$$\beta [P^r]\, q_s = \beta f(q_1, \ldots, q_{r+s}) = \sum (\partial f / \partial q_j)\, \beta q_j$$

can be expressed in the required form. Similarly

$$[P^r]\, \beta q_s = P^r \beta q_s + q_1 P^{r-1} \beta q_s + \cdots + q_r \beta q_s,$$

where the first term is zero since $r>ps$, and the remaining terms can be expressed in the required form by the induction hypothesis. This proves that $[P^h]\, \beta q_i$ can be expressed as a linear combination $\sum f_t \beta q_t$ with polynomial coefficients. Now using (1.7) and the induction hypothesis it follows that $P^h \beta q_i$ can also be expressed in this form. This completes the proof of Theorem 1.

The above proof is fairly effective for actual computation of the polynomials. As an example suppose that $h=1$. The Adem relations then take the simple form

$$P^1 P^i = (i+1)\, P^{i+1},$$
$$P^1 \beta P^i = i \beta P^{i+1} + P^{i+1} \beta;$$

and it follows that:

$$P^1 q_i = (i+1)\, q_{i+1} - q_1 q_i, \tag{1.9}$$
$$P^1 \beta q_i = i \beta q_{i+1} - q_1 \beta q_i. \tag{1.10}$$

(For example the following computation proves (1.10):

$$P^1 \beta q_i + q_1 \beta q_i = [P^1] \beta q_i = [P^1 \beta P^i] 1$$
$$= [i \beta P^{i+1} + P^{i+1} \beta] 1 = i \beta q_{i+1} + 0.)$$

Combining (1.9) and (1.10) for the special case $i=1$, we obtain a relation

$$\beta P^1 q_1 = 2 P^1 \beta q_1 \tag{1.11}$$

which will be needed in § 3.

Another fact which will be needed later is the following.

If $h<p$ then $P^h q_1$ is equal to $(h+1)\, q_{h+1}$ plus a polynomial in $q_1, \ldots, q_h$. (1.12)

This can be proved either by manipulating the binomial coefficient in Theorem 1, or by induction, starting with (1.9).

Remark. There is another, quite different method for computing the polynomial for $P^h q_i$. One can start with the particular bundle considered in Lemma 1 and use the theory of symmetric functions; following BOREL and SERRE [1]. I do not know whether a similar procedure will work for $P^h \beta q_i$. (Compare Appendix 3.)

§ 2. The classifying space B_G

Recall that $G(n+1)$ denotes the associative H-space consisting of maps of degree ± 1 from the sphere S^n to itself. Let $F(n) \subset G(n+1)$ denote the sub H-space consisting of those homotopy equivalences $S^n \to S^n$ which carry the north pole x_0 into itself. Clearly $F(n)$ is the fibre of a fibration

$$F(n) \to G(n+1) \to S^n .$$

Therefore

$$\pi_i F(n) \cong \pi_i G(n+1) \quad \text{for} \quad i < n-1 . \tag{2.1}$$

Each homotopy equivalence from the equator S^{n-1} to itself suspends to a homotopy equivalence of the pair (S^n, x_0). This defines an embedding $G(n) \subset F(n)$. The notation has been chosen so that

$$O(n) \subset G(n) \subset F(n)$$

where $O(n)$ is the orthogonal group.

According to HAEFLIGER [11] the pair $(F(n), G(n))$ is $(2n-3)$-connected. (Compare JAMES [13].) Hence

$$\pi_i G(n) \cong \pi_i F(n) \quad \text{for} \quad i < 2n-3 . \tag{2.2}$$

It follows that the homomorphisms

$$\pi_i G(n) \to \pi_i F(n) \to \pi_i G(n+1) \to \pi_i F(n+1) \to \pi_i G(n+2) \to \cdots$$

are all isomorphisms providing that $i<n-1$. These stable groups will sometimes be denoted by $\pi_i G=\pi_i F$.

The component of the identity in $G(n)$ or $F(n)$, consisting of maps of degree 1, will be denoted by $SG(n)$ or $SF(n)$ respectively.[5]) Thus

$$SO(n)\subset SG(n)\subset SF(n).$$

Clearly $SF(n)$ is homeomorphic to one component of the n-fold loop space $\Omega^n(S^n, x_0)$. Therefore

$$\pi_i F(n)\cong\pi_i\Omega^n S^n\cong\pi_{i+n}S^n \quad \text{for} \quad i>0. \tag{2.3}$$

Remark. It is important that the H-space structure in $SF(n)$ should come from the operation of composing mappings from (S^n, x_0) to itself. The loop space $\Omega^n(S^n, x_0)$ also has a natural H-space structure, which is not the one we want. This distinction is discussed further in Appendix 2.

Now let H be any topological space with a product operation

$$H\times H\to H$$

which is associative, has a 2-sided unit, and makes the set $\pi_0 H$ of path components into a group. DOLD and LASHOF [6] construct a "classifying space" B_H. As part of the construction they show that

$$\pi_i B_H\cong\pi_{i-1}H \quad \text{for} \quad i>0.$$

STASHEFF [21] has applied this construction to the particular H-space $G(n)$. He shows that there is an $(n-1)$-spherical fibre space γ^n over $B_{G(n)}$ which is *universal* in the following sense:

Given any $(n-1)$-spherical fibre space π over a CW-*complex X there exists a map*

$$f\colon X\to B_{G(n)}$$

so that π is fibre homotopy equivalent to the induced fibre space $f^*\gamma^n$. Furthermore f is unique up to homotopy.

STASHEFF also notes (p. 243) that the spaces $B_{G(n)}$ and $B_{F(n)}$ have the homotopy types of CW-complexes, say $B'_{G(n)}$ and $B'_{F(n)}$ respectively. Since the homotopy groups $\pi_i B_{G(n)}$ and $\pi_i B_{F(n)}$ are all countable, we may assume that $B'_{G(n)}$ and $B'_{F(n)}$ are countable CW-complexes.

The inclusions $G(1)\subset F(1)\subset G(2)\subset\cdots$ give rise to mappings $B'_{G(1)}\to B'_{F(1)}\to\cdots$. Using an iterated mapping cylinder construction we may replace these by inclusion mappings

$$B'_{G(1)}\subset B''_{F(1)}\subset B''_{G(2)}\subset B''_{F(2)}\subset\cdots$$

[5]) There is no general agreement on notation. For example our $SG(n)$ is denoted by G_n in reference [11] and by $G(n-1)$ in reference [10].

of countable CW-complexes. In practice, since these last complexes are the only ones we will actually use, we will drop the double primes and write simply

$$B_{G(1)} \subset B_{F(1)} \subset B_{G(2)} \subset \cdots .$$

The union CW-complex will be denoted by $B_G = B_F$. Clearly

$$\pi_i B_{G(n)} \cong \pi_i B_{F(n)} \cong \pi_i B_G \quad \text{for} \quad i < n .$$

A similar argument shows that

$$H^i B_{G(n)} \cong H^i B_{F(n)} \cong H^i B_G \quad \text{for} \quad i < n ,$$

using any coefficient group.

Note that $\pi_1 B_{G(n)} \cong \pi_1 B_{F(n)} \cong Z_2$. The two fold covering complex of $B_{G(n)}$ will be identified with $B_{SG(n)}$. This is a universal base space for *oriented* $(n-1)$-spherical fibrations.

THEOREM 2. (STASHEFF). *The classifying space $B_F = B_G$ has the structure of a homotopy associative, homotopy commutative H-space.*

Proof. The Whitney join operation for spherical fibre spaces gives rise to a product operation

$$B_{G(m)} \times B_{G(n)} \to B_{G(m+n)} .$$

which is clearly well defined, commutative, and associative, up to homotopy. Furthermore it is not difficult to check that this product is compatible, up to homotopy, with the inclusions $B_{G(n)} \subset B_{G(n+1)}$.

These partial products can easily be pieced together, using the homotopy extension theorem, so as to obtain a product operation $w: B_G \times B_G \to B_G$. In order to show that w is homotopy associative and commutative, we will make use of the fortunate fact that the homotopy groups

$$\pi_i B_G \cong \pi_{i-1} G \cong \pi_{n+i-1} S^n$$

are all finite.

LEMMA 2. *If X is a countable* CW-*complex and if $\pi_i Y$ is finite*[6]) *for all i, then the set $[X, Y]$ of homotopy classes of maps $X \to Y$ is equal to the inverse limit of $[K, Y]$ as K ranges over all finite subcomplexes of X.*

Proof. Let $K_1 \subset K_2 \subset \cdots$ be finite subcomplexes with union X and let $f, g: X \to Y$ be maps such that each $f|K_i$ is homotopic to $g|K_i$. Consider all possible homotopies $h: K_1 \times [0, 1] \to Y$ between $f|K_1$ and $g|K_1$. These fall into a finite number of homotopy

[6]) Here is an example to show that this finiteness assumption is necessary. Let X be the complement of a "solenoid" in S^3 (compare EILENBERG and STEENROD, *Foundations of Algebraic Topology*, p. 230), and let Y be either S^2 or $K(Z, 2)$. Then $[X, Y]$ is uncountably infinite. Yet X can be expressed as the union of subcomplexes $K_1 \subset K_2 \subset \cdots$ where each K_i is a solid torus; so that $[K_i, Y] = 0$.

classes, relative to $h|(K_1 \times \{0, 1\})$. Among this finite number there must be at least one which extends to a homotopy between $f|K_i$ and $g|K_i$ for all i. Choose a homotopy h_1 in this prefered class. Now consider homotopies between $f|K_2$ and $g|K_2$ which extend h_1. Again we can use the finiteness assumption to choose one, say h_2, which can be extended over $K_i \times [0, 1]$ for all i. Continuing inductively, we obtain the required homotopy between f and g. The rest of the proof of Lemma 2 is straightforward.

Theorem 2 now follows easily by applying Lemma 2 to the sets $[B_G \times B_G, B_G]$ and $[B_G \times B_G \times B_G, B_G]$. This completes the proof.

It follows that both $H_*(B_G; Z_p)$ and $H^*(B_G; Z_p)$ are commutative Hopf algebras. According to Borel's theorem, each must be isomorphic, as an algebra, to the tensor product of a Grassman algebra and a possibly truncated polynomial algebra. (Compare [17, § 7]).

Another useful consequence of Theorem 2 is the following. Let $P(R)$ denote the infinite real projective space.

COROLLARY 1. *B_G has the homotopy type of $B_{SG} \times P(R)$.*

Proof. The twisted S^0-bundle over $P(R)$ gives rise to a classifying map

$$P(R) \to B_{G(1)} \subset B_G.$$

Combining this with the natural map $B_{SG} \to B_G$ we obtain a map

$$B_{SG} \times P(R) \to B_G \times B_G \xrightarrow{w} B_G$$

which clearly induces isomorphisms of homotopy groups in all dimensions. This completes the proof.

Since the cohomology of $P(R)$ consists completely of 2-torsion, this implies:

COROLLARY 2. *For p odd the cohomology algebra $H^*(B_G; Z_p)$ is isomorphic to $H^*(B_{SG}; Z_p)$.*

We can now construct the *universal Wu class* $q_i \in H^{2i(p-1)}(B_G; Z_p)$ as follows. This group is canonically isomorphic to

$$H^{2i(p-1)}(B_{SG}; Z_p) = H^{2i(p-1)}(B_{SG(n)}; Z_p)$$

for $n > 2i(p-1)$; and in the last group one has the i-th Wu class of the universal oriented $(n-1)$-spherical fibration.

It follows incidentally that the characteristic class $q_i(\pi)$ of a spherical fibration π can be defined even if π is not orientable.[7] For π is classified by a map $f: B \to B_{G(n)} \subset B_G$, and we can define $q_i(\pi)$ to be the class $f^* q_i$.

---

[7] This could also be proved by constructing a Thom isomorphism with suitably twisted coefficients.

The diagonal mapping w^* in the Hopf algebra $H^*(B_G; Z_p)$ satisfies

$$w^* q_k = \sum_{i+j=k} q_i \otimes q_j, \tag{2.4}$$
$$w^*(\beta q_k) = \sum_{i+j=k} (\beta q_i \otimes q_j + q_i \otimes \beta q_j), \tag{2.5}$$

where q_0 is defined to be 1. This shows that the q_i and βq_i generate a sub Hopf algebra of $H^*(B_G; Z_p)$. But it follows from Lemma 1 that the q_k are algebraically independent. Applying Borel's theorem to this subalgebra, this proves:

COROLLARY 3. *The subalgebra of $H^*(B_G; Z_p)$ generated by the q_i and βq_i is free commutative, with all of the q_i and some subset of the βq_i as independent generators.*

Of course this "subset" may be the entire set. The question as to whether the βq_i are independent is discussed further in Appendix 3.

(*Remark*. No βq_i in $H^*(B_G; Z_p)$ is actually zero. This follows from (2.5) and the fact, which we will prove in § 3, that $\beta q_1 \neq 0$.)

These results leave many unanswered questions. For example consider the natural homomorphism

$$j^* : H^*(B_G; Z_p) \to H^*(B_O; Z_p).$$

Question 1. Is the image of j^* generated by the Wu classes $j^*(q_i)$?

Question 2. Does this image coincide with the set of all characteristic classes in $H^*(B_O; Z_p)$ which are invariants of fibre homotopy type?

Question 3. Does $H^*(B_G; Z_p)$ split as a tensor product of Hopf algebras:

$$H^*(B_G; Z_p) \cong (\text{Image}\, j^*) \otimes (?).$$

The second factor should presumably be equal (in the notation of [17, § 3]) to (Image $j^*)\backslash\backslash H^*(B_G; Z_p)$.

§ 3. The k-invariants of B_{SG}

First let me give a brief review of the concept of "k-invariant". Given a simply connected CW-complex B let $B^{[0,t-1]}$ denote some complex which is obtained from B by adjoining cells of dimension $\geq t+1$ so as to kill off all of the homotopy groups in dimensions $\geq t$. Clearly the pair $(B^{[0,t-1]}, B)$ is t-connected; so that:

$$\begin{aligned} H_i(B^{[0,t-1]}, B) &= 0 \quad \text{for} \quad i \leq t, \quad \text{and} \\ H_{t+1}(B^{[0,t-1]}, B) &\cong \pi_{t+1}(B^{[0,t-1]}, B) \cong \pi_t B. \end{aligned} \tag{3.1}$$

Thus the homology exact sequence of the pair $(B^{[0,t-1]}, B)$ in dimensions t, $t+1$ takes the form

$$H_{t+1} B \to H_{t+1} B^{[0,t-1]} \to \pi_t B \to H_t B \to H_t B^{[0,t-1]} \to 0. \tag{3.2}$$

The $(t+1)$-dimensional *k-invariant* of B is a cohomology class

$$k^{t+1} \in H^{t+1}(B^{[0,t-1]}; \pi_t B)$$

defined as follows. For any coefficient group Λ the universal coefficient theorem, together with (3.1) shows that $H^{t+1}(B^{[0,t-1]}, B; \Lambda)$ is isomorphic to $\mathrm{Hom}(\pi_t B, \Lambda)$. In particular, taking Λ to be $\pi_t B$ itself, we obtain a canonical cohomology class in

$$H^{t+1}(B^{[0,t-1]}, B; \pi_t B) \cong \mathrm{Hom}(\pi_t B, \pi_t B)$$

corresponding to the identity map of $\pi_t B$. *The image of this canonical class under the natural homomorphism*

$$H^{t+1}(B^{[0,t-1]}, B; \pi_t B) \to H^{t+1}(B^{[0,t-1]}; \pi_t B)$$

is called the k-invariant k^{t+1} of B.

Now consider a complex $B^{[0,t]}$. We may assume that $B^{[0,t]}$ is a subcomplex of $B^{[0,t-1]}$. (In fact starting with any given $B^{[0,t]}$ one can adjoin cells of dimension $\geq t+1$ so as to kill π_t; thus yielding a suitable $B^{[0,t-1]}$.) The inclusion

$$B^{[0,t]} \subset B^{[0,t-1]}$$

can be made into a fibre space

$$B_1^{[0,t]} \to B^{[0,t-1]}$$

by the usual construction. Here $B_1^{[0,t]}$ denotes the space of all paths in $B^{[0,t-1]}$ which end in the subcomplex $B^{[0,t]}$. The fibre will be denoted by K^t. Clearly K^t is an Eilenberg-MacLane space of type $K(\pi_t B, t)$.

We will apply these constructions to the simply connected complex $B = B_{SG}$; and will try to compute the mod$p$ cohomology of $B_{SG}^{[0,t]}$ by induction on t.

Let p be a fixed odd prime. *The integer* $2(p-1)$ *will always be denoted by* r. According to Toda [24] the p-primary component $\pi_j(B_{SG}; p)$ of the finite group

$$\pi_j B_{SG} \cong \pi_{j-1+n} S^n$$

is isomorphic to:

$$\begin{array}{lll} Z_p & \text{for} & j = r, 2r, 3r, \ldots, (p-1)r \\ Z_p & \text{for} & j = pr - 1 \\ Z_{p^2} & \text{for} & j = pr; \end{array}$$

and is zero for all other values of j less than $(p+1)r-2$. (Toda computes many further groups, but this will be more than enough for our purposes.)

If $\pi_t B$ has no p-primary component then clearly

$$H^*(B^{[0,t]}; Z_p) \cong H^*(B^{[0,t-1]}; Z_p).$$

So we only need to concentrate on the cases $t = r, 2r, 3r, \ldots, (p-1)r$. For the first case $t = r$, note that $B^{[0,r]}$ has the same mod$p$ cohomology as the Eilenberg-MacLane space K^r.

For each m between 2 and $p-1$ let

$$k_m \in H^{mr+1}(B_{SG}^{[0,mr-1]}; Z_p)$$

denote the cohomology class which corresponds to the p-primary component of the k-invariant

$$k^{mr+1} \in H^{mr+1}(B_{SG}^{[0,mr-1]}; \pi_{mr} B_{SG})$$

under some coefficient isomorphism[8])

$$\pi_{mr}(B_{SG}; p) \cong Z_p .$$

THEOREM 3. *If $2 \leq m < p$ then the algebra $H^* B_{SG}^{[0,mr-1]}; Z_p)$ is isomorphic, in dimensions less than $pr = 2p(p-1)$, to the free commutative algebra on generators*

$$P^i \bar{q}, \beta P^i \bar{q}, \quad (i \geqslant 0),$$

and

$$P^i k_m, \beta P^i k_m, \quad (i \geqslant 0);$$

where $\bar{q}$ corresponds to the first Wu class q_1 under the natural isomorphism

$$H^r(B_{SG}^{[0,mr-1]}; Z_p) \to H^r(B_{SG}; Z_p)$$

and where k_m is the p-primary component of the $(mr+1)$-dimensional k-invariant. Furthermore, if $m < p-1$ then k_m satisfies the relation:

$$((m+1) P^1 \beta - m \beta P^1) k_m = 0 . \tag{3.3}$$

The proof, which will occupy the rest of § 3, will be by induction on m.

Proof of Theorem 3 for $m=2$. Consider first the mod p cohomology of an Eilenberg-Maclane complex $K(Z_p, 2t)$. Let u denote a $2t$-dimensional generator. According to CARTAN [5]:

The algebra $H^*(K(Z_p, 2t); Z_p)$ is free commutative on the generators

$$P^i u, \beta P^i u, P^{i+1} \beta u, \beta P^{i+1} \beta u, \quad (0 \leqslant i < t);$$

together with other generators (such as $P^p P^1 u$) of dimension $\geq (p+1) r + 2t$.

(See also [9, p. 29]. The precise statement is that $H^*(K(Z_p, n); Z_p)$ is free commutative on generators

$$\beta^{\varepsilon_1} P^{i_1} \beta^{\varepsilon_2} P^{i_2} \beta^{\varepsilon_3} P^{i_3} \ldots u$$

where the integers $i_j \geq 0$ and $\varepsilon_j = 0$ or 1 are almost all zero, and are subject to the inequalities

$$\varepsilon_j + p i_j \leqslant i_{j-1} < \tfrac{1}{2} \dim (\beta^{\varepsilon_j} P^{i_j} \beta^{\varepsilon_{j+1}} P^{i_{j+1}} \ldots u)$$

for $j \geq 2$.)

---

[8]) Since this coefficient isomorphism is more or less arbitrary we will feel free to replace k_m by any non-zero multiple $\lambda \cdot k_m$, with $\lambda \in Z_p$.

Now recall that the space $B^{[0,2r-1]} = B_{SG}^{[0,2r-1]}$ has the mod p cohomology of an Eilenberg-MacLane complex $K(Z_p, r)$. Let $\bar{q} \in H^r(B^{[0,2r-1]}; Z_p)$ correspond to the Wu class q_1. It follows that:

The algebra $H^*(B^{[0,2r-1]}; Z_p)$ is free commutative on the generators

$$P^i \bar{q},\ \beta P^i \bar{q},\ P^{i+1} \beta \bar{q},\ \beta P^{i+1} \beta \bar{q}, \quad (0 \leqslant i \leqslant p-2);$$

together with other generators of dimension $\geq (p+2)\, r$.

Let us compute the k-invariant

$$k_2 \in H^{2r+1}(B^{[0,2r-1]}; Z_p).$$

This can be characterized, up to multiplication by units in Z_p, as the element which generates the kernel of the restriction homomorphism

$$i^*: H^{2r+1}(B^{[0,2r-1]}; Z_p) \to H^{2r+1}(B; Z_p).$$

But recall the identity (1.11):

$$2 P^1 \beta q_1 - \beta P^1 q_1 = 0,$$

in $H^{2r+1}(B; Z_p)$. Since $i^* \bar{q} = q_1$ this shows that

$$i^*(2 P^1 \beta \bar{q} - \beta P^1 \bar{q}) = 0.$$

Therefore k_2 must be a multiple of $2P^1 \beta \bar{q} - \beta P^1 \bar{q}$. By changing the isomorphism $\pi_{2r}(B; p) \to Z_p$ if necessary, we may assume that k_2 is equal to $2P^1 \beta \bar{q} - \beta P^1 \bar{q}$.

A short computation now shows that:

$$P^i k_2 = (i+2)\, P^{i+1} \beta \bar{q} - \beta P^{i+1} \bar{q}, \tag{3.4}$$
$$\beta P^i k_2 = (i+2)\, \beta P^{i+1} \beta \bar{q}, \quad \text{and} \tag{3.5}$$
$$P^i \beta k_2 = 2 \beta P^{i+1} \beta \bar{q}; \quad \text{for} \quad i < p. \tag{3.6}$$

(Here is a convenient subset of the Adem relations for this purpose. If $i<p$ then

$$P^i P^1 = (i+1)\, P^{i+1},$$
$$P^i \beta P^1 = i\, P^{i+1} \beta + \beta P^{i+1}.$$

Using these relations the three formulas above follow immediately.)

Formula (3.4) clearly shows that we can use $P^i k_2$ in place of $P^{iH+1} \beta \bar{q}$ as a free generator for the algebra $H^*(B^{[0,2r-1]}; Z_p)$; providing only that $i+2<p$. But if $i+2 \geq p$ then the element $P^{i+1} \beta \bar{q}$ has dimension $(i+2)\, r+1 > pr$. If we only attempt to describe the situation in dimensions $<pr$, then such elements can be ignored.

Similarly (3.5) shows that $\beta P^i k_2$ can be used in place of $\beta P^{i+1} \beta \bar{q}$ as a free generator. Thus:

The algebra $H^*(B^{[0,2r-1]}; Z_p)$ is free commutative, in dimensions $<pr$, on the required generators

$$P^i \bar{q}, \beta P^i \bar{q}, P^i k_2, \beta P^i k_2, \quad (i \geqslant 0).$$

Finally we must verify the relation

$$3 P^1 \beta k_2 - 2 \beta P^1 k_2 = 0.$$

But this follows immediately from (3.5) and 3.6). Thus we have proved Theorem 3 for $m=2$.

Remark. If we go slightly beyond the range of dimensions under consideration, it is interesting to note the relations $P^{p-1}\bar{q}=\bar{q}^p$ and

$$P^{p-2} k_2 = -\beta P^{p-1} \bar{q} = 0.$$

This last relation gives rise to a new cohomology class in $H^{pr}(B^{[0,2r]}; Z_p)$.

Now suppose inductively that Theorem 3 is true for a given value of m, with $2 \leq m < p-1$. Making the inclusion $B^{[0,mr]} \subset B^{[0,mr-1]}$ into a fibre space

$$K^{mr} \to B_1^{[0,mr]} \to B^{[0,mr-1]},$$

we recall that the fibre is an Eilenberg-MacLane space of type $K(\pi_{mr}B, mr)$. We will study the mod p cohomology spectral sequence of this fibration. (See for example SPANIER [19].) Let

$$u_m \in H^{mr}(K^{mr}; Z_p)$$

be the generator whose transgression $\tau u_m = d_{mr+1} u_m$ is equal to the k-invariant k_m. In dimensions less than $(p+m+1)\,r$ the cohomology of K^{mr} is free commutative on the generators

$$P^i u_m, \beta P^i u_m, P^{i+1} \beta u_m, \beta P^{i+1} \beta u_m;$$

where $0 \leq i < m(p-1)$.

It will be convenient to let θ denote the cohomology operation $(m+1)\,P^1\beta - m\beta P_1$ of degree $r+1$. Note the identity

$$\theta k_m = 0,$$

which is assumed (3.3) as part of the induction hypothesis. Let v_{m+1} denote the class θu_m of dimension $mr+r+1$ in $H^*(K^{mr}; Z_p)$. Since u_m is transgressive it is clear that v_{m+1} is transgressive[9]) and that

$$\tau(v_{m+1}) = \tau \theta u_m = \pm \theta \tau u_m = \pm \theta k_m = 0.$$

Similarly each of the classes $P^i v_{m+1}, \beta P^i v_{m+1}, P^i \beta v_{m+1}$ is transgressive. A short

---

[9]) At least this is clear if one uses the alternative definition: a cohomology class $u \in H^t F$ is *transgressive*, for a fibration $F \subset E \to B$, if $\delta u \in H^{t+1}(E, F)$ comes from a cohomology class in $H^{t+1}(B, b_0)$.

computation shows that

$$\begin{aligned} P^i v_{m+1} &= ((i+m+1)\, P^{i+1} \beta - m \beta P^{i+1})\, u_m, \\ \beta P^i v_{m+1} &= (i+m+1)\, \beta P^{i+1} \beta u_m, \quad \text{and} \\ P^i \beta v_{m+1} &= (m+1)\, \beta P^{i+1} \beta u_m; \quad \text{for} \quad i < p. \end{aligned}$$

Combining these last two equalities for $i=1$ we obtain the relation:

$$((m+2)\, P^1 \beta - (m+1)\, \beta P^1)\, v_{m+1} = 0. \tag{3.7}$$

From the first two we see that $P^i v_{m+1}$ and $\beta P^i v_{m+1}$ can be used in place of $P^{i+1} \beta u_m$ and $\beta P^{i+1} \beta u_m$ as free generators for the cohomology of the fibre K^{mr}; providing that that $i+m+1<p$. But the dimension of $P^{i+1} \beta u_m$ is $(i+m+1)\, r+1$; so those $P^{i+1} \beta u_m$ with $i+m+1 \geq p$ have dimensions $>pr$ and can be ignored. This proves:

ASSERTION A. *The cohomology of the fibre K^{mr} is free commutative in dimensions $<pr$ on transgressive generators*

$$P^i u_m,\, \beta P^i u_m,\, P^i v_{m+1},\, \beta P^i v_{m+1}, \qquad (i \geqslant 0).$$

The cohomology of the base $B^{[0,mr-1]}$ is free commutative in dimensions $<pr$ on generators

$$P^i \bar{q},\, \beta P^i \bar{q},\, P^i k_m,\, \beta P^i k_m, \qquad (i \geqslant 0);$$

and the transgression is given by:

$$\begin{aligned} \tau P^i u_m = P^i k_m,\ \tau \beta P^i u_m = \pm \beta P^i k_m, \\ \tau P^i v_{m+1} = 0,\ \tau \beta P^i v_{m+1} = 0. \end{aligned}$$

[It is important to note that the fibre has no cohomology in dimension $pr-1$. Hence we do not need to worry about the transgression

$$H^{pr-1}(K^{mr}; Z_p) \to H^{pr}(B^{[0,mr-1]}; Z_p)$$

where the group on the right is unknown.]

We are now in a position to completely describe our cohomology spectral sequence in the dimensions less than pr. The E_2 term is given of course by

$$E_2^{**} = H^*(B^{[0,mr-1]}; H^*(K^{mr}; Z_p));$$

and is free commutative, in dimensions less than pr, on the generators

$$\begin{aligned} &P^i u_m,\, \beta P^i u_m,\, P^i v_{m+1},\, \beta P^i v_{m+1}, \\ &P^i \bar{q},\, \beta P^i \bar{q},\, P^i k_m,\, \beta P^i k_m. \end{aligned}$$

The transgression

$$\tau = d_t : E_t^{0,t-1} \to E_t^{t,0}$$

is zero for $t \leq mr$; and it follows that d_t is identically zero for $t \leq mr$, so that

$$E_2 = E_3 = \cdots = E_{mr} = E_{mr+1}.$$

The boundary operator

$$d_{mr+1} : E_{mr+1}^{0,t} \to E_{mr+1}^{mr+1,t-mr}$$

carries u_m into k_m and is zero on the other free generators. It follows easily that the cohomology E_{mr+2} of (E_{mr+1}, d_{mr+1}) is free commutative, in dimensions $<pr$, on all of the generators listed above *except* u_m and k_m.

[*Proof.* The identity $d_{rm+1}(u_m^i) = i k_m u_m^{i-1}$ shows that the free subalgebra generated by u_m and k_m has trivial homology in positive dimensions less than pmr. But the entire algebra can be considered as the tensor product of this subalgebra and a complementary free algebra on which d_{rm+1} operates trivially. The Künneth theorem asserts that the cohomology of such a tensor product is equal to the tensor product of the cohomologies. Hence E_{rm+2} is free commutative on the remaining generators, in dimensions $<pr$].

Similarly the operator d_{rm+2} kills the generators βu_m and βk_m, the operator $d_{r(m+1)+1}$ kills $P^1 u_m$ and $P^1 k_m$, and so on, until all of the $P^i u_m$, $P^i k_m$, $\beta P^i u_m$ and $\beta P^i k_m$ have disappeared. This proves:

ASSERTION B. *The E_∞ term of this spectral sequence is free commutative in dimensions less than pr on the generators*

$$P^i \bar{q},\ \beta P^i \bar{q},\ P^i v_{m+1},\ \beta P^i v_{m+1}.$$

It now follows immediately that $H^*(B_1^{[0,mr]}; Z_p)$ is also free commutative in this range of dimensions. In fact, if

$$v'_{m+1} \in H^{(m+1)r+1}(B_1^{[0,mr]}; Z_p)$$

denotes any cohomology class whose restriction to the fibre is v_{m+1}, then we can take as free generators.

$$P^i \bar{q},\ \beta P^i \bar{q},\ P^i v'_{m+1},\ \beta P^i v'_{m+1}$$

The same description holds for $H^*(B^{[0,(m+1)r-1]}; Z_p)$, since this algebra is canonically isomorphic to $H^*(B^{[0,mr]}; Z_p)$. Now consider the next k-invariant

$$\begin{aligned} k_{m+1} &\in H^{(m+1)r+1}(B^{[0,(m+1)r-1]}; Z_p) \\ &\cong H^{(m+1)r+1}(B_1^{[0,mr]}; Z_p). \end{aligned}$$

LEMMA 3. *The restriction of k_{m+1} to the fibre K^{mr} of the fibration $B_1^{[0,mr]} \to B^{[0,mr-1]}$ is non-zero.*

This will be proved at the end of § 3. Assuming Lemma 3 for the time being, the proof of Theorem 3 proceeds as follows.

The restriction of k_{m+1} to the fibre K^{mr} is clearly some multiple of v_{m+1}. (For the only other basis element $\beta P^1 u_m$ in $H^{(m+1)r+1}(K^{mr}; Z_p)$ does not extend over the total space.) Since this restriction is non-zero, by Lemma 3, we may as well use k_{m+1} in place of v'_{m+1} as the extension of v_{m+1}. This proves:

ASSERTION C. *The algebra $H^*(B^{[0,(m+1)r-1]}; Z_p)$ is free commutative, in dimensions less than pr, on the generators*

$$P^i \bar{q},\ \beta P^i \bar{q},\ P^i k_{m+1},\ \beta P^i k_{m+1}, \qquad (i \geqslant 0).$$

Thus in order to complete the induction we need only prove that k_{m+1} satisfies the required relation (3.3). The following proof is due to STASHEFF.

Recall from § 2 that B is an H-space. It follows that each $B^{[0,t]}$ is an H-space, and hence that $H^*(B^{[0,t]}; Z_p)$ is a Hopf algebra. Furthermore the k-invariant

$$k_{m+1} \in H^*(B^{[0,(m+1)r-1]}; Z_p)$$

must be primitive, since it is the lowest dimensional element in the kernel of the homomorphism

$$H^*(B^{[0,(m+1)r-1]}; Z_p) \to H^*(B; Z_p).$$

of Hopf algebras. Hence all of our free generators

$$P^i \bar{q},\ \beta P^i \bar{q},\ P^i k_{m+1},\ \beta P^i k_{m+1}$$

for this algebra $H^*(B^{[0,(m+1)r-1)}; Z_p)$ are primitive. In fact:

These generators, and their linear combinations, are the only primitive elements in dimensions less than pr. (3.8)

This follows easily from the more general statement that the set $P(A_1 \otimes \cdots \otimes A_n)$ of primitive elements in a tensor product of Hopf algebras is equal to the direct sum $P(A_1) \oplus \cdots \oplus P(A_n)$ of the sets of primitive elements. (Compare [17, § 4.10].)

Now consider the element

$$P^1 \beta k_{m+1} \in H^{(m+2)r+2}(B_1^{[0,mr]}; Z_p).$$

This is also primitive. Hence (if we assume that $m+2<p$) there must be some linear relation of the form

$$P^1 \beta k_{m+1} = \lambda \cdot \beta P^1 k_{m+1}, \tag{3.9}$$

with $\lambda \varepsilon Z_p$. Restricting to the fibre K^{mr} this implies that

$$P^1 \beta v_{m+1} = \lambda \cdot \beta P^1 v_{m+1}.$$

But formula (3.7) shows that λ must be congruent to $(m+1)/(m+2)$ modulo p. Hence formula (3.9) must take the required form:

$$(m+2) P^1 \beta k_{m+1} - (m+1) \beta P^1 k_{m+1} = 0.$$

This completes the induction; except that we have not yet proved Lemma 3.

Proof of Lemma 3. Suppose that the restriction of the k-invariant k_{m+1} to the fibre K^{mr} were equal to zero. Using Assertion B it would follow easily that k_{m+1} must belong to the subalgebra of $H^*(B_1^{[0,mr]}; Z_p)$ generated by the classes $P^i\bar{q}$ and $\beta P^i\bar{q}$ which come from the base space.

On the other hand we know that k_{m+1} is a primitive element; and that the only primitive elements in this subalgebra are the generators and their multiples. (Compare (3.8).) In particular the primitive elements in dimension $(m+1)\,r+1$ are all multiples of the generator $\beta P^m\bar{q}$. Thus:

In order to prove Lemma 3 it is sufficient to prove that k_{m+1} *is not a multiple of* $\beta P^m\bar{q}$. (3.10)

Now consider the inclusion maps

$$U(n) \subset SO(2n) \subset SG(2n).$$

These give rise to a map

$$j: B_U \to B_{SG}.$$

We will use j to compare the k-invariants of $B=B_{SG}$ with those of B_U.

First consider the more general case of a map $f: X \to Y$. There is an induced map

$$f^{[0,t-1]}: X^{[0,t-1]} \to Y^{[0,t-1]},$$

which is well defined up to homotopy. Consider the diagram

$$\begin{tikzcd}[column sep=small] H^{t+1}(X^{[0,t-1]}; \pi_t X) \arrow[dr, "(f_*)_*"] & & H^{t+1}(Y^{[0,t-1]}; \pi_t Y) \arrow[dl, "f^{[0,t-1]*}"'] \\ & H^{t+1}(X^{[0,t-1]}; \pi_t Y), & \end{tikzcd}$$

where f_* denotes the coefficient homomorphism $\pi_t X \to \pi_t Y$. The relationship between the k-invariant $k^{t+1}(X)$ in the left hand group and the k-invariant $k^{t+1}(Y)$ on the right is now described by the identity

$$(f_*)_* k^{t+1}(X) = f^{[0,t-1]*} k^{t+1}(Y). \tag{3.11}$$

(Compare KAHN [14].)

In order to apply (3.11) we first need a description of the k-invariants of the space B_U. Recall that:

(a) The homotopy group $\pi_n B_U$ is infinite cyclic for n even and is zero for n odd.

(b) The integral homology group $H_n B_U$ is free abelian for n even and is zero for n odd.

(c) The Hurewicz homomorphism $\pi_{2t} B_U \to H_{2t} B_U$ carries a generator into an element which is divisible by precisely $(t-1)!$

(See BOTT [3, p. 88] and [2].) Using the exact sequence (3.2) it follows that

$H_{2t+1} B_U^{[0,2t-1]}$ is zero, and that

$$H_{2t} B_U^{[0,2t-1]} \cong Z \oplus \cdots \oplus Z \oplus Z_{(t-1)!}.$$

Therefore the cohomology group $H^{2t+1}(B_U^{[0,2t-1]}; Z)$ is cyclic of order $(t-1)!$. Furthermore:

The k-invariant k^{2t+1} of B_U is a generator of this finite cyclic group $H^{2t+1} B_U^{[0,2t-1]}$. (3.12)

This is clear since k^{2t+1} must generate the kernel of the homomorphism $H^{2t+1} B_U^{[0,2t-1]} \to H^{2t+1} B_U$. Also we will need:

The Bockstein homomorphisms

$$\beta: H^{2t}(B_U^{[0,2t-1]}; Z_p) \to H^{2t+1}(B_U^{[0,2t-1]}; Z_p) \tag{3.13}$$

is zero if $t \geq 2p+1$.

For β can be non-zero only if the corresponding homology group

$$H_{2t} B_U^{[0,2t-1]} \cong Z \oplus \cdots \oplus Z \oplus Z_{(t-1)!}$$

admits the cyclic group Z_p as a direct summand. If $t-1 \geq 2p$ then this is certainly not the case.

Now set $2t$ equal to $(m+1)r = 2(m+1)(p-1)$. Since $3 \leq m+1 < p$, the required inequality $t \geq 2p+1$ is easily verified; so the assertion (3.13) applies.

Next we must look at the coefficient homomorphism

$$j_*: \pi_{2t} B_U \to \pi_{2t} B_{SG}.$$

Again we assume that $2t = (m+1) r < pr$.

The image $j_* \pi_{2t} B_U$ contains the p-primary component $\pi_{2t}(B_{SG}; p) \cong Z_p$. (3.14)

For j_* can be described as a composition

$$\pi_{2t} B_U \to \pi_{2t} B_{SO} \xrightarrow{J} \pi_{2t} B_{SG};$$

where the first homomorphism maps a generator onto either a generator or twice a generator, according to Bott; and where the image $J(\pi_{2t} B_{SO})$ is known to contain the p-primary component of $\pi_{2t} B_{SG}$ for $2t < pr-1$. (See [16].)

It follows easily from (3.12) and (3.14) that the induced homomorphism

$$(j_*)_*: H^{2t+1}(B_U^{[0,2t-1]}; \pi_{2t} B_U) \to H^{2t+1}(B_U^{[0,2t-1]}; \pi_{2t} B_{SG})$$

carries the $(2t+1)$-dimensional k-invariant of B_U into a class $(j_*)_* k^{2t+1}(B_U)$ whose p-primary component is not zero.

Proof that k_{m+1} is not a multiple of $\beta P^m \bar{q}$. If k_{m+1} were a multiple of $\beta P^m \bar{q}$ then the image $j^{[0,\,2t-1]*}k_{m+1}$ would be a multiple of

$$j^{[0,\,2t-1]*}\beta P^m \bar{q} = \beta(P^m j^{[0,\,2t-1]*}\bar{q});$$

which is zero by (3.13). On the other hand $j^{[0,\,2t-1]*}k_{m+1}$ is equal to the p-primary component of $(j_*)_* k^{2t+1}(B_U)$ by (3.11); and we have just shown that this p-primary component is not zero.

Thus k_{m+1} cannot be a multiple of $\beta P^m \bar{q}$. According to (3.10) this proves Lemma 3, and completes the proof of Theorem 3.

§ 4. Conclusion: The cohomology of B_{SG}

Now let us carry the inductive calculation of $H^*(B_{SG}^{[0,\,t]}; Z_p)$ one stage further.

LEMMA 4. *The algebra $H^*(B_{SG}^{[0,\,(p-1)r]}; Z_p) \cong H^*(B_{SG}^{[0,\,pr-2]}; Z_p)$ is free commutative in dimensions less than* pr *on the generators $P^i\bar{q}$ and $\beta P^i \bar{q}$. Hence $H^*(B_{SG}; Z_p)$ is free commutative in dimensions less than $pr-1$ on the corresponding generators $P^i q_1$ and $\beta P^i q_1$.*

Proof. First consider the fibration

$$K^{(p-1)r} \to B_1^{[0,\,(p-1)r]} \to B^{[0,\,(p-1)r-1]}.$$

The fibre has only the cohomology classes u_{p-1} and βu_{p-1} in dimensions less than pr. According to Theorem 3 the base is free commutative on

$$P^i\bar{q},\ \beta P^i\bar{q},\ k_{p-1},\ \beta k_{p-1}$$

in this range of dimensions. Since

$$\tau(u_{p-1}) = k_{p-1},\ \tau(\beta u_{p-1}) = \beta k_{p-1}$$

it is clear that the cohomology of the total space $B_1^{[0,\,(p-1)r]}$ is obtained simply by eliminating these four generators.

The cohomology of $B^{[0,\,pr-2]}$ is the same since $\pi_t(B; p)=0$ for $(p-1)\,r<t\le pr-2$. But $B^{[0,\,pr-2]}$ can be obtained from $B=B_{SG}$ by attaching cells of dimension $\ge pr$. This operation may diminish the cohomology in dimension $pr-1$; but it certainly cannot change anything in dimensions $<pr-1$. This proves Lemma 4.

In order to prove Theorem 4, as stated in the introduction, it is only necessary to note that we can use q_{i+1} in place of $P^i q_1$ and βq_{i+1} in place of $\beta P^i q_1$ as free generators. Since this follows immediately from § 1.12; this completes the proof.

Appendix 1. The rational cohomology of $B_{G(n)}$

This appendix will prove Theorem 5, as stated in the introduction.

From the fibration

$$SF(2t-1) \to SG(2t) \to S^{2t-1},$$

and Serre's theorem that $\pi_i S^{2t-1}$ is finite for $i \neq 2t-1$, we see that $\pi_i SG(2t)$ is finite for $i \neq 2t-1$, and that $\pi_{2t-1}\, SG(2t)$ has rank 1. Passing to the classifying space $B_{SG(2t)}$ it follows that the rational cohomology $H^*(B_{SG(2t)}; Q)$ is a polynomial algebra on one generator of dimension $2t$. But the rational Euler class x lives in dimension $2t$, and is not zero. Thus:

LEMMA 5. *The algebra $H^*(B_{SG(2t)}; Q)$ is free commutative, with the Euler class x as generator.*

This space $B_{SG(2t)}$ is equal to the 2-fold covering of $B_{G(2t)}$. Let

$$f: B_{SG(2t)} \to B_{SG(2t)}$$

be the non-trivial covering transformation. Since f corresponds to the operation of reversing the orientation of a bundle, it is clear that

$$f^* x = -x.$$

Now $H^*(B_{G(2t)}; Q)$ can be identified with the subalgebra of $H^*(B_{SG(2t)}; Q)$ consisting of elements invariant under f^*. Hence:

The algebra $H^*(B_{G(2t)}; Q)$ is free commutative, generated by the class x^2.

If we map back to the cohomology of $B_{SO(2t)}$, it is well known that x^2 corresponds to the $4t$-dimensional Pontryagin class p_t. This proves half of Theorem 5.

Similar arguments show that the homotopy group $\pi_i SG(2t+1)$ is finite for $i \neq 4t-1$ and of rank 1 for $i = 4t-1$. Hence:

The algebra $H^*(B_{SG(2t+1)}; Q)$ is free commutative on one generator of dimension $4t$.

To complete the proof we must show that the natural homomorphism

$$i^*: H^{4t}(B_{SG(2t+1)}; Q) \to H^{4t}(B_{SG(2t)}; Q)$$

is not zero. Consider the diagram

$$\begin{array}{ccc} H^{4t}(B_{SG(2t+1)}; Q) & \xrightarrow{i^*} & H^{4t}(B_{SG(2t)}; Q) \\ \downarrow j^* & & \downarrow \\ H^{4t}(B_{SO(2t+1)}; Q) & \to & H^{4t}(B_{SO(2t)}; Q), \end{array}$$

where the bottom arrow is known to have kernel zero.

Comparing the fibrations $SO(2t+1) \to S^{2t}$ and $SG(2t+1) \to S^{2t}$ one sees that

$$j_*: \pi_{4t} B_{SO(2t+1)} \to \pi_{4t} B_{SG(2t+1)}$$

has rank 1. Hence the corresponding rational cohomology homomorphism j^* is certainly not zero. Looking at the diagram above it follows that $i^* \neq 0$.

The only possibility is clearly that j^* carries the generator in $H^{4t}(B_{SG(2t+1)}; Q)$

into a multiple of the Pontryagin class p_t. Since the Pontryagin classes of a bundle do not depend on orientation, it follows easily that

$$H^*(B_{G(2t+1)}; Q) \cong H^*(B_{SG(2t+1)}; Q).$$

This completes the proof.

Appendix 2. The two H-space structures on SF

Any loop space ΩX has a natural H-space structure, and the corresponding classifying space $B_{\Omega X}$ has the homotopy type of X. In particular the n-fold loop space $\Omega^n S^n$ has a classifying space which we can identify with $\Omega^{n-1} S^n$. The component $(\Omega^n S^n)_0$ of the constant path also has a classifying space which we can identify with the universal covering space $\tilde{\Omega}^{n-1} S^n$. (An argument similar to the proof of Corollary 1 in § 2 shows that $\Omega^{n-1} S^n$ has the homotopy type of $S^1 \times \tilde{\Omega}^{n-1} S^n$.)

Since the H-space $(\Omega^n S^n)_0$ has the homotopy type of $SF(n)$ one might conjecture that the classifying space $\tilde{\Omega}^{n-1} S^n$ has the homotopy type of $B_{SF(n)}$; but this is far from true. Thus $H^*(B_{SF(n)}; Z_p)$ contains the full polynomial algebra with independent generators $q_1, q_2, \ldots, q_{[n/2]}$. In contrast:

ASSERTION. *Every positive dimensional cohomology class y in $H^*(\tilde{\Omega}^{n-1} S^n; Z_p)$ satisfies the identity $y^p = 0$.*

Proof. DYER and LASHOF [8, § 5.2] show that the homology $H_*(\Omega^{n-1} S^n; Z_p)$ is a primitively generated Hopf algebra. This implies that every element of the dual Hopf algebra has height p. (Compare [17, § 4.20].)

Similar remarks hold modulo 2. Although the spaces $\tilde{\Omega}^{n-1} S^n$ and $B_{SF(n)}$ have the same homotopy groups, they are distinguished already by the first k-invariant in $H^4(K(Z_2, 2); Z_2)$.

It should be remarked however that modulo p, and in the range of dimensions considered in Theorem 4, one cannot distinguish between the two classifying spaces.

It seems likely that one could use these ideas to get a better grasp on the cohomology of $B_{SF} = B_{SG}$.

Appendix 3. Are the βq_i independent?

It follows from Theorem 4, together with Theorem 2 Corollary 3, that the cohomology classes

$$\beta q_1, \ldots, \beta q_{p-1} \in H^*(B_G; Z_p)$$

are *independent*. That is they freely generate a free commutative subalgebra. But the question as to whether all of the βq_i, $i \geq 1$, are independent remains open. One possible attack on this question is the following.

PROBLEM. *Does there exist a spherical fibre space ξ over some base B_ξ so that the two cohomology classes $q_1(\xi)$ and $\beta q_1(\xi)$ in $H^*(B_\xi; Z_p)$ are independent; but so that all of the $q_i(\xi)$ with $i \geq 2$ are zero?*

One candidate for such a ξ would be the canonical 3-spherical fibre space over $B_{SF(3)}$.

THEOREM 6. *Suppose that such a fibre space ξ exists. Then all of the universal classes q_i and βq_i, $i \geq 1$, in $H^*(B_G; Z_p)$ are independent.*

The proof, which is similar to that of Lemma 1, will require a graded version of the classical theory of symmetric functions, as follows.

Over a field of characteristic $\neq 2$ let A be the free commutative (graded) algebra on generators

$$x_1, \ldots, x_n \in A^{2t}$$

and generators

$$y_1, \ldots, y_n \in A^{2t+1}.$$

Let $\beta: A \to A$ be the derivation which carries each x_i into y_i and each y_i into 0. Let $\sigma_i \varepsilon A^{2ti}$ be the i-th elementary symmetric function of $x_1, \ldots, x_n$.

LEMMA 6. *The elements $\sigma_1, \ldots, \sigma_n$ and $\beta\sigma_1, \ldots, \beta\sigma_n$ freely generate a free commutative subalgebra of A.*

[In fact this subalgebra consists precisely of those elements of A which are "symmetric" in the sense that they are fixed under the action of the symmetric group of degree n, which acts on A by permuting the x_i and permuting the $y_i = \beta x_i$ correspondingly.]

The main step in the proof is the verification that the n-fold product

$$(\beta\sigma_1)(\beta\sigma_2)\ldots(\beta\sigma_n)$$

is non-zero[10]). This can best be checked by inserting the explicit formulas

$$\beta\sigma_1 = y_1 + \cdots + y_n$$
$$\beta\sigma_2 = (y_1 x_2 + x_1 y_2 + y_1 x_3 + \cdots + x_{n-1} y_n)$$
$$\beta\sigma_n = (y_1 x_2 \cdots x_n + \cdots + x_1 \cdots x_{n-1} y_n)$$

and then multiplying out and noting that the coefficient of $y_1 y_2 \ldots y_n x_1^{n-1} x_2^{n-2} \ldots x_{n-1}^1$ in the resulting expression is equal to $+1 \neq 0$.

The rest of the argument is purely formal. Let N denote the set $\{1, 2, \ldots, n\}$ and for each $S \subset N$ let

$$\tau_S = \textstyle\prod_{i \in S} (\beta\sigma_i).$$

---

[10]) Actually one has the following explicit identity: $(\beta\sigma_1)(\beta\sigma_2)\ldots(\beta\sigma_n) = y_1 \ldots y_n \prod_{1 \leq i < j \leq n} (x_i - x_j)$.

Since $\tau_N \neq 0$ we see that there is no relation of the form

$$f(x_1, \ldots, x_n)\, \tau_N = 0, \quad f \neq 0.$$

Hence a fortiori there is no relation of the form

$$g(\sigma_1, \ldots, \sigma_n)\, \tau_N = 0, \quad g \neq 0.$$

Now consider a relation

$$\sum g_S(\sigma_1, \ldots, \sigma_n)\, \tau_S = 0. \tag{1}$$

Multiplying by τ_N we see that $g_\phi \tau_N = 0$ and hence that $g_\phi = 0$. Suppose by induction that $g_T = 0$ for all proper subsets $T \subset S$. Then multiplying (1) by τ_{N-S} we obtain $\pm g_S \tau_N = 0$ and hence $g_S = 0$. This completes the proof of Lemma 6.

Remark. Here is an outline of a rather inelegant proof that the σ_i and $\beta\sigma_i$ generate the subalgebra consisting of all "symmetric" elements of A. It is convenient to assume that $t \gg n$. The number of linearly independent symmetric elements of A^{2tm+k}, where $0 \leq k \leq n$, turns out to be given by the expression

$$y = \sum_{i=0}^{d} p_k(i)\, p_{n-k}(d-i)$$

where

$$d = m - k(k+1)/2,$$

and where $p_k(i)$ denotes the number of partitions of i into at most k parts. On the other hand the number of linearly independent monomials of the form

$$\sigma_1^{r_1} \ldots \sigma_n^{r_n} (\beta\sigma_{j_1}) \ldots (\beta\sigma_{j_k})$$

in A^{2tm+k} turns out to be

$$y' = \sum_{i=0}^{d} p_n(i)\, p_{k,n-k}(d-i)$$

where $p_{k,l}(j)$ denotes the number of partitions of j into at most k parts each of which is less than or equal to l. Since $y = y'$ (compare the generating functions given in [15, p. 5]), the conclusion follows.

Proof of Theorem 6. Consider the product of n copies of B_ξ, and the n projection maps

$$\pi_i : B_\xi \times \cdots \times B_\xi \to B_\xi, \quad (1 \leqslant i \leqslant n).$$

Let η be the Whitney join of the n induced fibre spaces $\pi_1^* \xi, \ldots, \pi_n^* \xi$. It follows easily from Lemma 6 that the classes

$$q_1(\eta), \ldots, q_n(\eta), \beta q_1(\eta), \ldots, \beta q_n(\eta)$$

in $H^*(B_\xi \times \cdots B_\xi; Z_p)$ are independent. Since n can be arbitrarily large, this proves Theorem 6.

REFERENCES

[1] A. Borel and J. P. Serre, *Groupes de Lie et puissances réduites de Steenrod*, Amer. J. Math. *75* (1953), 409–448.
[2] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. *5* (1958), 35–61.
[3] R. Bott and J. Milnor, *On the parallelizability of the spheres*, Bull. Amer. Math. Soc. *64* (1958), 87–88.
[4] W. Browder, A. Liulevicius and F. P. Peterson, *Cobordism theories*, Annals of Math. *84* (1966), 91–101.
[5] H. Cartan, *Sur les groupes d'Eilenberg-MacLane* II, Proc. Nat. Acad. Sci. U.S.A. *40* (1954), 704–707.
[6] A. Dold and R. Lashof, *Principal quasi-fibrations and fibre homotopy equivalence of bundles*, Illinois J. Math. *3* (1959), 285–305.
[7] A. Dold, *Halbexakte Homotopiefunktoren*, Springer Lecture Notes *12* (1966).
[8] E. Dyer and R. Lashof, *Homology of iterated loop spaces*, Amer. J. Math. *84* (1962), 35–88.
[9] D. Epstein and N. Steenrod, *Cohomology operations*, Princeton Univ Press, 1962, Annals Study #50.
[10] S. Gitler and J. Stasheff, *The first exotic class of BF*, Topology *4* (1965), 257–266.
[11] A. Haefliger, *Differentiable embeddings of S^n in S^{n+q} for $q>2$*, Annals of Math. 83 (1966), 402–436.
[12] I. M. Hall, *The generalized Whitney sum*, Quart. J. Math. Oxford (2), 19 (1965), 360–384.
[13] I. M. James, *On the iterated suspension*, Quarterly J. Math. Oxford (2) *5* (1954), 1–10.
[14] D. W. Kahn, *Induced maps for Postnikov systems*, Trans. Amer. Math. Soc. *107* (1963), 432–450.
[15] P. A. McMahon, *Combinatory analysis*, 2, Cambridge Univ. Press 1916.
[16] J. Milnor, *On the Whitehead homomorphism* J., Bull. Amer. Math. Soc. *64* (1958), 79–82.
[17] J. Milnor and J. Moore, *On the structure of Hopf algebras*, Annals of Math. *81* (1965), 211–264.
[18] J. P. Serre, *Groupes d'homotopie et classes de groupes abéliens*, Annals of Math. *58* (1953), 258–294.
[19] E. H. Spanier, *Algebraic Topology*, McGraw Hill, 1966.
[20] M. Spivak, *Spaces satisfying Poincaré duality*, Topology *6* (1967), 77–101.
[21] J. Stasheff, *A classification theorem for fibre spaces*, Topology *2* (1963), 239–246.
[22] J. Stasheff, *More characteristic classes for spherical fibre spaces*, Comment. Math. Helv. *43* (1968), 78–86.
[23] R. Thom, *Espaces fibrés en sphères et carrés de Steenrod*, Ann. Sci. Ec. Norm. Sup., Paris *69* (1952), 109–182.
[24] H. Toda, *p-primary components of homotopy groups* III, *Stable groups of the sphere*, Memoirs Univ. of Kyoto [A] *31* (1958), 191–210.
[25] Wu Wen-Tsün, *Les i-carrés dans une variété Grassmanniénne*, C. R. Acad. Sci. Paris, *230* (1950), 918–920.
[26] Wu Wen-Tsün, *On Pontryagin classes* II, Sci. Sinica *4* (1955), 455–490.