Actions of R^n on manifolds

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We shall be concerned with smooth manifolds V^n, compact and without boundary, and actions of R^{n-1} on V all of whose orbits are $n-1$ dimensional. The rank of V is the largest k such that there is an action of R^k on V with k dimensional orbits; this is the same as the maximal number of linearly independent vector fields on V which pairwise commute. Elon Lima has proved the rank of S^3 is one [1], and the author proved the rank of $S^2 \times S^1$ is one [4]. One of our results is a generalization of Lima's theorem: the rank of a simply connected closed n manifold is less than $n-1$. Unfortunately, the author knows of no n-dimensional sphere whose rank is greater than one.

We also consider $M \times S^1$ where M is a closed two-dimensional manifold of genus greater than one. Our results are not complete; we do not know the rank of this space. We do prove, however, that if there is a locally free action of R^2 on $M \times S^1$, then it must have a torus orbit, embedded in a nontrivial way.[1])

Definitions and Notation

An action Φ of a Lie Group G on V is a differentiable map Φ: $G \times V \to V$ such that (i) $\Phi(gh, x) = \Phi(g, \Phi(h, x))$ for all $g, h \in G$ and $x \in V$, and (ii) $\Phi(e, x) = x$ for $x \in V$, e the identity of G. Given $x \in V$, the isotropy subgroup of x is $H_x = \{g \in G / \Phi_g(x) = x\}$, it is a closed subgroup of V. The orbit or leaf of x is $\{\Phi_g(x) / g \in G\}$. The action Φ induces a $1-1$ continuous map of G/H_x onto L_x, the orbit of x.

If $X_1, \ldots, X_k$ are vector fields on V, we say they pairwise *commute* if $[X_i, X_j] \equiv 0$ for all i and j. Let V be a closed manifold and $\xi^1, \ldots, \xi^k$ the integral curves of $X_1, \ldots, X_k$ respectively. We know $[X_i, X_j] \equiv 0$ is equivalent to $\xi^i_s \xi^j_t = \xi^j_t \xi^i_s$ for all real numbers s and t.

When $G = R^K$, an action of G on V is equivalent to K commuting vector field (we assume V is closed); the relation is

$$\Phi(t, x) = (\xi^1_{t_1} \circ \xi^2_{t_2} \circ \cdots \circ \xi^k_{t_k})(x), \qquad t = (t_1, \ldots, t_k) \in R^k.$$

We call Φ a *locally free* action if all the orbits are K-dimensional.

Suppose $n=3$ and $k=2$. The orbits of x are classified by their isotropy subgroups H_x and we have the following possibilities. If the dimension of H_x is two, then $H_x = R^2$ and $L_x = \dot{X}$. When H_x has dimension one we have $H_x = L + nv$, L a line through the origin and $v \in R^2$, $n = 0, \pm 1, \pm 2, \ldots$ L_x is then a line or circle (i.e., $1-1$ continuous

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1) Conversations with Elon Lima and André Haefliger were very useful in the preparation of this paper.

image of) depending on the direction of v. The case dimension $H_x=0$ gives three possible orbits. When $H_x=Z_u$, Z the group of integers, $u\in R^2$, we have $L_x=R^2$ or a cylinder depending on whether $u=0$ or $u\neq 0$. If $H_x=Zu+Zv$ with u and v independent, then H_x is a torus.

1. The Existence of Compact Leaves

THEOREM 1.1. (Reeb [2]). *Let V be a closed Riemannian manifold and ω a closed one form on V satisfying $\|\omega\|=1$. Let F be the foliation of V defined by $\omega=0$. Then the leaves of F are homeomorphic and if L is one leaf, there is a covering map $p: R\times L\to V$.*

Proof. Since $\|\omega\|=1$, the foliation is oriented, and we may choose a unit vector field on V orthogonal to the foliation.

The orthogonal trajectories to a leaf F are geodesics [3]. Let $\Psi_s(x)$ be a parametrization by arc length of the orthogonal trajectory through x. For each x, there is a neighborhood U of x, where we may define a smooth function $s(y)$ by $s(y)=$ the distance of the point y from the leaf containing x. Our assumptions imply $\omega=ds$ locally.

If L is a leaf of F and s a real number, ω vanishes on $\Psi_s(L)$. Thus Ψ_s carries leaves into leaves. The set $\{\Psi_s(L)|s\in R\}$ is open and closed in V, hence all of V. This proves the first assertion.

Let $x_0\in V$, and H be the subgroup of $\pi_1(V, x_0)$ of homotopy classes representable by closed curves h at x_0 such that

$$\int_h \omega = 0$$

Here we use the hypothesis $d\omega=0$.

Let W be the connected covering space of V over H. On W we have the one form $\omega^*=p^*\omega$ and a foliation F_0 defined by $\omega^*=0$. W inherits a Riemannian metric such that $\|\omega^*\|=1$; ω^* is never zero, and $d\omega^*=0$.

Let a be a closed curve in W based at some point in $p^{-1}(x_0)$. Since

$$\int_a \omega^* = \int_{pa} \omega$$

and pa represents an element of H, we have

$$\int_a \omega^* = 0.$$

It follows easily that the integral of ω^* about any closed curve in W is zero. Thus $\omega^*=df$ for some smooth function f on W. The level surfaces of f are precisely the leaves of F_0. Each orthogonal trajectory to F_0 is an embedding of R in W and each leaf meets an orthogonal trajectory in precisely one point. Hence W is homeomorphic to $R\times L_0$, where $L_0\in F_0$, and for each t, $t\times L_0$ corresponds to a leaf of F_0.

We observe that L_0 is homeomorphic to $p(L_0)=L$, a leaf of F. There is a map $L\to L_0$ defined as follows: fix $x_0\in L$ and $\bar{x}\in L_0$ such that $px=x_0$. For $x\in L$, let h be a path in L from x_0 to x. Lift h to a path a in L_0 starting at $\bar{x}$. We map $L\to L_0$ by sending x to $a(1)$, the endpoint of a. This map does not depend on the path h, since closed paths in L lift to closed paths in L_0. Thus V may be covered by $R\times L$.

THEOREM 1.2 (Sacksteder [5]). *Let Φ be a locally free action of R^{n-1} on a closed n manifold V, such that no orbit is compact. There is a Riemannian metric on V and a closed non-vanishing one form ω of norm one, such that the foliation defined by $\omega=0$ is the same as the foliation defined by Φ. This foliation admits a simple closed curve as an orthogonal trajectory.*

COROLLARY 1.3: *Let V be a closed n manifold with non-Abelian fundamental group. Then each locally free action of R^{n-1} on V has a non-simply connected leaf.*

Proof. Suppose the orbits of Φ are simply connected. Then theorems 1.1 and 1.2 imply V is covered by R^n and $H=\{[a]\in\pi_1(V)|\int_a\omega=0\}$ is isomorphic to $\pi_1(R^n)$ hence trivial. But H contains the commutator subgroup of V, hence $\pi_1(V)$ is abelian.

COROLLARY 1.4: *Let Φ be a locally free action of R^2 on $M\times S^1$ where M is a closed 2-dimensional manifold of genus greater than one. Then Φ has a compact orbit (a torus).*

Proof. Since $\pi_1(M\times S^1)$ is not abelian we know all the orbits of Φ cannot be R^2. If Φ has no compact orbit, all of the orbits are the one to one continuous image of $R\times S^1$, and each orbit is dense in $M\times S^1$. Let X and Y be linearly independent commuting vector fields on $M\times S^1$ such that X and Y span the orbits of Φ. Let $x_0\in V=M\times S^1$. The isotropy subgroup of R^2 at x_0 is a discrete group on one generator; hence, we may find real numbers a, b, c, d such that the vector fields $X'=aX+bY$, $Y'=cX+dY$ are linearly independent and the X' orbit through x_0 is a simple closed curve γ. Let ξt and η_τ be the integral curves of X' and Y'. Because X' and Y' commute, we have $\xi_t\eta_\tau=\eta_\tau\xi_t$ for all t and τ. Thus $\eta_\tau(\gamma)$ is also a simple closed curve for all τ. Since the Φ orbit of x_0 is dense in V, it follows from continuity that all the integral curves of X' are simple closed curves. Moreover, the foliation of V induced by Φ may be assumed oriented which implies the integral curves of X' have the same period. Consider the quotient space Y of V obtained by identifying each integral curve of X' to a point. Y is a closed two-dimensional orientable manifold. By choosing a non-zero normal vector field to the orbits of Φ we obtain a non-zero vector field on Y; hence Y must be a two-dimensional torus. But this means $M\times S^1$ is a circle bundle over a two torus which is easily seen to be a contradiction. Simply consider the homotopy exact sequence of this fibre bundle. Thus some orbit of Φ is compact.

THEOREM 1.4. *Let Φ be a locally free action of R^{n-1} on a closed n manifold V and assume Φ has no compact orbits. There is a covering map $p: R^{n-1} \times S^1 \to V$.*

Proof. We may apply 1.2 to obtain a metric on V and closed non-vanishing one form ω of norm one which defines the foliation induced by Φ. Let $j: I \to V$ be a parametrization by arc length of the closed orthogonal trajectory through x_0; i.e., $j(0)=j(1)=x_0, j(t_1)\neq j(t_2)$ if $t_1 \neq t_2, 0<t_1, t_2<1$ and $j(I)$ is orthogonal to Φ. It is no loss of generality to assume this orbit has length one.

Let L be the Φ orbit of x_0. By 1.1 we know V is covered by $R \times L$. If L is not simply connected, then $L=R^{n-i}\times T^{i-1}$ where T^{i-1} is the $i-1$ dimensional torus and $i>1$. In this case $R\times L$ is covered by $R^{n-1}\times S^1$. So we may assume L is the one to one continuous image of R^{n-1} which implies each orbit of Φ is of the same type. We state in [4] that these assumptions imply V is covered by $R^{n-1}\times S^1$. Since this was stated without proof, we give the proof here.

Let H be the subgroup of $\pi_1(V, x_0)$ generated by the homotopy class of j. Let W be the connected covering space of V over H with covering map p. We will prove W is homeomorphic to $R^{n-1}\times S^1$.

We may think of W as the quotient space of the space of paths $h: I\to V$ starting at x_0 where h_1 is identified with h_2 if $h_1(1)=h_2(1)$ and $h_1 h_2^{-1}$ represents an element of H. Parametrize j by arc length so that the distance of $j(t)$ to x_0 is t.

Define a path $h(\tau)$ at x_0 by $h(\tau)(t)=j(t\tau)$, $0\leq\tau\leq 1$. Let $U(\tau)=(h(\tau))=$ equivalence class of $h(\tau)$ in W. We have $U(0)=U(1)$ since $h(1)=j$, $h(0)=C_{x_0}=$ constant path at x_0, and $h(1)h(0)^{-1}=j$ represents an element of H. Also $U(\tau_1)\neq U(\tau_2)$ for $\tau_1\neq\tau_2$, $0<\tau_2, \tau_2<1$, since $h(\tau_1)\neq h(\tau_2)$. Hence U is a simple closed curve in W such that $pU=$j.

Let Φ_0 be a lifting of the action Φ to an action on W; that is, $p\Phi_0=\Phi(1\times p)$, $1=$ the identity map of R^{n-1}. The orbits of Φ_0 cover the orbits of Φ hence they are also the one to one continuous image of R^{n-1}. To complete the proof we will show each orbit of Φ_0 intersects the image of U is pre precisely one point.

Suppose some orbits A of Φ_0 meets U in two points $(h(\tau_1))$ and $(h(\tau_2))$. Let $\mu: I\to A$ be a path joining $(h(\tau_1))$ to $(h(\tau_2))$; $p\mu=\beta$ is a path from $j(\tau_1)$ to $j(\tau_2)$ contained in the orbit pA.

For $0\leq\tau\leq 1$, define $\eta(\tau)$: $I\to V$ by

$$\eta(\tau)(t)=\begin{cases} j(2t\tau_1), t\leq\frac{1}{2} \\ \beta(\tau(2t-1)), t\geq\frac{1}{2}\end{cases}$$

Then $\eta(0)=h(\tau_1)\circ C_{j(\tau_1)}$, $\eta(1)=h(\tau_1)\circ\beta$ so that $\eta h(\tau_1)^{-1}$ is homotopic to C_{x_0}. Let f be the path in W, $f(\tau)=(\eta(\tau))$. We have $pf(\tau)=\eta(\tau)(1)=\beta(\tau)$ and $f(0)=(\eta(0))=(h(\tau_1))$.Since $p\mu=\beta$ and $\mu(0)=(h(\tau_1))$, we have $\mu=f$; in particular $\mu(1)=f(1)$, $(h(\tau_2))=(\eta(1))=(h(\tau_1)\beta)$ so that $h(\tau_1)\beta h(\tau_2)^{-1}$ represents an element of H. Hence

$$\int_{h(\tau_1)\beta h(\tau_2)^{-1}} w$$

is an integer multiple of $\int_j w$. However,

$$\int_{h(\tau_1)\beta h(\tau_2)^{-1}} w = \int_{h(\tau_1)} w - \int_{h(\tau_2)} w + \int_\beta w = \tau_1 - \tau_2$$

i.e., $\int_\beta w = 0$ since β lies in one leaf. Consequently, $\tau_1 = \tau_2$ or $\tau_1 = 1$, $\tau_2 = 0$. In any case $(h(\tau_1)) = (h(\tau_2))$ and A meets U in at most one point.

Now we will show A meets U in at least one point. Let (h) be a point of A. We shall construct a map $G: I \times I \to V$ satisfying: $G(1, t) = h(t)$, $G(0, t) = h(a)(t)$ for some real number a, $G(s, 0) = x_0$ and $G(s, 1)$ is in the orbit through $h(1)$ for $0 \leq s \leq 1$. The map $s \to (G(s, \ ))$ is then a path in A joining (h) to $(h(a))$; where $G(s, \ )$ means the map $G(s, \ )(t) = G(s, t)$. Since (ha) is a point of V this will complete the proof. Observe that a curve h in V is homotopic to a curve consisting of segments such that each segment is an arc of an orthogonal trajectory or is entirely contained in one leaf. Therefore we may assume there exists numbers $0 = t_0 \leq t_1 \leq \cdots < t_k = 1$ such that for each i, the arc $h[t_i, t_{i+1}]$ is either a segment of an orthogonal trajectory or is contained in one leaf.

Let L be a leaf of Φ and $x \in L$; $C(t)$ a curve in L starting at x. The orthogonal trajectories are infinitely extendable, hence for any positive number s_0, the orthogonal trajectories of length s_0 along C define a map F: $I \times [0, s_0] \to V$ such that for fixed t, $F(t, s)$ is an orthogonal trajectory with $F(t, 0) = C(t)$, and $F(t, s)$ is the point a distance s from $C(t)$ along the orthogonal trajectory through $C(t)$. Moreover, the metric on V guarantees the points $F(t, s)$, for fixed s, are contained in the leaf through $F(0, s)$.

Now G is defined as follows. We may assume $h[t_0, t_1]$ is contained in the leaf L through x_0, and $h[t_1, t_2]$ is an orthogonal arc. Let C be the path $h[t_0, t_1]$ and s_0 the length of $h[t_1, t_2]$. Apply the last paragraph to obtain a map $F_1: I \times [0, s_0] \to V$ such that $F_1(0, s) = j(s)$, $F_1(1, s) = h(t_1 + s)$ and $F_1(t, s_0)$ is in the orbit through $j(s_0)$ for $0 \leq t \leq 1$. Repeat this construction with C the curve $F_1(t, s_0)$ followed by $h[t_2, t_3]$. Induction on k yields the desired map G. This completes the proof of 1.4.

COROLLARY 1.5. *Let V be a closed n manifold which cannot be covered by $R^{n-1} \times S^1$. Then a locally free action of R^{n-1} on V has a compact orbit.*

LEMMA 1.6. Let $D = \{(x_1, x_2, 0, \dots, 0) \in R^n | x_1^2 + x_2^2 \leq 1\}$, $\{e_1, \dots, e_{n-1}\}$ the $n-1$ frame on ∂D defined as follows: $e_1(x_1, x_2, 0, \dots, 0) = (-x_2, x_1, 0, \cdots, 0)$, $e_2 = (0, 0, 1, 0, \dots, 0), \dots, e_{n-1} = (0, 0, \dots, 0, 1)$. Then $\{e_1, \dots, e_{n-1}\}$ does not extend to an $n-1$ frame on D.

The frame $\{e_1, \dots, e_{n-1}\}$ represents the nonzero element of $\pi_1(SO(n))$. This is proved in Chevalley's book on Lie Groups.

THEOREM 1.7. *Let V be a simply connected closed n manifold. The rank of V is less than $n-1$.*

Proof. The case $n=3$ has been proved by Lima [1], and $n=4$ is trivial since a simply connected 4 manifold does not admit a foliation of codimension one; it does not admit a nonzero vector field. So we assume $n\geq 5$.

Let Φ be a locally free action of R^{n-1} on V. According to 1.5, Φ has a torus orbit T. Since V is simply connected, $i: T\subset V$, induces the zero homomorphism. Thus there is a simple closed curve C on T which bounds an embedded two-dimensional disk D in V such that D is transverse to T, (here we use $n\geq 5$). But this contradicts 1.6, (cf. [1]).

2. Locally Free Actions of R^2 on $M \times S^1$

(2.1) *Let D be a two-dimensional disk with k contours in the interior of D. Let $V=D\times I$ and S be an embedded sphere in V. Then S bounds an embedded ball.*

Proof. For $k=0$ this is Schoenflies Theorem. We consider the case $k=1$. Let C be an embedding of [0,1] in D with one endpoint on ∂D, the other on the contour, and interior $C\subset$interior D. If $S\cap A\neq\Phi$, $A=C\times I$, then we may cut V along A to obtain a 3 ball; this is the case $k=0$. Assume then, that $S\cap A\neq\Phi$ and the intersection is transverse. This is no loss of generality since S may be approximated by an embedded sphere which is transverse to A and then there is a diffeomorphism of V sending one sphere onto the other. Let $a_1 \ldots, a_k$ be the simple closed curves in $S\cap A$. Choose a_j so that a_j bounds a disk E on S and E contains no a_i in its interior. A is homeomorphic to $I\times I$ so a_j bounds a disk F on A. Consider the sphere $E\cup F$. For our purposes this sphere is disjoint from A, i.e., $E\cup F$ bounds a ball B in V. Now by an isotopy of B across A we obtain a sphere S_0 which intersects A in the curves $a_1\cup\ldots\cup \hat{a}_j\cup\ldots\cup a_k$ (cf. [4] for details). Continuing we see S is isotopic to a sphere which does not intersect A, hence bounds a ball. The general case is just as easy.

Suppose there are k contours with $k>1$. Let C be an embedding of I in D with both endpoints on distinct contours and interior $C\subset$interior D. If $S\cap A=\Phi$, $A=C\times I$, then by cutting V along A we reduce the problem to $k-1$ contours. Otherwise we take the intersection to be transverse and displace S off A as above.

(2.2) *Let M be a closed two-dimensional orientable manifold of connectivity $h>1$. Let S be a sphere embedded in $M\times S^1$. Then S bounds an embedded ball in $M\times S^1$.*

Proof. Let $a_1,\ldots, a_k$, $k=(h+1)/2$ be simple closed curves on M as indicated in figure 1.

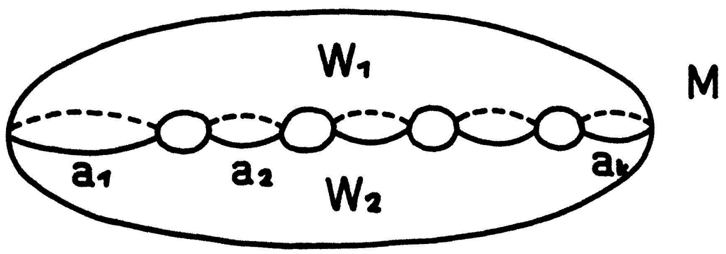

Fig. 1

Denote by $A_i = a_i \times S^1$, and $A = A_1 \cup \cdots \cup A_k$. A separates V into two connected components E_1 and E_2; $E_1 = W_1 \times S_1$, $E_2 = W_2 \times S_1$, where W_1, W_2 are the connected components of $M - (a_1 \cup \ldots \cup a_k)$. W_1 and W_2 are disks with $k-1$ contours. We may think of $M \times S^1$ as the quotient space of $M \times I$ where $(x, 0)$ is identified with $(x, 1)$, and we identify M with $M \times 0 \in M \times S^1$.

Suppose S is embedded in V so that S is disjoint from M. If S is also disjoint from A then S is contained in E_1 or E_2. Assume $S \subset E_1$. We have $E_1 = W_1 \times I$ where $W_1 \times 0$ is identified with $W_1 \times 1$. Since $S \cap M = \Phi$, S is really contained in a subspace of V homeomorphic to $W_1 \times I$ and by (2.1), S bounds a ball in this subspace, hence in V. Otherwise we may assume S meets A transversally. Let b be a simple closed curve in $S \cap A$ such that b bounds a disk E on S whose interior is disjoint from A. Since $S \cap M = \Phi$, b bounds a disk F contained in $a_i \times I$ for some i. Then $F \cup E$ is a spheric contained in $W_1 \times I$ or $W_2 \times I$ hence $F \cup E$ bounds a ball. Now by displacing E across this ball we see that S is isotopic to a sphere having one less circle of intersection with A. Continuing in this way, we obtain a sphere isotopic to S whose intersection with $A \cap M$ is void hence this sphere bounds a ball and S also bounds a ball.

It remains to consider the case $S \cap M \neq \Phi$. Let S meet M transversally, and b be a simple closed curve in $S \cap M$ which bounds a disk E on S whose interior is disjoint from M. Since the inclusion of M in V induces a monomorphism of $\pi_1(M)$ into $\pi_1(V)$, b must be null homotopic on M hence b bounds a disk F on M. The sphere $E \cup F$ is (for all practical purposes) disjoint from M hence bounds a ball in V. Then S may be displaced in V to a sphere having one less intersection curve with M and iterating the process removes S from M entirely. This completes the proof of 2.2.

(2.3) *Let T be a torus embedded in the interior of $M \times I$ where M is a closed orientable two-dimensional manifold of genus greater than one. Then T separates $M \times I$ into two connected components. Moreover $M \times 0$ and $M \times I$ are contained in the same connected component.*

Proof. Let i be the inclusion map of T into $M \times I$. The map $i_*: H_2(T) \to H_2(M \times I)$ is zero since $M \times I$ may be retracted onto $M \times 0 = M$, and M has genus greater than one so any map of T to M has degree zero. We must compute $H_0(M \times I - T)$ (all homology and cohomology groups are with Z_2 coefficients). By Lefshetz Duality $H_0(M \times I - T)$ is isomorphic to $H^3(M \times I; T)$. Consider the exact sequence in cohomology:

$$H^2(M \times I) \to H^2(T) \to H^3(M \times I; T) \to H^3(M \times I) \to H^3(T)$$

The first map is zero since it is the transpose of i_* and the last group is zero. The second and foürth groups are Z_2, hence $H^3(M \times I; T) = Z_2 + Z_2$. This proves the first part of 2.3.

Now we will prove $M \times 0$ and $M \times 1$ are in the same component. Let a_1 and a_2 be simple closed curves on $M \times 0$, as in 2.2.

Let T intersect $a_1 \times I$ and $a_2 \times I$ transversally. If T is disjoint from $a_1 \times I$ or $a_2 \times I$ then we may find a curve from $M \times 0$ to $M \times 1$ not meeting T. Assume then that $T \cap (a_1 \times I) = b_1 \cup \ldots \cup b_k$, $T \cap (a_2 \times I) = c_1 \cup \ldots \cup c_l$, where the b_i's and c_j's are pairwise disjoint simple closed curves.

If each b_i, or each c_j, is null homotopic in $M \times I$, then we can join $a_1 \times 0$ to $a_1 \times 1$ (or $a_2 \times 0$ to $a_2 \times 1$) by arcs in $a_1 \times I - T$ (or $a_2 \times I - T$). So we may suppose there is a b_i and c_j such that b_i and c_j are not homotopically trivial. Clearly b_i is homotopic to a_1 and c_j to a_2. Now b_i and c_j are disjoint simple closed curves on the torus T and both represent generators of $\pi_1(T)$, hence b_i and c_j are the boundary circles of a cylinder on T. This implies a_1 is homotopic to a_2 in M which is a contradiction. Thus $M \times 0$ and $M \times 1$ are in the same connected component of $M \times I - T$.

(2.4) *Let T be a torus embedded in $M \times S^1$ where M is a closed orientable two manifold of genus greater than one. If $T \cap (M \times x_0) = \Phi$ for some $x_0 \in S^1$, then T separates $M \times S^1$ into two connected components A and B. If h and g are the inclusion maps of T into A and B respectively, then $h_*: \pi_1(T) \to \pi_1(A)$ or $g_*: \pi_1(T) \to \pi_1(B)$ has a nonzero kernel.*

It remains to establish the latter assertion of 2.4. First we need an algebraic fact whose proof may be found in Kurosh, volume two.

(2.5) *Let G_1, G_2 and H be groups such that there are subgroups H_1 and H_2 of G_1, G_2 respectively each isomorphic to H. Denote by $G_1 {*H} G_2$ the free product of G_1 and G_2 with H amalgamated. Every element of $G_1 {*H} G_2$ can be written uniquely in the form*

$$h\, \bar{a}_1\, \bar{a}_2 \ldots \bar{a}_n$$

where $h \in H$, $n \geq 0$, $\bar{a}_i$ is a coset representative, other than the unit element, of a right coset of H_i in G_i, $i = 1,2$, and adjacent representatives $\bar{a}_i$, $\bar{a}_{i+1}$, $i = 1, \ldots, n-1$, lie in distinct G_i's.

From this it follows easily that the center of $G_1 {*H} G_2$ is contained in H.

Proof of 2.4. Suppose that h_* and g_* are both monomorphisms. Let $G_1 = \pi_1(A)$, $H_1 = h_* \pi_1(T)$, $G_2 = \pi_1(B)$, $H_2 = g_* \pi_1(T)$ and $H = \pi_1(T)$. According to Van Kampen's Theorem and (2.3) we have

$$\pi_1(M \times S^1) = G_1 {*H} G_2 .$$

Since $\pi_1(S^1)$ is contained in the center of $\pi_1(M \times S^1)$ and the center of $G_1 {*H} G_2$ is contained in H, we have $\pi_1(S^1)$ contained in $\pi_1(T)$. But T is disjoint from $M \times x_0$ for some $x_0 \in S^1$, hence no curve on T can represent a generator of $\pi_1(S^1)$. Thus h_* or g_* is not a monomorphism.

(3.1) *Let Φ be a locally free action of R^2 on $M \times S^1$ with M a closed two manifold* *of genus greater than one. Then Φ has a compact orbit, and each compact orbit of Φ intersects M.*

This follows immediately from 1.4, 2.4 and [1].

(3.2) *If T is a compact orbit of Φ, then $T \cap M$ contains a curve which is a generator of $\pi_1(T)$.*

Proof. Assume T is transverse to M and each curve in $T \cap M$ is trivial in $\pi_1(T)$. Let b be such a curve. Then b bounds a disk E on T, hence also bounds a disk F on M and the sphere $E \cup F$ bounds a ball in $M \times S^1$ by 2.2. Thus the intersection curve b may be removed from m by an isotopy of $M \times S^1$ and all intersection curves may be so removed. This gives rise to a new action which is locally free and has a compact orbit disjoint from M. But 3.1. contradicts this.

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Received December 18, 1965