Composition Functors and Spectral Sequences.

Composition Functors and Spectral Sequences

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1. Introduction

This paper constitutes an attempt at unification of the many spectral sequences current in algebraic topology. The basic notion is that, given a factorization

$$f = \dots j_{p+1} j_p j_{p-1} \cdots \tag{1.1}$$

of a morphism f in a given category $\mathfrak{C}$, and given a connected sequence of functors T_q, defined on the morphisms of $\mathfrak{C}$, then under certain very general assumptions it should be possible to obtain a spectral sequence relating $T(f)$ to the objects $T(j_p)$. If $\mathfrak{C}$ is the category consisting of based topological spaces and based continuous maps, and if the factorization (1.1) consists of two factors,

$$f = h g,$$

then it is fairly well-known that there is an exact sequence for homotopy, homology and cohomology functors which does relate $T(g)$ and $T(h)$ to $T(f)$. We regard such an exact sequence both as a special case of the result we aim at and as the axiomatic jumping-off point for the abstract algebraic theory which is developed and applied in this paper.

In a previous paper [10], to which this may be regarded as a sequel, we established the machinery of exact couples and spectral sequences in an abelian category $\mathfrak{A}$. In particular we studied the convergence problem in its fullest generality, and established the exact sequence (Theorem 4.16 of [10]),

$$0 \to \operatorname{coker} \alpha' \xrightarrow{\beta_*} E_\infty \xrightarrow{\gamma_*} \ker \alpha'' \to 0, \tag{1.2}$$

as a functor of the exact couple

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd} \tag{1.3}$$

where $U = D/\alpha^{-\infty}(0) = D/\bigcup_n \alpha^{-n}(0)$, $I = \alpha^\infty D = \bigcap_n \alpha^n D$, and $\alpha' : U \to U$, $\alpha'' : I \to I$ are induced by α. We obtained convergence theorems from (1.2), and used hypotheses about the grading of the couple (1.3) to derive properties of α' and α''. These convergence theorems naturally play a crucial role in the applications which appear in this paper. We also defined and discussed in [10] a special diagram in $\mathfrak{A}$ which we called a Rees system. This diagram also plays a key role in the present paper. Thus, both for the content of the earlier paper and for the notations introduced there and continued into the present paper, a certain familiarity with [10] must be assumed of the reader. This is especially true of section 5.

The plan of the paper is as follows. In section 2 we introduce the general concept of a connected sequence of functors defined on the morphisms of a category $\mathfrak{C}$. To avoid confusion with the (very closely related) concept of an exact sequence of functors defined on the objects of an abelian category[1]), we adopt the terminology *composition functor* for the concept introduced here. Thus a composition functor (T, ω) from $\mathfrak{C}$ to the abelian category $\mathfrak{A}$ consists of graded functor $T=(T_q, -\infty<q<\infty)$, $T_q: \mathfrak{C}^2 \to \mathfrak{A}$, together with a graded natural transformation $\omega=(\omega_q, -\infty<q<\infty)$, subject to the condition that, given $f=hg$ in $\mathfrak{C}$, the sequence

$$\cdots \to T_q(g) \xrightarrow{T_q\binom{1}{h}} T_q(f) \xrightarrow{T_q\binom{g}{1}} T_q(h) \xrightarrow{\omega_q(g,h)} T_{q-1}(g) \to \cdots \tag{1.4}$$

is exact. Properties of composition-functors are obtained; in particular, the case when $\mathfrak{C}$ is a *pointed* category is picked out and the familiar relation between the homotopy sequences of a pair and a triple is generalized and the logical relation between the two sequences is elucidated. In the course of this study a certain 'rolling stone' lemma from homological algebra (Lemma 2.14) is formulated; this may have some interest in its own right.

In section 3 we derive certain consequences from the associativity of composition in $\mathfrak{C}$, when a composition functor is applied. The main result, in essence, is the commutative diagram (3.3). This is really a special case of the exact couple which we obtain from an arbitrary factorization (1.1); in fact, the special case of three factors,

$$f = wvu.$$

Moreover, the various commutativity relations incorporated in (3.3) are crucial to establishing the properties of the associated Rees system in the general case (1.1). Thus we may say that we assume axiomatically the properties of the Rees system for a factorization into two factors – this is just (1.4); we then deduce the required properties for a factorization into three factors – this is (3.3); and we may then jump to the general case of an arbitrary infinite factorization.

In section 4 we digress from the programme outlined to enlarge the scope of

[1]) DOLD [5] introduced the notion of an exact sequence of functors. His notion and that of a composition functor are distinct but have overlapping domains where they agree. The relation between the two notions is studied by I. Pressman in his doctoral thesis (Cornell University, 1965). DOLD [7] has discussed *half-exact* functors. Again, there is overlap, but there is the crucial difference that Dold always requires some excision property for his functors, whereas we do not.

application of composition functors by introducing cut-off functors. Here we have principally in mind the case of the homotopy groups π_q. It is well-known that, for a map f, $\pi_q(f)$ is an abelian group if $q \geq 3$, a group if $q=2$, and a based set if $q=1$. In order, then, to apply the theory of composition functors it is necessary to modify the definition of π_q for $q<3$. Our main observation here is that there is a process which does *not* require that π_q had been defined at all for $q<3$ (let alone that exactness is in some sense preserved down to $q=1$). Thus we suppose that (T, ω) is defined down to dimension q. We then define T_{q-1} and ω_q in such a way that, if we take $T_r=0$, $r<q-1$, we obtain a composition-functor. This process is quite natural and may, of course, be applied to 'cut-off' any composition functor.

Section 5 contains our main theoretical results. In it we apply the theory of [10] to the case when a composition functor (T, ω) is applied to a factorization (1.1). The factorizations we consider are 'doubly-infinite'. We introduce this generality not because such factorizations have so far turned up in nature[1]) (though they can easily be synthesized in the laboratory), but because both left-finite, right-infinite factorizations (e.g., skeleton decompositions of infinite-dimensional polyhedra), and right-finite, left-infinite factorizations (e.g., Postnikov decompositions) do certainly arise and we wish to handle both simultaneously. We obtain (Theorem 5.5) a bigraded Rees system and pass to the limit 5.11. In the latter the graded groups associated with $T(f)$, $\varinjlim T(g_p)$, and $\varinjlim T(\bar{g}_p)$, all suitably filtered, appear and we then apply the convergence criteria, established in [10] and adapted to our situation, to relate these graded groups to the E_∞ term of the spectral sequence arising from either exact couple of the Rees system. Here $g_p = j_p j_{p-1} \cdots$, $\bar{g}_p = \cdots j_{p+1} j_p$; thus each is a familiar morphism in any of the standard factorizations arising in algebraic topology. The section closes with an explicit description of the objects ker α'', coker α' which occur in the crucial short exact sequence (1.2) and which measure, under suitable hypotheses, the deviation of E_∞ from the graded group associated with $T(f)$. For we wish to emphasize that we are not just concerned to give conditions under which E_∞ is isomorphic to this graded group; we regard it as a justification of the lengthy algebraical preliminaries that we are able immediately to identify the deviation when isomorphism does not hold.

In Section 6 we take up the contravariant case, that is, the case in which it is natural to regard each T_q as a contravariant functor into the abelian category $\mathfrak{A}$, say, rather than as a covariant functor into $\mathfrak{A}^{\mathrm{opp}}$. Of course, nothing new enters into the theory as a result of this changed point of view, but, as the authors have themselves discovered, there is a considerable pedagogical problem in effecting a smooth translation of the previous results into contravariant terms. We adopt, for the purpose of achiev-

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[1]) That is, in topology! A complete resolution can be interpreted as a doubly-infinite factorization arising in homological algebra.

ing such a smooth transition, the convention of relabelling the factorization (1.1) in such a way that the results of Section 5 continue to hold in the contravariant case without any substantial – and hence confusing – change of statement being required. In particular, the Rees systems (5.6) and (6.6) have identical appearance and practically identical properties, the one difference being that the q-degrees of the morphisms are changed into their negatives. Even this one difference could have been eliminated had we not been so conservative as to wish to retain the convention that the coboundary *raises* degree by 1!

We stress that, up to this point, we have introduced no special categories $\mathfrak{C}$, although our examples have been largely, but not entirely, drawn from topology. Thus our results certainly contain no reference to homotopy axioms or excision axioms, since these are naturally, though not exclusively, formulated in categories of topological spaces. Section 7, on the other hand, is devoted to a description of several applications of the theory to algebraic topology. The spectral sequences are set up then, using only the exactness axiom (1.4); but in computing with any given spectral sequence (for example, in identifying the differentials d_n) the homotopy and excision axioms, where they hold, are plainly of vital importance. The first example given is that of the Massey spectral sequence [19], here slightly generalized. This arises by applying a covariant composition functor to a skeleton decomposition. The second example is that of the Federer spectral sequence [14] and the treatment is similar; here, however, the composition functor is contravariant so that the conventions of section 6 are applied. These lead to some unusual degree conventions but no difficulty ensues.[1]) The differential d_0 is identified and convergence conditions are discussed. In particular, the point arises here of the *completeness* of the filtration of $T(f)$, referred to above. We emphasize that the question of completeness (see [11, 13]) is quite distinct from that of the relation of E_∞ to $\mathscr{G}T(f)$, but it does, of course, affect the extent to which the spectral sequence can provide information about $T(f)$.

In the third example the Federer spectral sequence is applied to a representable cohomology theory; the result is an Atiyah-Hirzebruch spectral sequence relating ordinary and extraordinary cohomology. Convergence questions are again taken up and the completeness of the filtration is discussed. In particular, we sketch the proof that a cohomology theory with dimension axiom on the category of CW-complexes coincides with the cellular theory, provided it satisfies a certain wedge axiom (see Brown [4]). We also discuss abstract cohomology theories (i.e., those not necessarily given by an Ω-spectrum).

In the final example the covariant truncated composition factor $\Pi_q(A, \ )$ is applied

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[1]) We point out that we are not recommending these degree conventions when a single one of the examples of Section 7 is under consideration. The conventions are designed solely to achieve a unification of the general theoretical results.

to a right-finite composition of fibrations. This case includes that of a Postnikov decomposition of a space X and that of the Adams spectral sequence [1]; the latter was discussed rather fully in [17], from the present point of view, and so has been omitted from the present paper.

The authors wish to acknowledge the important contribution of P. J. Huber to the early development of the ideas in this paper; indeed, ECKMANN and HUBER, in a more restricted and earlier approach to the problem, [12], developed much of the material of [10] in the category of abelian groups and applied it in connection with the functors $\Pi_q(A, \ )$, $\Pi_q(\ , B)$. It should also be remarked that DOLD [6, 7] has already dealt with the case when we are working in the category of finite CW-complexes and (1.1) is a skeleton decomposition. His results there go further in as much as he has also elucidated the product structure in the spectral sequence associated with a half-exact functor. It seems, however, that one should not always demand excision; and that one should look at other filtrations of complexes (e.g., the Milnor filtration, the James filtration of a reduced product complex); and that one should also have available 'cofiltrations' of complexes (e.g., the Postnikov decomposition). Thus, even if we confine attention to applications to algebraic topology, a case for generalization does exist.

A preliminary report on the contents of the present paper appeared in [16].

2. Composition functors

Let $\mathfrak{C}$ be an arbitrary category; let $\mathfrak{C}^2$ be the category of diagrams in $\mathfrak{C}$ based on the model category

$$\cdot \to \cdot\,,$$

and let $\mathfrak{C}^3$ be the category of diagrams in $\mathfrak{C}$ based on the model category

$$\cdot \to \cdot \to \cdot\,.$$

The objects of $\mathfrak{C}^2$ are thus morphisms of $\mathfrak{C}$, and the objects of $\mathfrak{C}^3$ are pairs of morphisms (f, g) such that gf is defined. There are functors

$$L, R: \mathfrak{C}^3 \to \mathfrak{C}^2$$

given by $L(f, g)=f$, $R(f, g)=g$.

A composition functor, (T, ω) from $\mathfrak{C}$ to the abelian category $\mathfrak{A}$ consists of

(i) a graded functor $T=(T_q, -\infty<q<\infty)$, T_q: $\mathfrak{C}^2 \to \mathfrak{A}$

and

(ii) a graded natural transformation $\omega=(\omega_q, -\infty<q<\infty)$, ω_q: $T_q R \to T_{q-1} L$: $\mathfrak{C}^3 \to \mathfrak{A}$ subject to the exactness condition:

for any $(f, g) \in \mathfrak{C}^3$, the sequence (in $\mathfrak{A}$)

$$\cdots \to T_q(f) \xrightarrow{T_q\binom{1}{g}} T_q(gf) \xrightarrow{T_q\binom{f}{1}} T_q(g) \xrightarrow{\omega_q(f,g)} T_{q-1}(f) \to \cdots \tag{2.1}$$

is exact.

Proposition 2.2 *If (T, ω) is a composition functor from $\mathfrak{C}$ to $\mathfrak{A}$ and if f is an equivalence in $\mathfrak{C}$, then $T_q(f)=0$ for all q.*

Proof. We apply (2.1) to the object $(f, 1)$ and obtain the exact sequence

$$\cdots \to T_q(f) \xrightarrow{T_q\binom{1}{1}} T_q(f) \xrightarrow{T_q\binom{f}{1}} T_q(1) \xrightarrow{\omega_q(f,1)} T_{q-1}(f) \to \cdots \tag{2.3}$$

Now $\binom{1}{1}$ and $\binom{f}{1}$ are equivalences in $\mathfrak{C}^2$. Thus $T_q\binom{1}{1}$ and $T_q\binom{f}{1}$ are isomorphisms, so that (2.3) is just the zero sequence.

Before proceeding further, we make explicit the naturally condition on ω. We suppose given the commutative diagrams

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{g_1} & Z_1 \\ \xi\downarrow & & \eta\downarrow & & \downarrow\zeta \\ X_2 & \xrightarrow{f_2} & Y_2 & \xrightarrow{g_2} & Z_2 \end{array} \tag{2.4}$$

in $\mathfrak{C}$; then, for each q, the diagram

$$\begin{array}{ccc} T_q(g_1) & \xrightarrow{\omega_q(f_1,g_1)} & T_{q-1}(f_1) \\ T_q\binom{\eta}{\zeta}\Big\downarrow & & \Big\downarrow T_{q-1}\binom{\xi}{\eta} \\ T_q(g_2) & \xrightarrow{\omega_q(f_2,g_2)} & T_{q-1}(f_2) \end{array} \tag{2.5}$$

commutes.

Now let us suppose that $\mathfrak{C}$ posseses a zero-object, o. Then there are unique morphisms $0_X\colon o\to X$, $0^X\colon X\to o$, and we obtain in this way embeddings

$$\underline{\imath}, \bar{\imath}\colon \mathfrak{C} \to \mathfrak{C}^2, \qquad \underline{j}, \bar{j}\colon \mathfrak{C}^2 \to \mathfrak{C}^3$$

given by

$$\underline{\imath}(X) = 0_X, \quad \bar{\imath}(X) = 0^X, \quad \underline{j}(f) = (0_X, f), \quad \bar{j}(f) = (f, 0^Y), \quad \text{for} \quad f\colon X \to Y.$$

We define graded functors $\underline{\tau}$, $\bar{\tau}$ from $\mathfrak{C}$ to $\mathfrak{A}$ by

$$\underline{\tau} = T\underline{\imath}, \qquad \bar{\tau} = T\bar{\imath}, \tag{2.6}$$

and graded natural transformation $\underline{\partial}$, $\bar{\partial}$ by

$$\bar{\partial} = \omega \underline{j}, \qquad \underline{\partial} = \omega \bar{j}. \tag{2.7}$$

Thus

$$\underline{\partial}_q(f) = \omega_q \underline{j}(f) = \omega_q(0_X, f)\colon T_q(f) \to \underline{\tau}_{q-1}(X), \qquad \text{and}$$
$$\bar{\partial}_q(f) = \omega_q \bar{j}(f) = \omega_q(f, 0^Y)\colon \bar{\tau}_q(Y) \to T_{q-1}(f).$$

Let ζ be the graded transformation given by $\zeta_q(X)=\omega_q(0_X, 0^X)$. We prove

THEOREM 2.8 (i) *ζ is a graded natural equivalence,* $\zeta_q : \bar{\tau}_q \cong \underline{\tau}_{q-1}$; (ii) *in the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \bar{\tau}_{q+1}(X) \arrow[r, "\bar{\tau}_{q+1}(f)"] \arrow[d, "\zeta_{q+1}(X)", "\cong"'] & \bar{\tau}_{q+1}(Y) \arrow[r, "\bar{\partial}_{q+1}(f)"] \arrow[d, "\zeta_{q+1}(Y)", "\cong"'] & T_q(f) \arrow[r, "\bar{\sigma}_q(f)"] \arrow[d, "="'] & \bar{\tau}_q(X) \arrow[r] \arrow[d, "\zeta_q(X)", "\cong"'] & \cdots \\
\cdots \arrow[r] & \underline{\tau}_q(X) \arrow[r, "\underline{\tau}_q(f)"] & \underline{\tau}_q(Y) \arrow[r, "\underline{\sigma}_q(f)"] & T_q(f) \arrow[r, "\underline{\partial}_q(f)"] & \underline{\tau}_{q-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

where $\underline{\sigma}_q(f) = T_q\binom{0_X}{1_Y}$, $\bar{\sigma}_q(f) = T_q\binom{1_X}{0_Y}$,

the squares commute and the rows are exact; (iii) *for any*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathfrak{C}^3$,

$$\omega_q(f, g) = T_{q-1}\binom{0_X}{1_Y} \circ \zeta_q(Y) \circ T_q\binom{1_Y}{0_Z} = \underline{\sigma}_{q-1}(f) \circ \zeta_q(Y) \circ \bar{\sigma}_q(g). \tag{2.9}$$

Proof (i) To see that ζ is natural, we apply (2.5) to the diagram

$$\begin{tikzcd}
o \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & o \arrow[d] \\
o \arrow[r] & Y \arrow[r] & o
\end{tikzcd}$$

To see that ζ is an equivalence, we construct the sequence (2.1) for $(0_X, 0^X)$ and apply Proposition 2.2.

(ii) The rows are exact, being simply (2.1) for $j(f)$, $\bar{j}(f)$ respectively. The first square commutes because ζ is natural; for the second we apply (2.5) to the diagram

$$\begin{tikzcd}
o \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "1"] & o \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r] & o\,;
\end{tikzcd}$$

and for the third we apply (2.5) to the diagram

$$\begin{tikzcd}
o \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "1"] & Y \arrow[d] \\
o \arrow[r] & X \arrow[r] & o\,.
\end{tikzcd}$$

(iii) In the light of (ii) it suffices to show that

$$\omega_q(f, g) = \underline{\sigma}_{q-1}(f) \circ \underline{\partial}_q(g). \tag{2.10}$$

This, however, follows immediately by applying (2.5) to the diagram

$$\begin{tikzcd}
o \arrow[r] \arrow[d] & Y \arrow[r, "g"] \arrow[d, "1"] & Z \arrow[d, "1"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Remarks (i) In most applications, the lower sequence of Theorem 2.8 (ii) is the first to become familiar, especially when f is an inclusion; the upper sequence was noticed (e.g. in [3]) when homology and homotopy functors were applied directly to maps without the intervention of a "mapping-cylinder" to reduce them effectively to inclusions.

(ii) Notice that (2.9) asserts that, for a pointed category, ω is entirely known when ζ is known; so, a fortiori, are $\underline{\partial}$ and $\bar{\partial}$. Further we have

$$\omega_q(f,g) = \underline{\sigma}_{q-1}(f)\circ\underline{\partial}_q(g) = \bar{\partial}_q(f)\circ\bar{\sigma}_q(g). \tag{2.11}$$

We now prove a converse of Theorem 2.8. Suppose given a graded functor $T\colon \mathfrak{C}^2\to\mathfrak{A}$ and a graded natural transformation ζ, $\zeta_q\colon\bar{\tau}_q\to\underline{\tau}_{q-1}\colon\mathfrak{C}\to\mathfrak{A}$. There are then sequences, and a map of sequences,

$$\begin{tikzcd}
\cdots \arrow[r] & \bar{\tau}_{q+1}(X) \arrow[r, "\bar{\tau}_{q+1}(f)"] \arrow[d, "\zeta_{q+1}(X)"] & \bar{\tau}_{q+1}(Y) \arrow[r, "\bar{\partial}_{q+1}(f)"] \arrow[d, "\zeta_{q+1}(Y)"] & T_q(f) \arrow[r, "\bar{\sigma}_q(f)"] \arrow[d, "="] & \bar{\tau}_q(X) \arrow[r] \arrow[d, "\zeta_q(X)"] & \cdots & (2.12a) \\
\cdots \arrow[r] & \underline{\tau}_q(X) \arrow[r, "\underline{\tau}_q(f)"] & \underline{\tau}_q(Y) \arrow[r, "\underline{\sigma}_q(f)"] & T_q(f) \arrow[r, "\underline{\partial}_q(f)"] & \underline{\tau}_{q-1}(X) \arrow[r] & \cdots & (2.12b)
\end{tikzcd}$$

for any $f\colon X\to Y$ in $\mathfrak{C}$, where $\bar{\sigma}$, $\underline{\sigma}$ are defined as in Theorem 2.8 and $\bar{\partial}$, $\underline{\partial}$ are defined to make the appropriate squares commute. Then we have

THEOREM 2.13 *If* (2.12a) *or* (2.12b) *is exact (for all f), then*
(i) *ζ is an equivalence*
(ii) (2.12a) *and* (2.12b) *are both exact (for all f)*;
(iii) *if ω is given by* (2.11), *then (T, ω) is a composition functor and*

$$\omega_q(0_X, 0^X) = \zeta_q(X)$$

Proof. (i) Let us suppose for definiteness that (2.12a) is exact. Setting $Y=o$, and noting that then $\bar{\sigma}_q(f)=1$, we deduce that $\bar{\tau}_q(o)=0$ for all q. But if we now set $X=o$ and look at the centre square we have

$$\begin{tikzcd}
\bar{\tau}_{q+1}(Y) \arrow[r, "\bar{\partial}_{q+1}(0_Y)", "\cong"'] \arrow[d, "\zeta_{q+1}(Y)"] & \underline{\tau}_q(Y) \arrow[d, "="] \\
\underline{\tau}_q(Y) \arrow[r, "1"'] & \underline{\tau}_q(Y).
\end{tikzcd}$$

Thus $\zeta_{q+1}(Y)$ is an isomorphism and so ζ is an equivalence. Similarly one deduces the conclusion from the assumption that (2.12b) is exact.

(ii) follows immediately from (i).

(iii) We first show that ω_q is natural, that is, that (2.5) holds. We note that, in the notation of (2.4),

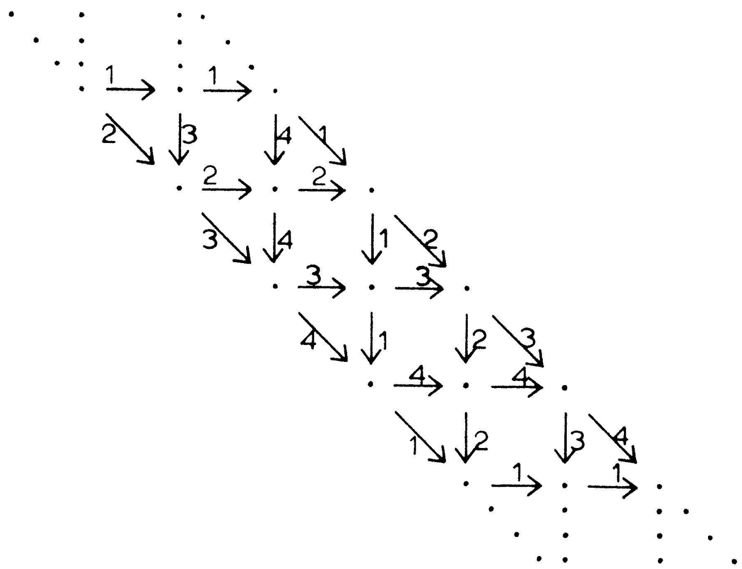
$$\bar{\sigma}_{q-1}(f_2)\circ\underline{\tau}_{q-1}(\eta) = {-}_{q-1}\binom{\xi}{\eta}\circ\underline{\sigma}_{q-1}(f_1), \quad \text{and}$$
$$\bar{\sigma}_q(g_2)\circ T_q\binom{\eta}{\zeta} = \bar{\tau}_q(\eta)\circ\bar{\sigma}_q(g_1),$$

just because T_{q-1}, T_q are functors. Thus

$$\begin{aligned}
\omega_q(f_2,g_2)\circ T_q\tbinom{\eta}{\zeta} &= \underline{\sigma}_{q-1}(f_2)\circ\zeta_q(Y_2)\circ\bar{\sigma}_q(g_2)\circ T_q\tbinom{\eta}{\zeta}\\
&= \underline{\sigma}_{q-1}(f_2)\circ\zeta_q(Y_2)\circ\bar{\tau}_q(\eta)\circ\bar{\sigma}_q(g_1)\\
&= \underline{\sigma}_{q-1}(f_2)\circ\underline{\tau}_{q-1}(\eta)\circ\zeta_q(Y_1)\circ\bar{\sigma}_q(g_1), \quad \text{since } \zeta \text{ is natural}\\
&= T_{q-1}\tbinom{\xi}{\eta}\circ\underline{\sigma}_{q-1}(f_1)\circ\zeta_q(Y_1)\circ\bar{\sigma}_q(g_1)\\
&= T_{q-1}\tbinom{\xi}{\eta}\circ\omega_q(f_1,g_1).
\end{aligned}$$

The relation $\omega_q(0_X, 0^X)=\zeta_q(X)$ is immediate from the definition of ω_q, so it remains only to prove the exactness of (2.1). We invoke the following general lemma.

LEMMA 2.14 *Let $\mathfrak{A}$ be an abelian category and let*



be a commutative diagram in $\mathfrak{A}$. We consider the four sequences of morphisms marked 1, 2, 3, 4 *and assume*

(i) *three of the four sequences are exact;*

(ii) *the fourth sequence is differential where it appears vertically or horizontally. Then the fourth sequence is exact.*

Proof. We may suppose that $\mathfrak{A}$ is the category of abelian groups. The argument now proceeds by standard diagram-chasing and is left to the reader. (Note the symmetry in the roles of the four sequences in the diagram; thus we may suppose that the sequences 1, 2 and 3 are exact in the proof, without loss of generality).

We apply Lemma 2.14 to the diagram

$$\begin{tikzcd}[column sep=large]
\tau_{q+1}(X) \arrow[r, "\tau_{q+1}(f)"] \arrow[dr, "\tau_{q+1}(gf)"'] \arrow[r, phantom, shift right=4ex, "\mathrm{a}"] & \tau_{q+1}(Y) \arrow[r, "\sigma_{q+1}(f)"] \arrow[d, "\tau_{q+1}(g)"] \arrow[dr, phantom, "\mathrm{b}"] & T_{q+1}(f) \arrow[d, "T_{q+1}\binom{1}{g}"'] \arrow[dr, "\partial_{q+1}(f)"] \arrow[dr, phantom, shift right=3ex, "\mathrm{c}"] & & \\
& \tau_{q+1}(Z) \arrow[r, "\sigma_{q+1}(gf)"] \arrow[dr, "\sigma_{q+1}(g)"'] \arrow[dr, phantom, shift left=3ex, "\mathrm{d}"] & T_{q+1}(gf) \arrow[r, "\partial_{q+1}(gf)"] \arrow[d, "T_{q+1}\binom{f}{1}"'] \arrow[dr, phantom, "\mathrm{e}"] & \tau_q(X) \arrow[d, "\tau_q(f)"'] \arrow[dr, "\tau_q(gf)"] \arrow[dr, phantom, shift right=3ex, "\mathrm{a}"] & \\
& & T_{q+1}(g) \arrow[r, "\partial_{q+1}(g)"] \arrow[dr, "\omega_{q+1}(f,g)"'] \arrow[dr, phantom, shift left=3ex, "\mathrm{f}"] & \tau_q(Y) \arrow[r, "\tau_q(g)"] \arrow[d, "\sigma_q(f)"] \arrow[dr, phantom, "\mathrm{b}"] & \tau_q(Z) \arrow[d, "\sigma_q(gf)"] \\
& & & T_q(f) \arrow[r, "T_q\binom{1}{g}"] & T_q(gf)
\end{tikzcd}$$

where we have omitted the underlines from the sequences (2.12b). The commutativity of a, b, and d follows from the functorial property of T. The commutativity of c follows by applying (2.5) to the diagram

$$\begin{tikzcd}
o \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "1"] & Y \arrow[d, "g"] \\
o \arrow[r] & X \arrow[r, "gf"] & Z
\end{tikzcd} \quad ;$$

the commutativity of e follows by applying (2.5) to the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[r, "gf"] \arrow[d, "f"] & Z \arrow[d, "1"] \\
o \arrow[r] & Y \arrow[r, "g"] & Z
\end{tikzcd} \quad ;$$

and the commutativity of f is just (2.10) – which followed from the naturality of ω. Thus, to apply Lemma 2.14 to prove the exactness of (2.1), it remains to show that (2.1) is differential at $T_q(gf)$.

Now $\binom{f}{1}\binom{1}{g}=\binom{f}{g}=\binom{1_Y}{g}\binom{f}{1_Y}$. But $T_q\binom{f}{1_Y}\colon T_q(f)\to T_q(1_Y)$, and the exactness of (2.12) immediately implies that $T_q(1_Y)=0$. Thus $T_q\binom{f}{1}T_q\binom{1}{g}=0$, and the lemma may be applied to complete the proof of the theorem.

We have therefore established that, in a pointed category $\mathfrak{C}$, a composition functor (T, ω) may be specified by (T, ζ), or, indeed, by $(T, \underline{\partial})$ or $(T, \bar{\partial})$. We will permit ourselves in the sequel to use whichever specification is convenient.

Examples. (i) $\mathfrak{C}=\mathfrak{T}$, the category of topological spaces, and $T_q=H_q$, the q^{th} singular homology group (with integer coefficients, say). Here $\mathfrak{A}$ is the category of abelian groups. We apply H_q directly to the singular chain complex of f itself. The existence of ω and the exactness of (2.1) then follow from standard homological algebra. We may also consider cohomology (essentially, by considering the category dual to the category of abelian groups). We obtain a pointed category by replacing $\mathfrak{T}$ by the category $\mathfrak{T}_0$ of based spaces and based maps and may then apply Theorems 2.8, 2.13.

(ii) $\mathfrak{C}$=category of Λ-modules, $T_q=\mathrm{Ext}_\Lambda^{-q}\,(A,\ )$, $q<0$, $T_q=\underline{\pi}_q(A,\ )$, $q\geq 0$. Here $\underline{\pi}_q$ stands for the projective homotopy group of [8, 15]. The category $\mathfrak{A}$ is the category of abelian groups. We note that our formulation requires that the functors $\mathrm{Ext}_\Lambda^q\,(A,\ )$ and $\underline{\pi}_q(A,\ )$ be applied to *morphisms* of $\mathfrak{C}$. This may be achieved, by defining Ext_Λ^q (A, φ) to be $\mathrm{Ext}_\Lambda^q\,(A, C_\varphi)$, where C_φ is the injective mapping cone of φ, $q>0$, and defining $\underline{\pi}_q(A, \varphi)$ to be $\underline{\pi}_{q-1}(A, K_\varphi)$, where K_φ is defined in a manner dual to C_φ, $q>0$. A special definition of $\underline{\pi}_0(A, \varphi)$, and a natural definition of ω, may then be given to make (T_q, ω) a composition functor. For details see [18].

3. Associativity of composition

Of course, the associativity of composition in $\mathfrak{C}$ is essential to the definition of composition in $\mathfrak{C}^2$ and $\mathfrak{C}^3$ (since, otherwise, the juxtaposition of commutative squares would not lead to a commutative square). In this section we draw some further elementary consequences from associativity under the application of a composition functor.

We consider the diagram

$$\cdot \xrightarrow{u} \cdot \xrightarrow{v} \cdot \xrightarrow{w} \cdot \tag{3.1}$$

in $\mathfrak{C}$, so that $wvu=w(vu)=(wv)u$. We expand the diagram (3.1) to the commutative diagram

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "1"] & \cdot \arrow[r, "v"] \arrow[d, "1"] & \cdot \arrow[d, "w"] \\
\cdot \arrow[r, "u"] \arrow[d, "1"] & \cdot \arrow[r, "wv"] \arrow[d, "v"] & \cdot \arrow[d, "1"] \\
\cdot \arrow[r, "vu"] \arrow[d, "u"] & \cdot \arrow[r, "w"] \arrow[d, "1"] & \cdot \arrow[d, "1"] \\
\cdot \arrow[r, "v"] & \cdot \arrow[r, "w"] & \cdot
\end{tikzcd} \tag{3.2}$$

Now let (T, ω) be a composition functor from $\mathfrak{C}$ to $\mathfrak{A}$. We apply (T, ω) to the diagram (3.2) and obtain the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & T_q(u) \arrow[r, "T_q\binom{1}{v}"] \arrow[d, "1"] & T_q(vu) \arrow[r, "T_q\binom{u}{1}"] \arrow[d, "T_q\binom{1}{w}"] & T_q(v) \arrow[r, "\omega_q(u,v)"] \arrow[d, "T_q\binom{1}{w}"] & T_{q-1}(u) \arrow[r] \arrow[d, "1"] & \cdots \\
\cdots \arrow[r] & T_q(u) \arrow[r, "T_q\binom{1}{wv}"] \arrow[d, "T_q\binom{1}{v}"] & T_q(wvu) \arrow[r, "T_q\binom{u}{1}"] \arrow[d, "1"] & T_q(wv) \arrow[r, "\omega_q(u,wv)"] \arrow[d, "T_q\binom{v}{1}"] & T_{q-1}(u) \arrow[r] \arrow[d, "T_{q-1}\binom{1}{v}"] & \cdots \\
\cdots \arrow[r] & T_q(vu) \arrow[r, "T_q\binom{1}{w}"] \arrow[d, "T_q\binom{u}{1}"] & T_q(wvu) \arrow[r, "T_q\binom{vu}{1}"] \arrow[d, "T_q\binom{u}{1}"] & T_q(w) \arrow[r, "\omega_q(vu,w)"] \arrow[d, "1"] & T_{q-1}(vu) \arrow[r] \arrow[d, "T_{q-1}\binom{u}{1}"] & \cdots \\
\cdots \arrow[r] & T_q(v) \arrow[r, "T_q\binom{1}{w}"] & T_q(wv) \arrow[r, "T_q\binom{v}{1}"] & T_q(w) \arrow[r, "\omega_q(v,w)"] & T_{q-1}(v) \arrow[r] & \cdots
\end{tikzcd} \tag{3.3}$$

Here the rows are, of course, exact: the commutativity of the squares on the right will be invoked in section 5, when we discuss the Rees system obtained by applying a composition functor to a morphism in $\mathfrak{C}$. Indeed, (2.1) and (3.3) may be regarded as the special cases of the Rees system, when the factorization involves two and three factors respectively.

We may apply the Barratt-Whitehead procedure [3] to (3.3) to obtain three "Mayer-Vietoris" sequences. However, it turns out that the first and third essentially coincide; thus we obtain the following result.

THEOREM 3.4 *Suppose given the diagram* (3.1) *in* $\mathfrak{C}$ *and a composition functor* (T, ω) *from* $\mathfrak{C}$ *to* $\mathfrak{A}$. *Then there are exact sequences*

$$\cdots \to T_{q+1}(wv) \xrightarrow{T_q\binom{1}{v}\omega_{q+1}(u,wv)} T_q(vu) \xrightarrow{\{T_q\binom{u}{1}, T_q\binom{1}{w}\}} T_q(v) \oplus T_q(wvu) \xrightarrow{\langle T_q\binom{1}{w}, -T_q\binom{u}{1}\rangle} T_q(wv) \to \cdots \tag{3.5}$$

and

$$\cdots \to T_q(vu) \xrightarrow{T_q\binom{u}{w}} T_q(wv) \xrightarrow{\{\omega_q(u,wv), T_q\binom{v}{1}\}} T_{q-1}(u) \oplus T_q(w) \xrightarrow{\langle T_{q-1}\binom{1}{v}, -\omega_q(vu,w)\rangle} T_{q-1}(vu) \to \cdots \tag{3.6}$$

(Notice that the sequence obtained from the last two rows of (3.3) coincides with (3.5) in the light of the commutativity of the right-hand centre square of (3.3).)

Examples (i) Consider a nested sequence of complexes $N \subset M \subset L \subset K$. Then applying the homology functor we obtain an exact sequence

$$\cdots \to H_{q+1}(K,M) \to H_q(L,N) \to H_q(L,M) \oplus H_q(K,N) \to H_q(K,M) \to \cdots$$

and an exact sequence

$$\cdots \to H_q(L,N) \to H_q(K,M) \to H_{q-1}(M,N) \oplus H_q(K,L) \to H_{q-1}(L,N) \to \cdots$$

(ii) Let K^n be the n-skeleton of the complex K. Then applying the homotopy functor we obtain an exact sequence

$$\cdots \to \pi_{q+1}(K^n, K^{n-2}) \to \pi_q(K^{n-1}, K^{n-3}) \to \pi_q(K^{n-1}, K^{n-2}) \oplus \pi_q(K^n, K^{n-3}) \to \pi_q(K^n, K^{n-2}) \to \cdots$$

and an exact sequence

$$\cdots \to \pi_q(K^{n-1}, K^{n-3}) \to \pi_q(K^n, K^{n-2}) \to \pi_{q-1}(K^{n-2}, K^{n-3}) \oplus \pi_q(K^n, K^{n-1}) \to \pi_{q-1}(K^{n-1}, K^{n-3}) \to \cdots$$

Remarks (i) (3.5) and (3.6), which were deduced from (2.1) are, in fact, each equivalent to (2.1); for we recover (2.1) from (3.5) or (3.6) by setting $v=1$ or $u=1$ (or $w=1$) respectively.

(ii) It might be interesting to study the interrelations of (3.5) and (3.6).

4. Cut-off functors

In the second example in section 3 we implicitly invoked the homotopy groups as a composition functor. Plainly this is not valid without some modification as π_2 is not a functor to abelian groups but to groups[1]), π_1 is a functor to sets, and π_q is not defined for $q<1$. Now MASSEY described in [19] a procedure whereby one may adapt the functor π so that its values are always abelian groups. We will simply describe this procedure in the setting in which we are developing the theory.

Let $\mathfrak{C}$ be a category and $\mathfrak{A}$ an abelian category. We will suppose throughout this section that $\mathfrak{C}$ is *pointed.* Then a *truncated* composition functor from $\mathfrak{C}$ to $\mathfrak{A}$ is a triple (T, ω, t) where $t \in Z$, $T=(T_q, q \geq t)$ is a graded functor from $\mathfrak{C}^2$ to $\mathfrak{A}$, and $\omega=(\omega_q, q \geq t+1)$, $\omega_q: T_q R \to T_{q-1} L: \mathfrak{C}^3 \to \mathfrak{A}$, is a graded natural transformation. Moreover, the triple is subject to the exactness condition: the sequence

$$\cdots \to T_q(f) \to T_q(gf) \to T_q(g) \to T_{q-1}(f) \to \cdots \to T_t(f) \to T_t(gf) \to T_t(g) \tag{4.1}$$

is *exact* for all $(f, g) \in \mathfrak{C}^3$. We may for further precision say that (T, ω, t) is *truncated below at t*; one may deal similarly with composition functors truncated *above,* but there is no necessity to deal with them separately since they may be brought within the compass of the case considered by considering composition functors from $\mathfrak{C}^{\text{opp}}$ to $\mathfrak{A}^{\text{opp}}$ and applying the sign-reversing trick. If a composition functor is truncated above and below then we may apply the technique described below first to the lower and then to the upper end of its domain of definition.

Our object, then, is to take a composition functor truncated below at t, and to construct from it a composition functor, over the whole range $-\infty < q < \infty$, which

[1]) Recall that, here, $\pi_q: \mathfrak{C}^2 \to \mathfrak{A}b$, $q > 2$; that is, we are considering the homotopy groups of *maps.*

coincides with the given functor where the latter is defined. We do this in a canonical way and, as is to be expected, it turns out that it is only necessary to find suitable definitions of T_{t-1}, ω_t.

Let, then, (T, ω, t) be a truncated composition functor from $\mathfrak{C}$ to $\mathfrak{A}$. Then $\bar{\tau}_q$, $\underline{\tau}_q$ are defined for $q \geq t$, and $\zeta_q : \bar{\tau}_q \to \underline{\tau}_{q-1}$, given by

$$\zeta_q(X) = \omega_q(0_X, 0^X), \tag{4.2}$$

is defined for $q \geq t+1$, and it follows, just as in section 2, that ζ_q is a natural equivalence. For the analogue of Proposition 2.2 plainly holds if we replace (T, ω) by (T, ω, t) and only ask for the conclusion for $q \geq t$.

We now proceed to extend T so that it is defined for each q. Specifically we set, for $f \in \mathfrak{C}^2$,

$$\begin{aligned} T_{t-1}(f) &= \operatorname{coker} \bar{\tau}_t(f), \\ T_q(f) &= 0, \qquad q < t-1 \end{aligned} \tag{4.3}$$

Since $\bar{\tau}_t$ is a functor it is plain that T_{t-1} is indeed a functor $\mathfrak{C}^2 \to \mathfrak{A}$. Moreover since (as was observed above) $\bar{\tau}_t(o) = T_t(1_0) = 0$, it follows that

$$\begin{aligned} \underline{\tau}_{t-1}(X) &= T_{t-1}(0_X) = \operatorname{coker} \bar{\tau}_t(0_X) = \operatorname{coker}(0 \to \bar{\tau}_t(X)) = \bar{\tau}_t(X), \\ \bar{\tau}_t(X) &= \underline{\tau}_{t-1}(X); \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \bar{\tau}_{t-1}(X) &= T_{t-1}(0^X) = \operatorname{coker} \bar{\tau}_t(0^X) = \operatorname{coker}(\bar{\tau}_t(X) \to 0) = 0, \\ \bar{\tau}_{t-1}(X) &= 0. \end{aligned} \tag{4.5}$$

We further extend $\zeta_q : \bar{\tau}_q - \underline{\tau}_{q-1}$ from (4.2), to the whole range by setting

$$\begin{cases} \zeta_t = 1 : \bar{\tau}_t \to \underline{\tau}_{t-1} \\ \zeta_q = 1 (= 0), \qquad q < t. \end{cases} \tag{4.6}$$

Definition (4.6) is valid in the light of (4.4), (4.5).

THEOREM 4.7. *The pair (T, ζ) satisfies the hypothesis of Theorem 2.13. Moreover, the associated composition functor extends the original truncated composition functor over the whole range.*

Proof. We must verify the exactness of one of the sequences

$$\begin{tikzcd} \cdots \arrow[r] & \bar{\tau}_t(X) \arrow[r, "\bar{\tau}_t(f)"] \arrow[d, "="', "\zeta_t(X)"] & \bar{\tau}_t(Y) \arrow[r, "\bar{\partial}_t(f)"] \arrow[d, "="', "\zeta_t(Y)"] & T_{t-1}(f) \arrow[r, "\bar{\sigma}_{t-1}(f)"] \arrow[d, "="'] & \bar{\tau}_{t-1}(X) = 0 \\ \cdots \arrow[r] & \underline{\tau}_{t-1}(X) \arrow[r, "\underline{\tau}_{t-1}(f)"] & \underline{\tau}_{t-1}(Y) \arrow[r, "\underline{\sigma}_{t-1}(f)"] & T_{t-1}(f) \arrow[r, "\bar{\partial}_{t-1}(f)"] & \underline{\tau}_{t-2}(X) = 0. \end{tikzcd}$$

Let $\kappa(f) : \bar{\tau}_t(Y) \twoheadrightarrow \operatorname{coker} \bar{\tau}_t(f) = T_{t-1}(f)$ be the natural projection. It then suffices to show that $\kappa(f) = \underline{\sigma}_{t-1}(f) = T_{t-1}\binom{0_X}{1_Y}$. We have, however, the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & \bar{\tau}_t(X) \arrow[d, "\bar{\tau}_t(f)"] \\
\bar{\tau}_t(Y) \arrow[r] \arrow[d, two heads, tail, "="'] & \bar{\tau}_t(Y) \arrow[d, two heads, "\kappa(f)"] \\
\underline{\tau}_{t-1}(Y) \arrow[r, "\underline{\sigma}_{t-1}(f)"] & T_{t-1}(f)
\end{tikzcd}$$

and this establishes the result. The second assertion is quite evident in view of (4.2). Note that

$$\omega_t(f,g) = \kappa(f) \circ \bar{\sigma}_t(g) \colon T_t(g) \to T_{t-1}(f).$$

Remark In [19], MASSEY modifies the homotopy group functor by replacing $\pi_2(f)$ by the subgroup Im $\pi_2(Y) \subset \pi_2(f)$. This procedure is essentially equivalent to ours, but in our formulation it is superfluous to suppose that $\pi_2(f)$ is already defined in a larger category than $\mathfrak{Ab}$, namely in the category $\mathfrak{G}$ of groups. We now proceed to describe this example in detail.

Example 4.8 Let $\mathfrak{T}_0$ be the category of based spaces and based maps. We take $A \in \mathfrak{T}_0$ and we then have a truncated composition functor $(T, \omega, 3)$, where $T_q = \Pi_q(A, \,)$, and ω_q is the natural boundary homomorphism in the homotopy sequence of a triple. Then $\bar{\tau}_3(X) = \Pi_3(A, 0^X) = \Pi_2(A, X)$ and $\bar{\tau}_3(f) = f_* \colon \Pi_2(A, X) \to \Pi_2(A, Y)$. Thus the procedure we have described leads to the definition $T_2(f) = \operatorname{coker} f_*$. In fact, of course, $\operatorname{coker} f_* \cong \mathit{Im}\, \Pi_2(A, Y) \subset \Pi_2(A, f)$; but it was not necessary to our purpose that $\Pi_2(A, f)$ had already been defined as a group. Indeed it was not even necessary that $\Pi_2(A, X)$ had already been defined as an abelian group. For, given the functor $\Pi_3(A, \,)$, we could have defined T_2 as in the general theory and we would then have had $\underline{\tau}_2(X) = \bar{\tau}_3(X)\ (= \Pi_2(A, X))$.

This example serves to underline a general question. We plainly lose information in passing from $\Pi_2(A, \,)$ to T_2. We should therefore investigate the question to what extent we need the structure of an abelian category for the range category of T. This question has been considered (from a slightly different viewpoint) by Dold; and we ourselves hope to return to it in a subsequent paper. Meanwhile we think it worth mentioning that we cannot apply the procedure described in this section when, for example, we replace $\mathfrak{A}$ by the category of groups. For

$$\pi_1(X) \xrightarrow{f_*} \pi_1(Y) \to \operatorname{coker} f_* \to 0$$

is not always exact in $\mathfrak{G}$.

We should point out how the general argument we have given would proceed if we began with a truncated system (T, ζ, t). That is, we might have assumed the functors T_q defined for $q \geq t$, the natural transformation $\zeta \colon \bar{\tau}_q \to \underline{\tau}_{q-1}$ for $q \geq t+1$ and then that (2.12a) or (2.12b) is exact so far as it is defined. However there is then a difference of an essential nature between these two sequences since (2.12b) terminates at $\to T_t(f)$, whereas (2.12a) goes on for two more terms. If we assume (2.12a) exact then we infer (setting $X = Y = o$) that $\bar{\tau}_q(o) = 0$, $q \geq t$, and hence, as in the proof of Theorem 2.13(i), that ζ_q is an equivalence, $q \geq t+1$. Thus we may proceed to define T_{t-1}, ζ_t as in (4.3), (4.6) and so extend (T, ζ, t) to a full system (T, ζ). If on the other hand we assumed (2.12b) exact, we would then have no reason to assume the exactness of $\underline{\tau}_t(Y) \to T_t(f) \to \bar{\tau}_t(X)$ or, equivalently, to suppose ζ_{t+1} an equivalence. Thus, the hypothesis of the exactness of (2.12b) would need to be supplemented by a condition equivalent to that of supposing ζ_{t+1} to be a natural *equivalence* in order that the truncated system could be completed.

5. Factorizations of morphisms

Let $\mathfrak{C}$ be a category (not necessarily pointed) and let f be a morphism in $\mathfrak{C}$. A *finite factorization* of f is then an expression for f as

$$f = j_{P_0} \cdots j_{p+1} j_p j_{p-1} \cdots j_{P_1}, \tag{5.1}$$

where each j_p, $P_1 \leq p \leq P_0$, is a morphism in $\mathfrak{C}$. The *length* of the factorization is $P_0 - P_1 + 1$.

We wish to consider *infinite factorizations* of f. By an infinite factorization of f we understand a triple of sequences of morphisms in $\mathfrak{C}$, $(j_p, g_p, \bar{g}_p, -\infty < p < \infty)$, satisfying the relations

$$f = \bar{g}_p g_{p-1}, \quad j_p g_{p-1} = g_p, \quad \bar{g}_{p+1} j_p = \bar{g}_p, \quad -\infty < p < \infty. \tag{5.2}$$

Formally we may write

$$\begin{aligned} f &= \cdots j_{p+1} j_p j_{p-1} \cdots, \\ g_p &= j_p j_{p+1} \cdots, \\ \bar{g}_p &= \cdots j_{p+1} j_p, \end{aligned}$$

and regard f as factorized as an (infinite) product of the morphisms j_p. The case (5.1) of the finite factorization of f is then subsumed under the definition of an infinite factorization by taking

$$g_p = 1, \quad p < P_1, \tag{5.3}$$

$$\bar{g}_p = 1, \quad p > P_0. \tag{5.4}$$

Notice that (5.3) implies $j_p = 1, p < P_1$, and (5.4) implies $j_p = 1, p > P_0$. If (5.3) holds we say that the factorization of f is *left-finite*, and if (5.4) holds we say that the factorization of f is *right-finite*.

Examples (i) The skeleton decomposition of a simplicial complex is a left-finite factorization by inclusions. It is finite if the complex is finite-dimensional.

(ii) The Postnikov decomposition of a 1-connected space is a right-finite factorization of fibrations.

(iii) An injective resolution of a module gives rise to a left-finite factorization; a projective resolution of a module to a right-finite factorization.

Now let (T, ω) be a composition-functor from $\mathfrak{C}$ to $\mathfrak{A}$ and let (5.2) be an infinite factorization of f. We then utilize the exactness axiom (2.1) to prove

THEOREM 5.5 *There is a diagram in* $\mathfrak{A}^{Z\times Z}$

$$\begin{tikzcd}
 & D \arrow[rr, "\alpha"] \arrow[dl, "\varphi"'] & & D \arrow[dl, "\beta"'] \arrow[dr, "\varphi"] & \\
F \arrow[dr, "\bar\varphi"'] & & E \arrow[ul, "\gamma"'] \arrow[dl, "\bar\gamma"] & & F \arrow[dl, "\bar\varphi"] \\
 & \bar D \arrow[uu, "\xi"] \arrow[rr, "\bar\alpha"'] & & \bar D \arrow[uu, "\xi"'] \arrow[ul, "\bar\beta"] &
\end{tikzcd} \tag{5.6}$$

in which

(*i*) $D=(D^{p,q})=(T_q(g_p))$; $\bar D=(\bar D^{p,q})=(T_q(\bar g_p))$; $E=(E^{p,q})=(T_q(j_p))$; $F=(F^{p,q})=(T_q(f))$;

(*ii*) $\alpha^{p,q}=T_q\binom{1}{j_{p+1}}$; $\beta^{p,q}=T_q\binom{g_{p-1}}{1}$; $\gamma^{p,q}=\omega_q(g_{p-1},j_p)$; $\bar\alpha^{p,q}=T_q\binom{j_p}{1}$; $\bar\beta^{p,q}=\omega_q(j_{p-1},\bar g_p)$; $\bar\gamma^{p,q}=T_q\binom{1}{\bar g_{p+1}}$; $\xi^{p,q}=\omega_q(g_{p-1},\bar g_p)$; $\varphi^{p,q}=T_q\binom{1}{\bar g_{p+1}}$; $\bar\varphi^{p,q}=T_q\binom{g_{p-1}}{1}$;

(iii) $\deg\alpha=\deg\bar\alpha=(1,0)$; $\deg\beta=\deg\bar\gamma=(0,0)$; $\deg\gamma=\deg\bar\beta=\deg\xi=(-1,-1)$; $\deg_q\varphi=\deg_q\bar\varphi=0$; $\deg_p\bar\varphi\varphi=1$;

(iv) (α,β,γ) *and* $(\bar\alpha,\bar\beta,\bar\gamma)$ *are exact couples*;

(v) $(\xi,\varphi,\bar\varphi)$ *is an exact triangle*;

(vi) *the commutativity relations* $\alpha\xi=\xi\bar\alpha$, $\beta\xi=\bar\beta$, $\gamma=\xi\bar\gamma$ hold;

(vii) $\varphi^{p+1}\alpha^p=\varphi^p$, $\bar\alpha^p\bar\varphi^p=\bar\varphi^{p+1}$ *for all* p, q;

(viii) $\bar\varphi^p\varphi^p=\bar\gamma^p\beta^p$ *for all* p, q;

(ix) *the spectral sequences associated with* (α,β,γ), $(\bar\alpha,\bar\beta,\bar\gamma)$ *coincide and*

$$d^{p,q}=\omega_q(j_{p-1},j_p)\colon E^{p,q}\to E^{p-1,q-1}. \tag{5.7}$$

Proof Statements (i)–(v) are immediate consequences of the exactness axiom (2.1). Statement (vi) ist just the commutativity of the three right-hand squares in (3.3), where $u=g_{p-1}$, $v=j_p$, $w=\bar g_{p+1}$.

Statements (vii) and (viii) follow from the functorial nature of T_q. Now statements (iv)–(vi) assert that (5.6) is a Rees system [10]; thus Theorem 7.10 of [10] may be applied to show that the spectral sequences associated with (α,β,γ), $(\bar\alpha,\bar\beta,\bar\gamma)$ coincide. Finally (5.7) follows from the commutativity of the right-hand bottom square in (3.3), where

$$u=g_{p-2},\ v=j_{p-1},\ w=j_p.$$

(Recall that $d^{p,q}=\beta^{p-1,q-1}\gamma^{p,q}$)

Let $\Phi \in \mathfrak{A}^{\mathbf{Z}}$ be given by $\Phi^q = T_q(f)$. Then

$$F = \Delta\Phi$$

where $\Delta : \mathfrak{L} \to \mathfrak{L}^{\mathbf{Z}}$ is the functor, defined for any category $\mathfrak{L}$, given by

$$(\Delta B)^p = B, \quad \text{(for all } p \in \mathbf{Z}).$$

Let $\theta : \mathfrak{L}^{\mathbf{Z}} \to \mathfrak{L}^{\mathbf{Z}}$ be the functor given by

$$(\theta M)^p = M^{p-1}, \; M \in \mathfrak{L}^{\mathbf{Z}}.$$

Then

$$\theta\Delta = \Delta. \tag{5.8}$$

Let $\vartheta : 1 \to \theta$ be the natural transformation given by: $\vartheta_M : M \to \theta M$ is the morphism of degree 1 which is the identity on each component. Then, in particular,

$$\vartheta_{\Delta B} : \Delta B \to \Delta B$$

is an automorphism; we write it simply as ϑ.

With this convention, we note that F in (5.6) admits the automorphism ϑ and that, in terms of this automorphism, statements (vii) and (viii) of Theorem 5.5 may be reexpressed as

$$\varphi\alpha = \vartheta\varphi, \; \bar{\alpha}\bar{\varphi} = \bar{\varphi}\vartheta, \; \bar{\varphi}\vartheta^{-1}\varphi = \bar{\gamma}\beta, \tag{5.9}$$

where we regard $\varphi, \bar{\varphi}$ as having some fixed, but arbitrary, p-degree, subject only to the condition that the sum of their p-degrees is 1 (see Theorem 5.5(iii)). The degree of ϑ is $(1, 0)$.

Comparing with (7.9) of [10], we conclude

THEOREM 5.10 *The diagram* (5.6) *is a special Rees system, where* $\vartheta : F \cong F$ *is given by*

$$\vartheta^{p,q} = 1 : F^{p,q} \to F^{p+1,q}.$$

The general theory of special Rees systems thus applies to (5.6). We may, in particular, pass to the limit and obtain the diagram (see Theorem 7.32 and subsequent remarks in [10])

$$\begin{tikzcd}
 & & \mathcal{G}D^{\infty} \arrow[d, two heads, "\mathcal{G}^{+}\vartheta"] & \\
\operatorname{coker}\bar{\alpha}' \arrow[r, tail, "\xi'"] \arrow[d, "="'] & \operatorname{coker}\alpha' \arrow[r, two heads, "\varphi'"] \arrow[d, tail, "\beta_*"'] \arrow[ur, "\alpha_0'"] & \mathcal{G}F \arrow[d, tail, "\bar{\varphi}''"'] \arrow[dr, two heads, "\mathcal{G}^{-}\vartheta"] & \\
\operatorname{coker}\bar{\alpha}' \arrow[r, tail, "\bar{\beta}_*"] & E_{\infty} \arrow[r, two heads, "\bar{\gamma}_*"] \arrow[d, two heads, "\gamma_*"'] & \ker\bar{\alpha}'' \arrow[d, two heads, "\xi''"'] & \mathcal{G}\bar{D}^{-\infty} \arrow[l, tail, "\bar{\alpha}_0''"'] \\
 & \ker\alpha'' \arrow[r, "="] & \ker\alpha'' &
\end{tikzcd} \tag{5.11}$$

where

(i) commutativity holds in all squares and triangles;
(ii) the horizontal and vertical sequences are exact;
(iii) ξ' and ξ'' are induced by ξ and have degree $(-1, -1)$;
(iv) $\beta_*, \gamma_*, \bar{\beta}_*, \bar{\gamma}_*$ are induced by $\beta, \gamma, \bar{\beta}, \bar{\gamma}$ and have their respective degrees;
(v) $\varphi', \bar{\varphi}''$ are induced by $\varphi, \bar{\varphi}$ and have degrees (0, 0);
(vi) $\mathscr{G}F$ is the p-graded object associated with the filtration

$$\ldots \subset \varphi^{p-1} T(g^{p-1}) \subset \varphi^p T(g^p) \subset \ldots \subset T(f)$$

which coincides with the filtration

$$\ldots \subset \ker \bar{\varphi}^p \qquad \subset \ker \bar{\varphi}^{p+1} \subset \ldots \subset T(f);$$

(vii) $\mathscr{G}D^\infty$ is the p-graded object associated with the filtration

$$\ldots \subset \pi^{p-1} T(g^{p-1}) \subset \pi^p T(g^p) \subset \ldots \subset \varinjlim_p (T(g_p), \alpha^p);$$

(viii) $\mathscr{G}\bar{D}^{-\infty}$ is the p-graded object associated with the filtration

$$\ldots \subset \ker \pi^p \qquad \subset \ker \bar{\pi}^{p+1} \subset \ldots \subset \varprojlim_p (T(\bar{g}_p), \bar{\alpha}^p);$$

(ix) $\alpha_o', \bar{\alpha}_o''$ are of degree (0, 0);
(x) $\mathscr{G}^+ \vartheta, \mathscr{G}^- \vartheta$ are induced by $\varphi, \bar{\varphi}$ and have degree (0, 0).

Here $(D^\infty, \pi^p) = \varinjlim_p (T(g_p), \alpha^p)$, $(\bar{D}^{-\infty}, \bar{\pi}^p) = \varprojlim_p (T(g_p), \bar{\alpha}^p)$.

REMARKS (i) The interpretation of $\mathscr{G}F$ is simply taken from (6.18) in [10]; we have here written $\mathscr{G}F$ rather than F^+ in order to stress that it is a graded object associated with a filtration.

(ii) When we identify F^- with F^+ (Theorem 7.26 of [10]), we do so via an automorphism induced by ϑ which has degree (1,0). This explains why $\bar{\varphi}'' \varphi'$ has p-degree 0 while $\bar{\varphi}\varphi$ has p-degree 1. This remark is also brought out in the two descriptions above of the filtration of $T(f)$.

(iii) The diagram (5.11) has been supplemented by the objects $\mathscr{G}D^\infty, \mathscr{G}\bar{D}^{-\infty}$ and their associated morphisms in order to emphasize that it is impossible to obtain from the spectral sequence information about F not relevant to D^∞ or $\bar{D}^{-\infty}$.

(iv) Consider the morphism $\psi_{p,p'}: D^p \to \bar{D}^{p'}$, given by

$$\psi_{p,p'} = \bar{\varphi}^p \varphi^{p'}.$$

Then $\bar{\alpha}^{p'} \psi_{p,p'} = \psi_{p,p'+1}$ and $\psi_{p,p'} \alpha^{p-1} = \psi_{p-1,p'}$. Thus the family $\psi_{p,p'}$ determine a morphism $\psi: D^\infty \to \bar{D}^{-\infty}$ with $\bar{\pi}^{p'} \psi \pi^p = \psi_{p,p'}$. Moreover ψ is filtration preserving; for

$$\bar{\pi}^{p+1} \psi \pi^p = \psi_{p,p+1} = \bar{\varphi}^{p+1} \varphi^p = 0,$$

so that ψ induces $\psi^p: \pi^p D^p \to \ker \bar{\pi}^{p+1}$. The morphisms ψ^p induce a morphism

$\mathcal{G}\psi : \mathcal{G}D^{\infty} \to \mathcal{G}\bar{D}^{-\infty}$ of the associated graded objects which is just the composite $(\mathcal{G}^{-}\vartheta)(\mathcal{G}^{+}\vartheta)$ in (5.11).

We now draw the evident conclusions from (5.11). We first state the formal conclusions and then interpret them in terms of the factorization (5.2) and the composition functor (T, ω).

THEOREM 5.12 (i) *If* $E_{\infty}=0$, *then all objects in* (5.11) *are zero; in particular* $\mathcal{G}F=0$.

(ii) *If* coker $\bar{\alpha}'=0$, *then* coker $\bar{\alpha}' \cong \mathcal{G}D^{\infty} \cong GF$, $E_{\infty} \cong \ker \bar{\alpha}''$, *and there is an exact sequence*

$$\mathcal{G}F \rightarrowtail E_{\infty} \twoheadrightarrow \ker \alpha''. \tag{5.13}$$

(iii) *If* $\ker \alpha''=0$, *then* $\mathcal{G}F \cong \mathcal{G}\bar{D}^{-\infty} \cong \ker \bar{\alpha}''$, coker $\alpha' \cong E_{\infty}$, *and there is an exact sequence*

$$\operatorname{coker} \bar{\alpha}' \rightarrowtail E_{\infty} \twoheadrightarrow \mathcal{G}F. \tag{5.14}$$

(iv) *If* coker $\bar{\alpha}' = \ker \alpha''=0$, *then all other objects in* (5.11) *are isomorphic; in particular*

$$E_{\infty} \cong \mathcal{G}F. \tag{5.15}$$

It should be noted that we are concerned here with morphisms of bigraded objects; thus the isomorphisms of this theorem are not necessarily of degree (0, 0). Indeed, the degrees of the morphisms above may be readily inferred from the statements following (5.11); in particular, when (5.15) holds the isomorphism in question has degree (0, 0).

We now discuss conditions under which the hypotheses of Theorem 5.12 are verified. Let us say that the factorization (5.2) is *left-T-finite* if, for each q, there is a $P_1(q)$ such that

$$T_q(g_p) = 0, \quad p < P_1.$$

Thus (5.2) is left-T-finite if and only if, in (5.6), D is *positively graded* in the sense of [10]; and plainly (5.2) is left-T-finite if it is left-finite. We also introduced in [10] the concept of D being *ultimately* positively graded; by this we understood that D_n is positively graded for some n. There is a more general – and, in a sense, more natural – definition of this concept, in which the index n is itself allowed to depend on[1] q. We are led, then, to make the following definition. The factorization (5.2) is *ultimately* left-T-finite if, for each q, there exist $n(q)$, $P_1(q)$ such that

$$\alpha^n T_q(g_{p-n}) = 0, \quad p < P_1,$$

where α^n is the n^{th} power of the morphism α.

[1]) This generalization was suggested by R. Greenblatt.

There is evidently the corresponding notion of an *ultimately right-T-finite* factorization. We then have

COROLLARY 5.16 (i) *If the factorization* (5.2) *is ultimately right-T-finite then there is an exact sequence*

$$\mathscr{G}F \rightarrowtail E_\infty \twoheadrightarrow \ker \alpha''.$$

(ii) *If the factorization* (5.2) *is ultimately left-T-finite then there is an exact sequence*

$$\operatorname{coker} \bar{\alpha}' \rightarrowtail E_\infty \twoheadrightarrow \mathscr{G}F.$$

(iii) *If the factorization* (5.2) *is ultimately T-finite then*

$$E_\infty \cong \mathscr{G}F.$$

We recall that $\mathscr{G}F$ is the bigraded object explained in (5.11 i); E_∞ is the limit of the spectral sequence associated with each of the couples (α, β, γ), $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ of Theorem 5.5; $\alpha'':I\to I$ is the endomorphism induced by $\alpha:D\to D$, where $I=\bigcap_n \alpha^n D$; and $\bar{\alpha}':\bar{U}-\bar{U}$ is the endomorphism[1]) induced by $\bar{\alpha}:\bar{D}\to\bar{D}$, where $\bar{U}=\bar{D}/\bigcup_n \ker\bar{\alpha}^n$. We also remark that the condition on (5.2) imposed in Corollary 5.16(i) implies not simply that $\operatorname{coker}\bar{\alpha}'=0$ but even that $\bar{U}=0$; it is therefore very far from a necessary condition for the validity of (5.13); similar remarks hold for (ii) and (iii) above.

In [10] we also introduced the notion of *stationarity* for the endomorphism $\alpha:D\to D$. Generalizing as for positive and negative grading, we say that α is *ultimately positively stationary* if, given any q, there exist $n(q)$, $p_0(q)$ such that $\alpha_n^{p,q}$ is an isomorphism for $p\geq p_0$. Similarly we define ultimate negative stationarity; and we say that the factorization (5.2) is *ultimately right-T-stationary* if $\alpha:T(g)\to T(g)$ is ultimately positively stationary; and that it is *ultimately left-T-stationary* if $\bar{\alpha}:T(\bar{g})\to T(\bar{g})$ is ultimately negatively stationary. We have the following implications.

PROPOSITION 5.17 (i) *If* (5.2) *is ultimately right-T-finite it is ultimately right-T-stationary;*

(ii) *if* (5.2) *is ultimately left-T-finite it is ultimately left-T-stationary;*

(iii) *if* (5.2) *is ultimately right-T-stationary then* $\operatorname{coker}\alpha'\cong\mathscr{G}D^\infty$;

(iv) *if* (5.2) *is ultimately left-T-stationary then* $\mathscr{G}\bar{D}^{-\infty}\cong\ker\bar{\alpha}''$;

(v) α *is ultimately positively (negatively) stationary if and only if* $\bar{\alpha}$ *is ultimately positively (negatively) stationary.*

Proof. (i) and (ii) are trivial; (iii) and (iv) are translations of Theorem 6.10 of [2]) [10]. To prove (v), we consider the Rees system (5.6). First we remark that if

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[1]) In the notation of [10], we have $\alpha^n = \nu_n\eta_n : D \twoheadrightarrow D_n \rightarrowtail D$, and $I = \varprojlim(D_n, \nu_n)$, $\bar{U} = \varinjlim(\bar{D}_n, \bar{\eta}_n)$.

[2]) The slight generalization of the concept 'ultimate', compared with [10], complicates the proof imperceptibly.

$\alpha_m^{p,q}: D_m^{p,q} \cong D_m^{p+1,q}$, $p \geq p_0$, and if $n \geq m$, then $\alpha_n^{p,q}: D_n^{p,q} \cong D_n^{p+1,q}$, $p \geq p_0 + n - m$. Let us then suppose that α is ultimately positively stationary. If we fix q, then $\exists n_0, n_1, p_0, p_1$ such that

$$\begin{array}{lll} \alpha_{n_0}^{p,q} & \text{is an isomorphism,} & p \geq p_0, \\ \alpha_{n_1}^{p,q-1} & \text{is an isomorphism,} & p \geq p_1. \end{array}$$

Let $n = \max(n_0, n_1)$, $P = \max(p_0 + n - n_0, p_1 + n - n_1)$. Then

$$\alpha_n^{p,q} \text{ and } \alpha_n^{p,q-1} \quad \text{are isomorphisms,} \quad p \geq P.$$

We study the n^{th} derived system of (5.6). Adopting the convention that $D_n^{p,q}$ is a sub-object of $D^{p,q}$, we have the exact sequence

$$\cdots \to D_n^{p+n-1,q} \xrightarrow{\alpha_n^{p+n-1,q}} D_n^{p+n,q} \to E_n^{p,q} \to D_n^{p-1,q-1} \xrightarrow{\alpha_n^{p-1,q-1}} D_n^{p,q-1} \to \cdots.$$

This shows that $E_n^{p,q} = 0$ for $p \geq P+1$. Here n and P are functions of q. Let $N = \max(n(q), n(q-1))$ and consider the following exact sequence extracted from the lower couple of the n^{th} derived system of (5.6):

$$E_N^{p,q} \to \bar{D}_N^{p,q} \xrightarrow{\bar{\alpha}_N^{p,q}} \bar{D}_N^{p+1,q} \to E_N^{p-N,q-1}$$

Then $E_N^{p,q} = 0$ if $p \geq P+1$, $E_N^{p-N,q-1} = 0$ if $p \geq P+N+1$. Thus if $p \geq P+N+1$, $\bar{\alpha}_N^{p,q}$ is an isomorphism, so $\bar{\alpha}$ is ultimately positively stationary. The converse is proved by the same argument and the case of negative stationarity similarly.

REMARKS (i) We may suppress the word 'ultimately' from any of the statements of Proposition 5.17.

(ii) Proposition 5.17(v) shows that we could have defined right- and left-stationarity for (5.2) in terms of α or $\bar{\alpha}$. We chose the definitions given because the stated hypotheses seem to arise naturally in applications.

THEOREM 5.18 *If α (or $\bar{\alpha}$) is ultimately stationary, then the spectral sequence of (5.6) converges finitely. That is, given p, q, there exists N, such that $E_N^{p,q} = E_{N+1}^{p,q} = \ldots$ $\ldots = E_\infty^{p,q}$.*

Proof. This was proved in [10], Theorem 6.13, under more general assumptions on the degrees of β and γ, but with a more restricted definition of ultimate stationarity. However the essential observation in the proof is that, for each q, there exist P', P'', n such that

$$E_n^{p,q} = 0 \quad \text{except perhaps if} \quad P'' \leq p \leq P';$$

and the argument for this was given in the proof of Proposition 5.17(v) above. We then conclude that

$$E_N^{p,q} = E_{N+1}^{p,q}, \tag{5.19}$$

provided $N > \{P'(q+1) - p - 1, p + 1 - P''(q-1), n(q+1), n(q-1)\}$.

Plainly (5.19) implies that, with such a value of N,

$$E_N^{p,q} = E_{N+1}^{p,q} = \cdots = E_\infty^{p,q}.$$

Proposition 5.17 and Theorem 5.18 enable us to strengthen Corollary 5.16 in the following way.

THEOREM 5.20 (i) *If the factorization* (5.2) *is ultimately right-T-finite then* $\ker \alpha'' \cong \mathscr{G} D^{-\infty}$ *and there is an exact sequence*

$$\mathscr{G} F \rightarrowtail E_\infty \twoheadrightarrow \mathscr{G} D^{-\infty}$$

(ii) *If the factorization* (5.2) *is ultimately left-T-finite then* $\operatorname{coker} \bar{\alpha}' \cong \mathscr{G} \bar{D}^\infty$ *and there is an exact sequence*

$$\mathscr{G} \bar{D}^\infty \rightarrowtail E_\infty \twoheadrightarrow \mathscr{G} F$$

(iii) *If the factorization* (5.2) *is ultimately T-finite then the spectral sequence of* (5.6) *converges finitely and*

$$E_\infty \cong \mathscr{G} F.$$

Proof. We prove (i) and (ii) by invoking Proposition 5.17(v), and Theorem 6.10 of [10]. (iii) just restates Theorem 5.18 and Corollary 5.16 (iii).

REMARKS (i) In Theorem 5.10 (i, ii) we may replace $\mathscr{G} F$ by $\mathscr{G} D^\infty$, $\mathscr{G} \bar{D}^{-\infty}$respectively. However the real interest of this theorem is likely to be precisely that $\mathscr{G} D^{-\infty}$, $\mathscr{G} \bar{D}^\infty$ measure the deviation of E_∞ (which is 'too big') from $\mathscr{G} F$.

(ii) It is evident, when the argument of Theorem 5.18 is analysed, that if (5.2) is ultimately right-T-stationary, then, given (p, q), there is an N_1 such that

$$E_{N_1}^{p,q} \supset E_{N_1+1}^{p,q} \supset \cdots \supset E_\infty^{p,q};$$

and that, if (5.2) is ultimately left-T-stationary, then, given (p, q), there is an N_2 such that

$$E_{N_2}^{p,q} \twoheadrightarrow E_{N_2+1}^{p,q} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{p,q}.$$

(iii) If $\mathfrak{A}$ is a category of modules then $\operatorname{coker} \bar{\alpha}' \cong \mathscr{G} \bar{D}^\infty$ without further hypothesis on the factorization; but one cannot infer the short exact sequence of Theorem 5.20 (ii).

Diagram (5.11) demonstrates the crucial role played by $\operatorname{coker} \bar{\alpha}'$ and $\ker \alpha''$ in studying the relation between E_∞ and $\mathscr{G} F$. It is therefore worthwhile giving alternative characterizations of these objects which may be better adapted to our applications (see section 7). We study the case of $\ker \alpha''$ in detail and appeal to the duality principle for the interpretation of $\operatorname{coker} \bar{\alpha}'$.

Referring back to the factorization (5.1), let us set

$$j_{p,n} = j_p j_{p-1} \cdots j_{p-n}, \; n \geq 0. \tag{5.21}$$

Then plainly

$$j_{p+1,n}\, g_{p-n} = g_{p+1}\,,$$

and, more generally,

$$j_{p+k,n+k-1}\, g_{p-n} = g_{p+k} \tag{5.22}$$

We now apply the centre part of the diagram (3.2) with $u = g_{p-n}$, $v = j_{p,n-2}$, $w = j_{p+1}$, to obtain the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & T_{q+1}(j_{p+1,n}) \arrow[r, "\gamma(n)"] \arrow[d, "T_{q+1}\binom{j_{p,n-1}}{1}"] & T_q(g_{p-n}) \arrow[r, "\alpha(n)"] \arrow[d, "\alpha^n"] & T_q(g_{p+1}) \arrow[r, "\beta(n)"] \arrow[d, "="] & T_q(j_{p+1,n}) \arrow[r] \arrow[d, "T_q\binom{j_{p,n-1}}{1}"] & \cdots \\
\cdots \arrow[r] & T_{q+1}(j_{p+1}) \arrow[r, "\gamma"] & T_q(g_p) \arrow[r, "\alpha"] & T_q(g_{p+1}) \arrow[r, "\beta"] & T_q(j_{p+1}) \arrow[r] & \cdots
\end{tikzcd} \tag{5.23}$$

where the rows are exact, α^n is the n^{th} iterate of α, the dimension symbols on the morphisms have been suppressed, and $\alpha(n)$, $\beta(n)$, $\gamma(n)$ are merely ad hoc notations for the morphisms of the exact sequence obtained by applying (T, ω) to the object $(j_{p+1,n}, g_{p-n})$ of $\mathfrak{C}^3$. Of course, $\alpha(n) = \alpha^{n+1}$.

THEOREM 5.24 $(\ker \alpha'')^{p,q} = \bigcap_n Im\, \alpha^n \gamma(n)$.

Proof. This theorem holds in any abelian category $\mathfrak{A}$ when we interpret the right hand side as $\varprojlim \mu_n$, where $\alpha^n\gamma(n)$ splits as $\mu_n \varepsilon_n$. However, since this theorem is quoted mainly with a view to certain specific applications, we will be content to give a proof when $\mathfrak{A}$ is a category of modules.

Now $x \in (\ker \alpha'')^{p,q}$ if and only if $x \in \gamma T_{q+1}(j_{p+1})$ and $x \in \alpha^n T_q(g_{p-n})$ for every n. Moreover, it follows from the exactness of the rows of (5.23) that $x \in \gamma T_{q+1}(j_{p+1}) \cap \cap \alpha^n T_q(g_{p-n})$ if and only if $x \in \alpha^n \gamma(n) T_{q+1}(j_{p+1,n})$. This proves the theorem if $\mathfrak{A}$ is a category of modules.

We note that, for x to be a *non-zero* element of $(\ker \alpha'')^{p,q}$, it is necessary and sufficient that x should be in the image of every $T_{q+1}(j_{p+1,n})$ without being in the image of $T_{q+1}(g_{p+1})$, or, as we may write g_{p+1} for this purpose as $j_{p+1,\infty}$, without being in the image of $T_{q+1}(j_{p+1,\infty})$. This formulation expresses most clearly the situation under which we will arrive at a non-zero object $\ker \alpha''$.

We now pass briefly to a discussion of coker $\bar{\alpha}'$. We consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & T_q(j_{p-1}) \arrow[r, "\bar{\gamma}"] \arrow[d, "T_q\binom{1}{j_{p+n-1,n-1}}"] & T_q(g_{p-1}) \arrow[r, "\bar{\alpha}"] \arrow[d, "="] & T_q(g_p) \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{\alpha}^n"] & T_{q-1}(j_{p-1}) \arrow[r] \arrow[d, "T_{q-1}\binom{1}{j_{p+n-1,n-1}}"] & \cdots \\
\cdots \arrow[r] & T_q(j_{p+n-1,n}) \arrow[r, "\bar{\gamma}(n)"'] & T_q(g_{p-1}) \arrow[r, "\bar{\alpha}(n)"'] & T_q(g_{p+1}) \arrow[r, "\bar{\beta}(n)"'] & T_{q-1}(j_{p+n-1,n}) \arrow[r] & \cdots
\end{tikzcd} \tag{5.25}$$

This diagram may be brought into closer relation to (5.23) by writing

$$\bar{j}_{p,n} = j_{p+n,n} = j_{p+n} \cdots j_p,\quad n \geq 0. \tag{5.26}$$

(It would thus be reasonable to write $\bar{j}_p$ for j_p also). Then if $\bar{\beta}(n)\bar{\alpha}^n$ splits as $\bar{\mu}_n \bar{\varepsilon}_n$, we have (compare the interpretation of Theorem 5.24)

$$(\operatorname{coker} \bar{\alpha}')^{p,q} = \varinjlim \bar{\varepsilon}_n. \tag{5.27}$$

Assuming $\mathfrak{A}$ to be a category of modules, we may say that $x \in \bar{D}^{p,q} = T_q(\bar{g}_p)$ represents a non-zero element of $(\operatorname{coker} \bar{\alpha}')^{p,q}$ if and only if it has non-zero image under every morphism $\bar{\beta}(n)\bar{\alpha}^n$, $n \geq 0$.

6. The contravariant case

Since all our results have been obtained for composition functors into arbitrary abelian categories there has been no logical necessity to deal separately with covariant and contravariant functors. However, in the applications we make in the next section we are concerned with categories of modules and it will therefore be convenient to reformulate our results for contravariant functors. All that is involved, then, is the mechanical process of interpreting our results, obtained for composition functors from $\mathfrak{C}$ to $\mathfrak{A}$, in $\mathfrak{A}^{\text{opp}}$. However it will be convenient to reformulate some of our definitions, too: for it is realistic to suppose given a factorization of a morphism in $\mathfrak{C}$, together with a composition functor, as defined in section 2, from $\mathfrak{C}$ to $\mathfrak{A}^{\text{opp}}$, with the interest residing in the interpretation in $\mathfrak{A}$.

DEFINITION *6.1* A *contravariant* composition functor from $\mathfrak{C}$ to $\mathfrak{A}$ is a composition functor from $\mathfrak{C}$ to $\mathfrak{A}^{\text{opp}}$.

REMARK It is perfectly acceptable to regard a composition functor from $\mathfrak{C}^{\text{opp}}$ to $\mathfrak{A}$ as a contravariant composition functor from $\mathfrak{C}$ to $\mathfrak{A}$, since we may identify $(\mathfrak{C}^{\text{opp}})^2$ with $(\mathfrak{C}^2)^{\text{opp}}$ in an obvious way. However, this point of view leads to the convention that the 'connecting homomorphism' ω reduces degree by 1. Since in our applications it is natural (or customary, according to viewpoint) to have ω raise degree by 1 in the contravariant case, we prefer to adopt Definition 6.1 rather than that indicated in this remark; of course the latter could be brought into line by changing the sign of the degree index on the graded functor T.

Let (T, ω) be a contravariant composition functor from $\mathfrak{C}$ to $\mathfrak{A}$ throughout this section. Our notational conventions are sufficiently indicated by the following proposition.

PROPOSITION *6.2 Corresponding to each object (f, g) of $\mathfrak{C}^3$ there is an exact sequence*

$$\cdots \to T^q(g) \xrightarrow{T^q\binom{f}{1}} T^q(gf) \xrightarrow{T^q\binom{1}{g}} T^q(f) \xrightarrow{\omega^q(f,g)} T^{q+1}(g) \to \cdots$$

in $\mathfrak{A}$, which is natural in the evident sense.

We will not stop to describe explicitly the contravariant forms of the theorems of sections 2, 3, 4 since these are obtained immediately. Our real concern in this section is to translate the principal results of section 5.

We will find it convenient to reindex the factors in a factorization (5.2) as follows. We set

$$f = g^p \bar{g}^{p+1}, \; j^p \bar{g}^{p+1} = \bar{g}^p, \; g^{p-1} j^p = g^p, \; -\infty < p < \infty . \tag{6.3}$$

Formally we may write

$$\begin{aligned} f &= \cdots j^{p-1} j^p j^{p+1} \cdots, \\ g^p &= \cdots j^{p-1} j^p, \\ \bar{g}^p &= \qquad\quad j^p j^{p+1} \cdots, \end{aligned}$$

and we may identify (6.3) with (5.2) by the rule

$$j_p = j^{-p}, \; g_p = \bar{g}^{-p}, \; \bar{g}_p = g^{-p}. \tag{6.4}$$

This rule is important since, in practice, we often wish to consider a single factorization and apply to it various functors, covariant and contravariant. However, the advantage of the adoption of (6.3) is that it assigns to j, g, $\bar{g}$, and the index p, under the contravariant composition functor (T, ω), the same roles as they played originally under a (covariant) composition functor. In other words, (6.3) *could* be interpreted as expressing (5.2) in the category $\mathfrak{C}^{\mathrm{opp}}$ (see the Remark above).

Now let the contravariant composition-functor from $\mathfrak{C}$ to $\mathfrak{A}$ be applied to the factorization (6.3). We obtain the basic theorem:

THEOREM 6.5 *There is a diagram in* $\mathfrak{A}^{\mathbf{Z}\times\mathbf{Z}}$

$$\begin{tikzcd} & D \arrow[rr, "\alpha"] \arrow[dl, "\varphi"'] & & D \arrow[dl, "\beta"'] \arrow[dr, "\varphi"] & \\ F \arrow[dr, "\bar{\varphi}"'] & & E \arrow[ul, "\gamma"'] \arrow[dl, "\bar{\gamma}"] & & F \arrow[dl, "\bar{\varphi}"] \\ & \bar{D} \arrow[uu, "\xi"'] \arrow[rr, "\bar{\alpha}"'] & & \bar{D} \arrow[uu, "\xi"] \arrow[ul, "\bar{\beta}"] & \end{tikzcd} \tag{6.6}$$

in which

(i) $D = (D^{p,q}) = (T^q(g^p))$; $\bar{D} = (\bar{D}^{p,q}) = (T^q(g^p))$: $E = (E^{p,q}) = (T^q(j^p))$;
$F = (F^{p,q}) = (T^q(f))$;

(ii) $\alpha^{p,q} = T^q\binom{j^{p+1}}{1}$; $\beta^{p,q} = T^q\binom{1}{g^{p-1}}$; $\gamma^{p,q} = \omega^q(j^p, g^{p-1})$;
$\bar{\alpha}^{p,q} = T^q\binom{1}{j^p}$; $\bar{\beta}^{p,q} = \omega^q(\bar{g}^p, j^{p-1})$; $\bar{\gamma}^{p,q} = T^q\binom{g^{p+1}}{1}$;
$\xi^{p,q} = \omega^q(\bar{g}^p, g^{p-1})$; $\varphi^{p,q} = T^q\binom{\bar{g}_{p+1}}{1}$; $\bar{\varphi}^{p,q} = T^q\binom{1}{g_{p-1}}$;

(iii) $\deg \alpha = \deg \bar{\alpha} = (1,0)$; $\deg \beta = \deg \bar{\gamma} = (0,0)$;
$\deg \gamma = \deg \bar{\beta} = \deg \xi = (-1,1)$; $\deg_q \varphi = \deg_q \bar{\varphi} = 0$; $\deg_p \bar{\varphi}\, \varphi = 1$;

(iv) (α, β, γ) *and* $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ *are exact couples;*

(v) $(\xi, \varphi, \bar{\varphi})$ *is an exact triangle;*

(vi) *the commutativity relations* $\alpha\xi = \xi\bar{\alpha}$, $\beta\xi = \bar{\beta}$, $\gamma = \xi\bar{\gamma}$ *hold;*

(vii) $\varphi^{p+1}\alpha^p=\varphi^p$, $\bar{\alpha}^p\bar{\varphi}^p=\bar{\varphi}^{p+1}$ *for all* p, q;
(viii) $\bar{\varphi}^p\varphi^p=\bar{\gamma}\beta^p$ *for all* p, q;
(ix) *the spectral sequences associated with* (α, β, γ), $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ *coincide and*

$$d^{p,q} = \omega^q(j^p, j^{p-1}) \colon E^{p,q} \to E^{p-1,q+1}. \tag{6.7}$$

Note that the only formal difference between diagrams (5.6) and (6.6) is that the q-degrees of the morphisms have been changed into their negatives. The other difference is one of interpretation, based on the device (6.4) of renaming the factors in the factorization of f. Since all our subsequent theorems in section 5 are based on Theorem 5.5, they therefore remain valid (with the replacement only of q by $-q$ where appropriate) in the new interpretation. In particular

THEOREM 6.8 *Let* (T, ω) *be a contravariant composition functor from* $\mathfrak{C}$ *to* $\mathfrak{A}$ *and let* (6.3) *be a factorization of* f *in* $\mathfrak{C}$. *Then the limit diagram* (5.11) *holds with the sole change that the morphisms* ξ', ξ'' *have degrees* $(-1,1)$. *The interpretation of the morphisms of* (5.11) *is made through Theorem* 6.5. *Theorem* 5.12 *remains valid.*

We say that the factorization (6.3) is left-T-finite if D is positively graded in (6.6) with the evident generalization to ultimate left-T-finiteness. Similarly we say that (6.3) is (ultimately) right-T-stationary if $\alpha : T(g) \to T(g)$ is (ultimately) positively stationary. Then Corollary 5.16, Proposition 5.17, Theorems 5.18 and 5.20 all continue to hold, without any change of wording. However it is particularly necessary at this stage to stress the difference of interpretation. It would be absurd to change the notion of a left-finite or right-finite factorization according to whether we anticipated the application of a covariant or a contravariant functor. Thus we must remember that, if (T, ω) is a *covariant* composition-functor then a *left*-finite factorization is *left-T*-finite; but if (T, ω) is a *contravariant* composition functor then a *left*-finite factorization is *right-T*-finite.

Also we must change the interpretations given at the close of section 5 of the objects $\ker \alpha''$, coker $\bar{\alpha}'$, which play so crucial a role in the passage from E_∞ to $\mathscr{G}F$. The diagram corresponding to (5.23) is

$$\begin{tikzcd}
\cdots \arrow[r] & T^{q-1}(j^{p+1,n}) \arrow[r, "\gamma(n)"] \arrow[d, "T^{q-1}\binom{1}{j^{p,n-1}}"] & T^q(g^{p-n}) \arrow[r, "\alpha(n)"] \arrow[d, "\alpha^n"] & T^q(g^{p+1}) \arrow[r, "\beta(n)"] \arrow[d, "="] & T^q(j^{p+1,n}) \arrow[r] \arrow[d, "T^q\binom{1}{j^{p,n-1}}"] & \cdots \\
\cdots \arrow[r] & T^{q-1}(j^{p+1}) \arrow[r, "\gamma"] & T^q(g^p) \arrow[r, "\alpha"] & T^q(g^{p+1}) \arrow[r, "\beta"] & T^q(j^{p+1}) \arrow[r] & \cdots
\end{tikzcd} \tag{6.9}$$

where

$$j^{p,n} = j^{p-n} \ldots j^p, n \geq 0. \tag{6.10}$$

Then Theorem 5.24 holds:

$$(\ker \alpha'')^{p,q} = \bigcap_n Im\, \alpha^n \gamma(n); \tag{6.11}$$

and, as in the covariant case, we find that $(\ker \alpha'')^{p,q}$ is non-zero if and only if there is an element[1]) in $T^q(g^p)$ which lies in the image of every $T^{q-1}(j^{p+1,n})$ without being in the image of $T^{q-1}(g^{p+1})$.

The diagram corresponding to (5.25) is

$$\begin{tikzcd}
\cdots \arrow[r] & T^q(\bar{j}^{p-1}) \arrow[r, "\bar{\gamma}"] \arrow[d, "T^q\binom{\bar{j}^{p,n-1}}{1}"] & T^q(\bar{g}^{p-1}) \arrow[r, "\bar{\alpha}"] \arrow[d, "="] & T^q(\bar{g}^p) \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{\alpha}^n"] & T^{q+1}(\bar{j}^{p-1}) \arrow[r] \arrow[d, "T^{q+1}\binom{\bar{j}^{p,n-1}}{1}"] & \cdots \\
\cdots \arrow[r] & (\bar{j}^q T^{p-1,n}) \arrow[r, "\bar{\gamma}(n)"] & T^q(\bar{g}^{p-1}) \arrow[r, "\bar{\alpha}(n)"] & T^q(\bar{g}^{p+n}) \arrow[r, "\bar{\beta}(n)"] & T^{q+1}(\bar{j}^{p-1,n}) \arrow[r] & \cdots
\end{tikzcd} \tag{6.12}$$

where
$$\begin{aligned}\bar{j}^{p,n} &= j^p \ldots j^{p+n}, n \geq 0, \\ &= j^{p+n,n}\end{aligned}$$

(and $\bar{j}^p = j^p$). Then (5.27) remains true with the same formal definition of $\bar{\varepsilon}_n$; and, when $\mathfrak{A}$ is a category of modules, we conclude that $x \in \bar{D}^{p,q} = T^q(\bar{g}^p)$ represents a non-zero element of (coker $\bar{\alpha}')^{p,q}$ if and only if $\bar{\beta}(n)\bar{\alpha}^n x \neq 0 \in T^{q+1}(j^{p-1,n})$ for all n.

7. Applications to algebraic topology

We confine ourselves to applications of the general theory to various spectral sequences of algebraic topology, all obtained from suitable factorizations of maps by applying homotopy or cohomology functors. In establishing the existence of the spectral sequences under consideration, only the axioms for composition functors are used, and excision and homotopy play no rôle. However they are essential in obtaining properties of the spectral sequences; e.g., in computing the first terms and in discussing convergence. The factorizations are either skeleton decompositions of CW-complexes or successive fibrations. Some of the examples are only sketched; complete arguments and further developments will be or have been given elsewhere.

EXAMPLE 7.1. Let $\mathfrak{C}$ be the category of based CW-complexes; for $X \in \mathfrak{C}$, X_p will denote the p-skeleton of X, for $p \geq 0$, X_p = base-point o for $p < 0$, $j_p = j^{-p}$ the embedding $X_{p-1} \subset X_p$, $f = \ldots j_{p+1} j_p j_{p-1} \ldots$ the embedding $o \in X$. If X is finite-dimensional, of dimension N, $j_p = 1$ for $p > N$. The maps g_p, $\bar{g}_p$ of the former sections are

$$\begin{aligned} g_p &= \bar{g}^{-p} = \text{inclusion } o \to X_p, \\ \bar{g}_p &= g^{-p} = \text{inclusion } X_{p-1} \to X. \end{aligned}$$

The factorization is left-finite, and finite if X is finite-dimensional.

Further let T be the truncated composition functor of Example 4.8: T_q is the homotopy group functor $\Pi_q(A, \ ) = \Pi(\Sigma^q A, \ )$ for $q \geq 3$, $T_q = 0$ for $q \leq 1$, and $T_2(f)$ for $f\colon X \to Y$ in $\mathfrak{C}$ is the cokernel of $f_*\colon \Pi_2(A, X) \to \Pi_2(A, Y)$. The "lower end" of the

[1]) We assume here that $\mathfrak{A}$ is a category of modules, so that we may talk of elements.

exact triple sequence for the triple gf is

$$\cdots \to \Pi_3(A, gf) \to \Pi_3(A, g) \to T_2(f) \to T_2(gf) \to T_2(g) \to 0 \to \cdots$$

Applying this functor to the skeleton factorization of $f: o \to X$, we obtain a Rees system and, the factorization being left-T-finite, a spectral sequence (E_n, d_n) whose limit E_∞ is related to the $F_q = \Pi_q(A, X)$ by the exact sequence (cf. Theorem 5.20)

$$\mathscr{G}\bar{D}^\infty \rightarrowtail E_\infty \twoheadrightarrow \mathscr{G}F.$$

Here F is the graded Abelian group whose q-component F_q is $\Pi_q(A, X)$ for $q \geq 2$, 0 for $q < 2$, filtered by the images of the $\Pi_q(A, X_p)$. The spectral sequence starts with $E_o = (E_o^{p,q})$, $E_o^{p,q} = \Pi_q(A, j_p)$ for $q \geq 3$, $E_o^{p,q} = 0$ for $q < 2$, and $E_o^{p,2}$ is the cokernel of the map $\Pi_2(A, X_{p-1}) \to \Pi_2(A, X_p)$ induced by the embedding. The differential $d_o^{p,q}$ is the boundary homomorphism $\Pi_q(A, j_p) \to \Pi_{q-1}(A, j_{p-1})$ in the triple sequence for $j_p j_{p-1}$ $(q > 3)$; it is the homomorphism $\Pi_3(A, j_p) \to \mathrm{Coker}\,(\Pi_2(A, X_{p-2}) \to \Pi_2(A, X_{p-1}))$ for $q = 3$. This is a mild generalization of the Massey spectral sequence [19] which we obtain by putting $A = S_o$.

The groups $\bar{D}^{p,q} = \Pi_q(A, \bar{g}_p)$ for $q \geq 3$, $= \mathrm{Coker}\,(\Pi_2(A, X_{p-1}) \to \Pi_2(A, X))$ for $q = 2$, and 0 for $q < 2$. If X is finite-dimensional, we know that $\mathscr{G}\bar{D}^\infty$ is 0. For general X, if we assume A to be a finite-dimensional CW-complex, we have $\Pi_q(A, \bar{g}_p) = 0$ for $p > P$ depending on q and the dimension of A; thus again $\mathscr{G}\bar{D}^\infty = 0$: *If A or X are finite-dimensional, E_∞ is related to the $\Pi_q(A, X)$ by*

$$\begin{aligned} E_\infty^{p,q} &= \mathscr{G}^p \Pi_q(A, X), \quad & q \geq 2, \\ &= 0 \quad &, \quad q < 2. \end{aligned}$$

EXAMPLE 7.2. Here we apply the truncated contravariant homotopy group functor $\Pi_q(\ , B)$ to the skeleton factorization of Ex. 7.1; i.e., $T_q = \Pi_q(\ , B)$ for $q \geq 3$, $T_q = 0$ for $q < 2$ and $T_2(f) = $ cokernel of $f^*: \Pi_2(Y, B) \to \Pi_2(X, B)$ for $f: X \to Y$. The functor being contravariant, the index p in the bigrading of E, D etc. will refer to the notation j^p, g^p and $\bar{g}^p$ of the factorization, according to the conventions laid down in section 6 (note that the q-index in the triple sequence for T_q is decreasing, which is not customary for contravariant functors; this, however, does not affect the general arguments of section 6).

In the Rees system and the spectral sequence obtained here, the relevant terms can be described as follows (we do not insist here on the small values of q, but simply note that for $q < 2$ everything is 0): $E_o = (E_o^{p,q})$ with[1] $E_o^{-p,q} = \Pi_q(j^{-p}, B) = \Pi_q(j_p, B)$; by excision this is naturally isomorphic to $\Pi_{q-1}(X_p/X_{p-1}, B) = \Pi(X_p/X_{p-1}, \Omega^{q-1}B)$,

[1]) We could translate our statements about E_r into the usual statements by replacing $-p$ by p. But this would force us also to change the statements of Section 6. A similar remark also applies to the subsequent examples.

which is equal, under an obvious identification, to the cellular cochain group $C^p(X;\ \pi_{p+q-1}(B))$. The differential $d_o^{-p,q}\colon E_o^{-p,q} \to E^{-p-1,q+1}$ can be computed explicitely: it is the combinatorial coboundary

$$C^p(X;\pi_{p+q-1}(B)) \to C^{p+1}(X;\pi_{p+q-1}(B))$$

of the CW-complex X. Without giving the proof in detail, we simply remark that the only non-trivial argument to be used is the homotopy addition theorem, when X is assumed to be a simplicial complex. Thus $E_1^{-p,q}$ is equal to the cellular cohomology group $H^p(X;\pi_{p+q-1}(B))$. For the groups $D=(D^{p,q})$, $\bar{D}=(\bar{D}^{p,q})$ and $F=(F^q)$ we have

$$\begin{aligned} D^{-p,q} &= \Pi_q(g^{-p}, B) = \Pi_q(\bar{g}_p, B) \cong \Pi(X/X_{p-1}, \Omega^{q-1} B), \\ \bar{D}^{-p,q} &= \Pi_q(\bar{g}^{-p}, B) = \Pi_q(g_p, B) \cong \Pi(X_p, \Omega^{q-1} B), \\ F^q \quad &= \Pi_q(f, B) \quad = \Pi(X, \Omega^{q-1} B), \end{aligned}$$

and the filtration $(F^q)^{-p}$ of F^q is by the images of the groups $\Pi(X/X_{p-1}, \Omega^{q-1} B)$ under the maps $X \to X/X_{p-1}$. The $(-p)$-component of $\mathscr{G} F^q$ is

$$\mathscr{G}^{-p} F^q = (F^q)^{-p}/(F^q)^{-p-1} = Im\ \Pi(X/X_{p-1}, \Omega^{q-1} B)/Im\ \Pi(X/X_p, \Omega^{q-1} B).$$

Since $\bar{D}^{-p,q}=0$ for $p<0$, i.e. for $-p>0$, the factorization is *right-T-finite*, and hence there is the exact sequence

$$\mathscr{G} F \rightarrowtail E_\infty \twoheadrightarrow \mathscr{G} D^{-\infty} = \ker \alpha''.$$

Furthermore, if X is finite-dimensional, $D^{-p,q}=0$ for large p, i.e., for small $-p$, and thus the factorization is also left-T-finite, which yields $\mathscr{G} F \cong E_\infty$. The same conclusion holds for arbitrary X, if B has only a finite number of homotopy groups $\pi_r \neq 0$; for there is, in that case, for each q an integer $P(q)$ such that $D^{-p,q} = \Pi(X/X_{p-1}, \Omega^{q-1} B) = 0$ for $p>P(q)$, in other terms, for $-p<-P(q)$.

In conclusion we have a spectral sequence starting with $E_1^{-p,q} = H^p(X;\pi_{p+q-1}(B))$ and converging to

$$E_\infty^{-p,q} = \mathscr{G}^{-p}\Pi(X, \Omega^{q-1}(B))$$

if X has finite dimension or if B has only a finite number of $\pi_r \neq 0$; in the general case $\mathscr{G}^{-p}\Pi(X, \Omega^{q-1} B)$ is isomorphic to a subgroup of $E_\infty^{-p,q}$, the factor group being isomorphic to $(\ker \alpha'')^{-p,q}$ as described in 5.24. – The spectral sequence above is due to FEDERER [14].

It should be noted that in general the filtration of $F^q = \Pi(X, \Omega^{q-1} B)$ is *not complete* [13]. The direct limit $(-p \to \infty$, i.e. $p \to -\infty)$ is, of course, equal to $F^q = \Pi(X, \Omega^{q-1} B)$; the inverse limit $(p \to \infty)$ of the corresponding cofiltration $F^q/(F^q)^{-p} = \Pi(X, \Omega^{q-1} B)/Im\ \Pi(X/X_{p-1}, \Omega^{q-1} B)$ is $\cong F^q/\tilde{F}^q$, where $\tilde{F}^q = \tilde{\Pi}(X, \Omega^{q-1} B)$ is the *subgroup consisting of those homotopy classes* $X \to \Omega^{q-1} B$ *which vanish on all skeleta* X_{p-1}, and this subgroup need not be 0. Since we are not interested here in the comparison theorem for spectral sequences, completeness is not of main importance. However, the subgroup $\tilde{F}^q$ is relevant to the information on F^q obtained from the spectral sequence. Since $\mathscr{G}F^q = \mathscr{G}(F^q/\tilde{F}^q)$, the information, given by the spectral sequence and ker α'', can only help us in reconstructing $\Pi(X, \Omega^{q-1}B)/\tilde{\Pi}(X, \Omega^{q-1}B)$. In the two special cases above which yield $E_\infty \cong \mathscr{G}F$ (X finite-dimensional or B having a finite number of $\pi_r \neq 0$), the group $\tilde{\Pi}(X, \Omega^{q-1}B)$ is 0.

EXAMPLE 7.3. The Federer spectral sequence can be applied to *representable cohomology theories*, as follows. Let $\{B_m, \sigma_m\}$ be an Ω-spectrum; i.e., B_m, $m \in \mathbf{Z}$, is a sequence of based topological spaces and the σ_m are homotopy equivalences $B_m \to \Omega B_{m+1}$. Then, for $f: X \to Y$ the groups $\Pi_q(f, B_{m+q-1})$ are defined and abelian for all $q \geq 1$ and independent of q (up to natural isomorphisms); they can thus be identified with each other and denoted by $h^m(f)$. If $f: o \to X$, the group $h^m(f)$ is, according to usual conventions, written $h^m(X)$. The functor h^m is a (reduced) cohomology functor fulfilling the exactness, excision and homotopy axioms, but in general not the dimension axiom $h^m(S_o) = 0$ for $m \neq 0$. The dimension axiom is fulfilled if and only if the spectrum is the Eilenberg-Maclane spectrum $B_m = K(G, m)$ for a fixed abelian group G, $m \geq 0$, $B_m = o$ for $m < 0$.

In order to apply the Federer sequence, we can fix the value of q, e.g. by taking $q = 3$, and choose $B = B_{m+2}$; then $h^m(\ ) = \Pi_3(\ , B_{m+2})$ is a contravariant composition functor, where now m plays the role of q in the general theory. The various groups in the Rees system and the spectral sequence for the skeleton decomposition of x are:

$$E_o^{-p,m} = C^p(X; \pi_{p+2}(B_{m+2})) = C^p(X; \pi_o(B_{m-p})) = C^p(X; h^{m-p}(S_o)),$$

$d_o^{-p,m}$ being the cellular coboundary $C^p(X; h^{m-p}(S_o)) \to C^{p+1}(X; h^{m-p}(S_o))$;

$$E_1^{-p,m} = H^p(X; h^{m-p}(S_o));$$
$$D^{-p,m} = h^m(X/X_{p-1}),$$

(obtained as $\Pi(X/X_{p-1}, \Omega^2 B_{m+2}) = \Pi(X/X_{p-1}, B_m)$),

$$\bar{D}^{-p,m} = h^m(X_p),$$
$$F^m = h^m(X), \qquad \text{filtered by} \quad (F^m)^{-p} = Im\ h^m(X/X_{p-1}).$$

The factorization is always *right-h-finite*. It is also *left-h-finite* (a) if X is finite dimensional or (b) if $h^m(S_o) = 0$ for $m < M$, for some $M \in \mathbf{Z}$. The proof of (a) is immediate, since $D^{-p,m} = 0$ for $p > \dim X$. In the case (b), we have $\pi_r(B_m) = h^{m-r}(S_o) = 0$ for $m - r < M$, i.e. for $r > m - M$; in other words, there is an integer $P(m)$ such that $\Pi(X/X_{p-1}, B_m) = 0$ for $p > P(m)$, hence $D^{-p,m} = 0$ for $-p < -P(m)$.

We thus obtain a spectral sequence, generalizing the Atiyah-Hirzebruch spectral sequence [2], *relating ordinary and extraordinary cohomology on CW-complexes*: $E_1^{-p,m}$ is the cellular cohomology group $H^p(X; h^{m-p}(S_o))$; and $E_\infty^{-p,m}$ the graded group $\mathscr{G}^{-p}h^m(X)$ associated with the fibration $(h^m(X))^{-p}$ = subgroup of $h^m(X)$

consisting of those elements which vanish on X_{p-1}, if (*a*) X is finite-dimensional or if (*b*) $h^m(S_o)=0$ for small m. In the general case, not assuming (*a*) or (*b*), $\mathscr{G}^{-p}h^m(X)$ is a subgroup of $E_\infty^{-p,m}$, the factor group being $=(\ker \alpha'')^{-p,m}$. This group, measuring the "deviation from convergence" of the spectral sequence, has been described in (6.11): it consists of those elements of $D^{-p,m}=h^m(X/X_{p-1})$ which are images under $\alpha^n\gamma(n)\colon h^{m-1}(X_{p-1+n}/X_{p-2})\to h^m(X/X_{p-1})$ for all n (note that for the index p in (6.9)–(6.11) we have taken here $-p$). A non-zero element of $(\ker \alpha'')^{-p,m}$ is thus an $x\in h^m(X/X_{p-1})$ such that there is, for each n, an element $y_n\in h^{m-1}(X_{p-1+n}/X_{p-2})$ mapped under the "coboundary" $\alpha^n\gamma(n)$ to x, but where no y_n can be chosen so as to come from a $y\in h^{m-1}(X/X_{p-2})$ under the restriction from X to X_{p-1+n}.

As before in 7.2, the filtration of $F^m=h^m(X)$ is not complete; the information given by the spectral sequence will refer to the factor group $h^m(X)/\tilde{h}^m(X)$ only, $\tilde{h}^m(X)$ being the subgroup consisting of elements of $h^m(X)$ which vanish on all skeleta X_p. The filtration of $h^m(X)/\tilde{h}^m(X)$ is then complete, and every element of $h^m(X)/\tilde{h}^m(X)$ is actually represented in E_∞ (but E_∞ may be "too big"). In the two convergence cases (*a*) and (*b*) above, where $E_\infty^{-p,m}\cong\mathscr{G}^{-p}h^m(X)$, the group $\tilde{h}^m(X)$ is automatically 0; this is obvious in case (*a*), and follows, in case (*b*), from the relation $\pi_r(B_m)=0$ for $r>m-M$: an element $x\in\tilde{h}^m(X)$ is represented by a map $g_x\colon X\to B_m$ which is null-homotopic on all skeleta, and it is possible to choose the nullhomotopies such that they are coherent, and hence yield a nullhomotopy of g_x.

If, in particular, a *representable cohomology theory* h^m *with dimension axiom* is considered, condition (*b*) is fulfilled. We thus have $E_\infty^{-p,m}\cong\mathscr{G}^{-p}h^m(X)$, the filtration being complete. Moreover $E_1^{-p,m}=H^p(X;h^{m-p}(S_o))=0$ for $m\neq p$, $=H^p(X;G)$ for $m=p$, and thus all differentials d_n, $n\geq 1$, vanish; thus $E_\infty^{-p,m}=H^p(X;G)$ for $m=p$, and $=0$ for $m\neq p$, which yields

$$h^m(X)\cong H^m(X;G);$$

Thus a representable cohomology theory with dimension axiom coincides with cellular cohomology on the category of all CW-complexes.

REMARK 7.4. Example 7.3 (representable cohomology functors) can, of course, be dealt with directly, without passing through the Federer spectral sequence 7.2. The arguments are then exactly the same, except that the general theory is applied to the (non-truncated) composition functor h^m, with $h^m(f)=\Pi_1(f,B_m)$, $h^m(X)=$ $=\Pi(X,B_m)$, for the given Ω-spectrum $\{B_m,\sigma_m\}$. In particular, the identification of $E_o^{-p,m}=\Pi(X_p/X_{p-1},B_m)$ with the cochain group $C^p(X;\pi_p(B_m))=C^p(X;h^{m-p}(S_o))$ uses the fact that the homotopy functor $\Pi(\ ,B_m)$ transforms an arbitrary wedge into a product; and the computation of $d_o^{-p,m}=$cellular coboundary is based on the homotopy addition theorem.

REMARK 7.5. In the case of an abstract cohomology theory h^m (not given by an

Ω-spectrum), fulfilling the exactness, excision and homotopy axioms, the spectral sequence can be set up for an arbitrary CW-complex X exactly as in 7.4. However, in order to identify $E_o^{-p,m}$ with the cochain group $C^p(X; h^{m-p}(S_o))$, we need an extra axiom saying that h^m of an arbitrary wedge of spheres of the same dimension is the product of the h^m of the spheres[1]). The identification of $d_o^{-p,m}$ with the cellular coboundary cannot be based on homotopy addition, but would use the degrees of the attaching maps of the cells of X.

Thus all the previous results hold for *a cohomology theory with the "special wedge axiom"* above on the category of all based CW-complexes, with the exception that nothing can be said about $\tilde{h}^m(X)$ and it is not always true that the filtration of $h^m(X)/\tilde{h}^m(X)$ is complete. It should, however, be remarked that according to Brown [4], a cohomology theory with a *general* wedge axiom is representable; thus in the latter case we can recover all the above results via the representation theorem. However, it is clear from examples that the *special* wedge axiom does not imply representability.[2])

Example 7.6. Let $\mathfrak{C}$ be the category of based topological spaces, and $f = \ldots j_{p+1} j_{p-1} \ldots$, where $f\colon X \to o$ is a composition of fibrations $j_p\colon X_{p-1} \to X_p$, with $X_p = o$ for $p \geq 0$. The factorization is *right-finite*. We have

$$\begin{aligned} j^p &= j_{-p}\colon X_{-p-1} \to X_{-p}, \\ \bar{g}^p &= g_{-p}\colon X \to X_{-p}, \\ g^p &= \bar{g}_{-p}\colon X_{-p-1} \to o. \end{aligned}$$

We take T to be the covariant truncated composition functor $\Pi_q(\mathrm{A},\ )$ of Example 4.8 and Example 7.1; we will not describe here explicitly the modifications for low values of q. We obtain a spectral sequence with

$$\mathscr{G}F \rightarrowtail E_\infty \twoheadrightarrow \ker \alpha''.$$

The groups in the Rees system are

$$E_o^{-p,q} = T_q(j_{-p}) = \Pi_q(A, j_{-p}) = \Pi_{q-1}(A, Y_p),$$

Y_p being the fibre of $j_{-p}\colon X_{-(p+1)} \to X_{-p}$;

$$D^{-p,q} = T_q(g_{-p}) = \Pi_q(A, g_{-p}) = \Pi_{q-1}(A, Z_p)$$

Z_p being the fibre of $g_{-p}\colon X \to X_{-p}$;

$$\bar{D}^{-p,q} = T_q(\bar{g}_{-p}) = \Pi_q(A, \bar{g}_{-p}) = \Pi_{q-1}(A, X_{-(p+1)}).$$

The differential $d_o^{-p,q}$ is the boundary homomorphism $\Pi_q(A, j_{-p}) \to \Pi_{q-1}(A, j_{-(p+1)})$

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[1]) No extra axiom is needed if X is finite in each dimension.

[2]) (Added in proof) We discuss this question in detail in Example 7.28 of [11].

of the triple sequence for $j_p j_{p-1}$; we can regard $d_o^{-p,q}$: $\Pi_{q-1}(A, Y_p) \to \Pi_{q-2}(A, Y_{p+1})$ as the ordinary homotopy boundary homomorphism of the fibration $\bar{Y}_p \to Y_p$ with fibre Y_{p+1} ($\bar{Y}_p$=fiber of $j_{-p}\, j_{-p-1}$).

Furthermore $F_q = \Pi_q(A, f) = \Pi_{q-1}(A, X)$, filtered by the images $(F_q)^{-p}$ of $\Pi_{q-1}(A, Z_p)$. The spectral sequence converges to $E_\infty \cong \mathscr{G}F$ (a) if the factorization is left-finite, i.e., if $g_{-p}: X \to X_{-p}$ is the identity for large p; (b) if A is a finite-dimensional CW-complex and the connectivity of Z_p tends to infinity with p. The case (a) is obvious. In the case (b) there is, for a given q, an integer $P(q)$ such that $D^{-p,q} = \Pi(A, \Omega^q Z_p) = 0$ for $-p < -P(q)$; hence the factorization is left-T-finite.

We thus obtain a spectral sequence starting with $E_o^{-p,q} = \Pi_{q-1}(A, Y_p)$, Y_p being the fibre of $j_{-p}: X_{-(p+1)} \to X_{-p}$, and converging to $E_\infty^{-p,q} = \mathscr{G}^{-p}\Pi_{q-1}(A, X)$, either

(a) in the case of a finite composition $X = X_{-N} \to \ldots \to X_{-1} = o$ of fibrations, or

(b) in the case of an infinite composition of fibrations, provided A is a finite-dimensional CW-complex and the connectivity of the fibres Z_p of $X \to X_{-p}$ tends to infinity with p.

The connectivity condition in (b) is fulfilled, e.g., if the composition of fiberings is the Postnikov decomposition of X (assumed, for simplicity, 1-connected), with

$$Y_p = \text{fiber of } j_{-p} = K(\pi_p(X), p).$$

Then

$$\begin{aligned} E_o^{-p,q} &= \Pi_{q-1}(A, K(\pi_p(X), p) = \Pi(A, \Omega^{q-1} K(\pi_p(X), p)) \\ &= \Pi(A, K(\pi_p(X), p-q+1) = H^{p-q+1}(A; \pi_p(X)), \end{aligned}$$

and the $d_o^{-p,q}$ are easily deducible from the Postnikov invariants of X. For a finite-dimensional CW-complex, $E_\infty^{-p,q} \cong \mathscr{G}^{-p}\Pi_{q-1}(A, X)$.

As another application of the present example one may also obtain the Adams spectral sequence [1], as described in [17]. In this case we again have $\ker \alpha'' = 0$ although neither condition (a) nor (b) is, in general, fulfilled.

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