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On a class of conformal metrics, with application to differential geometry in the large

by ROBERT FINN

To CHARLES LOEWNER, on the occasion of his seventieth birthday, and in token of my esteem.

1. Introductory Remarks

One of the most striking justifications for the concept of a *complete, open two-dimensional surface*, as introduced by HOPF and RINOW [1], is the theorem of COHN-VOSSEN [2], that if $\mathfrak{S}$ is such a surface, and if the Gaussian curvature K is absolutely integrable over $\mathfrak{S}$, then

$$C \leq 2\pi\chi \tag{1}$$

where C is the *curvatura integra*, or total curvature of $\mathfrak{S}$, and χ is the EULER Characteristic[1,2]. The theorem is best illustrated by some simple examples:

i) $\mathfrak{S}$ is an infinite cylinder. Then $C = 0, \chi = 0$, so that equality is attained.

ii) $\mathfrak{S}$ is a semi-infinite cylinder, closed at one end by a spherical cap. Then $C = 2\pi$, $\chi = 1$. Again equality is attained.

iii) $\mathfrak{S}$ is a circular cone of vertex half-angle α. Then $C = 2\pi(1 - \sin\alpha)$, $\chi = 1$. Strict inequality prevails.

In the last example, the loss of equality is not caused by the singularity at the vertex, as C and χ remain unchanged if $\mathfrak{S}$ is smoothed near its vertex. In fact, a little reflection shows that equality occurs only under special conditions, while in general there is a wide divergence between the two sides of COHN-VOSSEN's inequality.

One of the objects of this paper is to characterize, in terms of elementary intrinsic geometrical quantities on $\mathfrak{S}$, the difference between the two sides of (1). To fix the ideas, consider example iii) above. Let $\mathfrak{L}(h)$ be the length of a circular section at distance h from the vertex along the axis, and let $\mathfrak{A}(H; h)$ be the surface area on $\mathfrak{S}$ bounded between two such sections, $H < h$. One computes $\mathfrak{L}(h) = 2\pi h \tan\alpha$, $\mathfrak{A}(H; h) = \pi(h^2 - H^2)\sec\alpha\tan\alpha$.

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1) A surface $\mathfrak{S}$ is *complete* in the sense of HOPF-RINOW if every divergent path on $\mathfrak{S}$ has infinite length. A path on $\mathfrak{S}$ is said to be *divergent* if it is the topological image $p = p(t)$ of a half-open interval $0 \leq t < 1$, and if $p(t)$ lies outside any given compact set on $\mathfrak{S}$ for all t sufficiently close to 1.

2) COHN-VOSSEN's theorem was later improved and clarified in important ways by HUBER [3].

Thus, $\frac{\mathfrak{L}^2(h)}{4\pi\mathfrak{A}(H;\,h)} = \sin\alpha + o(1)$, as $h\to\infty$. Hence, setting $\nu = \lim_{h\to\infty}\frac{\mathfrak{L}^2(h)}{4\pi\mathfrak{A}(H;\,h)}$, we obtain $C = 2\pi(1 - \sin\alpha)$, $\chi = 1$, $\nu = \sin\alpha$, so that

$$C = 2\pi(\chi - \nu)$$

in this case.

Another example is obtained by rotating about an axis a circular arc of radius 1 whose center lies at distance 2 from the axis. If the arc is allowed to turn through the angles α_1, α_2 on each side of a perpendicular to the axis from the center, and is then continued indefinitely by straight lines, the total curvature of the resulting surface of revolution is easily computed. If r denotes distance to the axis and ϑ is the angle through which the arc has turned, then $d\mathfrak{A} = 2\pi r\, d\vartheta$, $K = -\frac{\cos\vartheta}{r}$, $K d\mathfrak{A} = -2\pi\cos\vartheta\, d\vartheta$, hence $C = -2\pi(\sin\alpha_1 + \sin\alpha_2)$. Computing the limits ν_1, ν_2 on each side, we have as before, $\nu_1 = \sin\alpha_1$, $\nu_2 = \sin\alpha_2$. Here $\chi = 0$, hence we find the relation

$$C = 2\pi(\chi - \Sigma\nu_j). \tag{2}$$

In § 2 I shall show that a relation of the form (2) holds for any abstract surface of finite connectivity on which the metric has a property of rotational symmetry near each ideal boundary component, and for which C exists (finite or infinite) in the sense of a principal value. I had hoped, by suitably defining the $\{\nu_j\}$, to obtain (2) without a symmetry requirement, for the surfaces treated by COHN-VOSSEN and HUBER in their studies leading to (1). I am at present able to do so only under the presumably superfluous hypothesis that the region $K > 0$ has compact support on $\mathfrak{S}$. There is, however, an interesting intermediate case which contains many of the essential features of the problem. This is the case of *normal metrics*, which are conformal metrics $ds = e^{u(z)}|dz|$, such that when $u(z)$ is represented as the potential of a mass distribution, the additive harmonic function assumes a particular degenerate form. Normal metrics are defined in § 3. Their significance for the problem at hand is illustrated in § 4, in which the theory is worked out completely in the greatly simplified case for which the measure has compact support.

In § 5 inequalities from below for length and area are derived, which hold for an arbitrary conformal metric over a plane region. They are very simple and are based on known techniques, but – so far as I could determine – they are not available in the literature. Some of this material is essential for subsequent sections, in which normal metrics are studied under the single assumption that the absolute variation of the measure is finite. It is necessary to extend the definition of the $\{\nu_j\}$ to the case of such a metric defined in a neighborhood of an isolated boundary point p. This is done by enclosing p in concentric circumferences Γ, γ. $\mathfrak{L}(\gamma)$ is then taken to be the length, in the metric, of γ, and $\mathfrak{A}(\Gamma; \gamma)$ the area, in the metric, of the enclosed annular region. With this definition, asymptotic formulas for $\mathfrak{L}(\gamma)$, $\mathfrak{A}(\Gamma; \gamma)$ are derived, and the relation (2) is then obtained for this case in the full generality in which (1) was derived in the papers of COHN-VOSSEN and of HUBER (Theorem 12).

When $\chi = 1$, (2) exhibits a suggestive formal similarity with an isoperimetric inequality, due to HUBER [4]. This is discussed in § 6, following the statement of Theorem 12.

There is also a connection with the theory of minimal surfaces. R. OSSERMAN has shown [7, p. 358] that if $\mathfrak{S}$ is a complete minimal surface of total curvature C and EULER Characteristic χ, then[3]) $C = 2\pi[\chi - \sum_1^k (\eta_j - 1)]$, where η_j is the order of the pole of a certain analytic differential ω at the conformal image p_j (necessarily a point) of a boundary component, and k is the number of such components. By proving that $\eta_j \geq 2$, OSSERMAN obtained the relation $C \leq 2\pi(\chi - k)$. The results of the present paper yield the geometric interpretation, $\eta_j = 1 + \lim_{\gamma \to p_j} \frac{\mathfrak{L}^2(\gamma)}{4\pi\mathfrak{A}(\Gamma; \gamma)}$, for the quantities $\{\eta_j\}$ of OSSERMAN.

General estimates on length and stretching ratio near the $\{p_j\}$ are given under varying assumptions. They will be found in §§ 6, 7 and 8. A particular consequence is the demonstration that, in a certain loose sense, the $\{\gamma_j\}$ are approximate geodesic circles in the metric. This result permits an *a posteriori* interpretation of (2) in terms of the explicitly given geometry. It seems, however, remarkable that the $\{\gamma_j\}$, which evidently play a distinguished role in the metric theory of complete surfaces, are themselves completely characterized by the conformal geometry, in the determination of which the metric properties are of subsidiary importance.

The estimates of §§ 6, 7, 8 show also that for complete metrics of the type considered, the asymptotic growth of the length ratio is – at least in an average sense – characterized completely by the quantities $\{\nu_j\}$ (Theorem 11). Under a hypothesis on the rate of decay of curvature at a boundary component, this characterization can be made considerably more precise (Theorem 14), and under assumptions on the asymptotic sign of the curvature, pointwise bounds from either side can be obtained (Theorem 16). However, local estimates both above and below cannot in general be expected under assumptions of this sort, as is pointed out in § 8.

[3]) To avoid confusion in notation, I have replaced OSSERMAN's symbol ν_j by η_j.

In § 9 I apply the method to general abstract surfaces of finite connectivity, which are complete and have finite *curvatura integra*. The definition and essential properties of such surfaces are only briefly indicated; for a more extended discussion, cf. [3] and the references cited in that work. For purposes of grasping the essential content of the result, it suffices to envisage a surface embedded in 3-space, which is of finite connectivity and which may have a number of branches extending to infinity. I reduce the study of such surfaces to previous considerations by showing that a neighborhood of infinity on each branch can be mapped conformally to a plane domain so as to yield a normal metric at the image of the ideal boundary. I am unfortunately able to do this only under an additional supposition, as indicated above, and in this sense my result cannot yet be considered to be established in its natural context.

As corollaries of the method, independent demonstrations are obtained, in the cases considered here, of certain of HUBER's results, notably his Theorems 1 and 15 in [3]. HUBER's results hold, however, also in a more general situation.

It seems likely that the quantities $\{\nu_j\}$ are extremal in the sense that the corresponding inferior limit taken for any other system of curves surrounding p_j would be not less than ν_j. I have, however, not proved this. There are also evidently connections with extremal length, which should be investigated.

In this paper I have deliberately avoided dwelling on questions of local regularity, and I have chosen to assume at each step that all functions which enter have the smoothness properties indicated in the context. All results hold, however, under the conditions assumed by HUBER [3, p. 16], and an inspection of the text will convince the reader that there is no danger in applying the results to certain more general situations, e.g., to polyhedral surfaces. Apparently this does not begin to exhaust the possibilities, as is indicated by the recent profound investigations of REŠETNJAK (see, e.g., [5]) on the existence of isothermal parameters in a general case.

My thanks are due many of my colleagues for informal conversations which have contributed much to my understanding in a subject with which I was initially unfamiliar. I am indebted especially to Professor P. MALLIAVIN for a suggestion which has led to a significant improvement of some of my original results.

2. Rotationally Symmetric Metrics

Any function $u(x, y)$ defined over a region $\mathfrak{G}$ in the (x, y) plane determines a conformal metric

$$ds^2 = e^{2u}(dx^2 + dy^2) = e^{2u}|dz|^2. \tag{3}$$

If $\mathfrak{G}$ is of finite connectivity, one may always suppose that $\mathfrak{G}$ lies interior to a circumference Σ_0 about the origin (which might consist of the single point at infinity) and that $\mathfrak{G}$ is bounded by Σ_0 and by n other circumferences (or points) $\Sigma_1, \ldots, \Sigma_n$. The metric (3) will be said to be *rotationally symmetric* if there is a neighborhood Δ_j of each Σ_j such that in a system of polar coordinates ϱ, ϑ with origin at the center of Σ_j, $u(x, y)$ is independent of ϑ in Δ_j.

For points near the Σ_j, it is easy to compute the various geometrical quantities associated with a rotationally symmetric metric. For the *curvatura integra* of the annulus bounded by circular arcs Γ_0, γ_0 of radii r, ϱ, which lie in Δ_0 and are concentric with Σ_0, one has

$$C(\Gamma_0, \gamma_0) = -2\pi \int_r^\varrho \Delta u \cdot \varrho d\varrho = -2\pi \int_r^\varrho (\varrho u_\varrho)_\varrho d\varrho = -2\pi(\varrho u_\varrho(\varrho) - r u_\varrho(r)) .$$

Let us suppose that $C(\Gamma_0, \gamma_0)$ remains bounded from below as ϱ tends to the radius ϱ_0 of Σ_0. Thus, there holds $\overline{\lim}_{\varrho \to \varrho_0} \varrho u_\varrho = \Phi_0 \neq +\infty$. It follows that if $\varrho_0 \neq \infty$, then $u < M < \infty$ near Σ_0, so that $\int_r^{\varrho_0} e^u d\varrho < \infty$. A similar discussion applies to each of the other boundary components Σ_j, as one sees by transforming by inversion with respect to the center of Σ_j. We conclude:

Theorem 1. *Let* $u(x, y)$ *define a rotationally symmetric complete metric* (3) *in a region* $\mathfrak{G}$ *bounded by a finite number of points or circular arcs* $\{\Sigma_j\}$. *Let* $\{\gamma_j\}$ *be a system of concentric arcs tending to the* $\{\Sigma_j\}$, *and let* $\mathfrak{G}_\gamma$ *be the corresponding subregion determined by the* $\{\gamma_j\}$. *Suppose that the curvatura integra* $C(\mathfrak{G}_\gamma) = C_\gamma > C > -\infty$ *as* $\gamma_j \to \Sigma_j$, $j = 0, \ldots, n$. *Then the metric* (3) *is conformally parabolic, so that each* Σ_j *is a single point*[4]).

Let us now discard the assumption $\Phi_0 \neq +\infty$, and assume instead that $\lim_{\varrho \to \varrho_0} \varrho u_\varrho = \Phi_0$ exists and that the metric is complete and conformally parabolic. In particular, Σ_0 as defined above is the point at infinity. One sees immediately that if $\Phi_0 = -\infty$, the metric could not be complete. If $\Phi_0 \neq \pm\infty$, then $e^u = A\varrho^{\Phi_0 + o(1)}$ some constant A. Hence, completeness of the metric implies $\Phi_0 \geq -1$. Repeating this discussion for each boundary point, we find $\Phi_j \geq 1$ if $j \neq 0$, and *we are led to the* Cohn-Vossen *relation*, $C \leq 2\pi\chi$, *where* χ *is the* Euler *Characteristic of* $\mathfrak{G}$.

The length of the circumference γ_0 in the given metric is $\mathfrak{L}_0(\varrho) = 2\pi\varrho e^u$. The area of the annular region between Γ_0 and γ_0 is

$$\mathfrak{A}_0(r; \varrho) = 2\pi \int_r^\varrho \sigma e^{2u} d\sigma .$$

[4]) cf. Huber [3], Theorem 15, which this result overlaps. The essential character of the result is really local, and shows that each boundary component at which the metric is complete must degenerate to a point.

We have, if $\Phi_0 \neq \pm\infty$, $\dfrac{d}{d\varrho}\,\mathfrak{L}_0^2 = 4\pi(1 + \Phi_0 + o(1))\,\dfrac{d}{d\varrho}\,\mathfrak{A}_0$.

Since $\dfrac{d}{d\varrho}\,\mathfrak{A}_0 \neq 0$, we may write

$$\frac{d\,\mathfrak{L}_0^2}{d\,\mathfrak{A}_0} = 4\pi(1 + \Phi_0 + o(1)) \tag{4}$$

from which follows, since $\Phi_0 \geq -1$, that whenever $\mathfrak{A}_0 \to \infty$, in particular whenever $\Phi_0 > -1$, the limit $\nu_0 = \lim\limits_{\gamma_0 \to \Sigma_0} (\mathfrak{L}_0^2/4\pi\mathfrak{A}_0)$ exists, and $\nu_0 = (1 + \Phi_0)$. On the other hand, if $\Phi_0 = -1$ and $\mathfrak{A}_0(r;\varrho) \to \mathfrak{A}_0^* \neq \infty$, (4) implies that $\mathfrak{L}_0(\varrho) \to \mathfrak{L}_0^* \neq \infty$. But

$$\mathfrak{A}_0\,(r;\varrho) = \frac{1}{2\pi}\int_r^{\varrho} \frac{1}{\sigma}\,\mathfrak{L}_0^2(\sigma)\,d\sigma$$

hence $\mathfrak{L}_0^* = 0$, the limit ν_0 exists also in this case, and $\nu_0 = (1 + \Phi_0) = 0$.

Similarly one obtains the limit $\nu_j = \Phi_j - 1$ for each other boundary point Σ_j. Referring back to the determination of the *curvatura integra* in terms of the quantities $\{\Phi_j\}$ and noting that if one of the $\{\Phi_j\}$ is infinite, then $C = -\infty$, and we find the following result:

Theorem 2. *Let $u(x, y)$ define a rotationally symmetric complete metric* (3) *in a region $\mathfrak{G}$ bounded by a finite number of points. Let Γ_j, γ_j be circular arcs centered at the boundary point Σ_j, let $\mathfrak{A}_j$ be the area of the corresponding annular region and let $\mathfrak{L}_j$ be the length of γ_j in the given metric. Let $\mathfrak{G}_\gamma$ be the subregion defined by the $\{\gamma_j\}$ and $C(\mathfrak{G}_\gamma)$ the corresponding curvatura integra. Suppose $C(\mathfrak{G}_\gamma) \to C$ (finite or infinite) as each $\gamma_j \to \Sigma_j$. Then $\nu_j = \lim\limits_{\gamma_j \to \Sigma_j} \dfrac{\mathfrak{L}_j^2}{4\pi\mathfrak{A}_j}$ exists for each j, and $C = 2\pi(\chi - \sum\limits_0^n \nu_j)$ in the sense that $\Sigma\nu_j = \infty$ whenever $C = -\infty$.*

Remark. The significance of the hypothesis $C(\mathfrak{G}_\gamma) \to C$ is made evident by considering a surface obtained by revolving a curve $x = x(z)$ about the z-axis in (x, y, z) space. By introducing small irregularities in the function $x(z)$, the area of any part of the surface can be made arbitrarily large, without appreciably changing the circumference of any section. In this way, any such surface can be modified so that $\nu_j = 0$, each j, while $C(\mathfrak{G}_\gamma)$ will remain bounded without approaching a limit.

In the above result, the assumed completeness and parabolicity of the metric were used only to show that $\Phi_0 \geq -1$ and $\Phi_j \geq 1$, $j \neq 0$, from which followed the formulae for the $\{\nu_j\}$. It is possible by the same methods to give an analogous result under much more general conditions.

Theorem 3. *Let* $u(x, y)$, $\mathfrak{G}$, $\{\gamma_j\}$, $\{\Sigma_j\}$, $\mathfrak{G}_\gamma$, C_γ *be as in Theorem 1. Suppose* $C_\gamma \to C$ *(finite or infinite) as all* $\gamma_j \to \Sigma_j$. *Then* $\eta_j = \frac{1}{4\pi} \lim\limits_{\gamma_j \to \Sigma_j} \frac{d\mathfrak{L}_j^2}{d\mathfrak{A}_j}$ *exists for each* j, *and* $C = 2\pi(\chi - \sum\limits_{0}^{n} \eta_j)$.

Here there is no restriction on the sign or finiteness of C or of the $\{\eta_j\}$. It should be noted, however, that $\eta_j \geq 0$ whenever $\mathfrak{A}_j \to \infty$.

It is instructive to compute the geodesic curvature k of the radial and circumferential lines in a complete rotationally symmetric metric for which $C_\gamma \to C \neq \infty$ as all $\gamma_j \to \Sigma_j$. Evidently, $k = 0$ on each radial line $\vartheta = \text{const.}$, near Σ_j. From the relation

$$k = e^{-u}\left(k_e + \frac{\partial u}{\partial n}\right) \tag{5}$$

where k_e is the Euclidean curvature in the z-plane (see, e.g., [3, p. 13]), we find for the geodesic curvature of γ_0,

$$k_{\gamma_0} = [\nu_0 + o(1)]\,\varrho^{-[\nu_0 + o(1)]}$$

with a similar result for each γ_j. Hence:

Theorem 4. *In a rotationally symmetric metric, all radial lines are geodesics near the* $\{\Sigma_j\}$. *If the metric is complete, and if* $C_\gamma \to C \neq \pm\infty$ *as* $\gamma_j \to \Sigma_j$, *then near any boundary point for which* $\nu_j \neq 0$, *the circumferential lines behave asymptotically as geodesics.*

Remark 1. Whenever $C < 2\pi\chi$, at least one of the $\{\nu_j\}$ must differ from zero.

Remark 2. Also in the case $\nu_j = 0$ it is possible to give conditions ensuring that the $\{\gamma_j\}$ are asymptotically geodesic; however, the requirement $\nu_j \neq 0$ cannot simply be eliminated, as one sees from the example of the complete conformal metric

$$ds = \frac{|dz|}{(|z| + 1)\log(|z| + 2)}, \text{ for which } k \to -1 \text{ (cf. Huber [3, p. 61]).}$$

The applications of the above results to surface theory are immediate, as any surface of revolution[5]) can be mapped conformally to the z-plane so as to yield

[5]) This example is chosen in the interest of simplicity. It would be possible by these methods to obtain an analogous result for any abstract surface which can be realized by a surface having a finite number of ideal boundary components, near each of which a rotationally symmetric metric is prescribed (cf. the considerations in § 8).

a rotationally symmetric metric. Consider such a surface $\mathfrak{S}: r = f(\zeta)$, $r^2 = \xi^2 + \eta^2$, defined in the interval $-\infty \leq A_1 \leq \zeta < A_2 \leq \infty$, and such that $f(\zeta) \neq 0$ in the open interval $A_1 < \zeta < A_2$. On any such surface one has either $\chi = 1$ or $\chi = 0$. For the *curvatura integra* $C(\zeta_1, \zeta_2)$ of the part of $\mathfrak{S}$ for which $\zeta_1 < \zeta < \zeta_2$, there holds $-2\pi \leq C \leq 4\pi$. Hence, by Theorem 1, *every complete surface of revolution is conformally parabolic.*

The hypothesis of completeness is evidently verified in the simply-connected case ($\chi = 1$) if $A_2 = \infty$, and in the doubly-connected case ($\chi = 0$) if $A_1 = -\infty$, $A_2 = +\infty$.

In the mapping to a rotationally symmetric metric, the circles $\zeta =$ const. correspond to the circles $\varrho =$ const. The length of such a circle is $\mathfrak{L}(\zeta) = 2\pi f(\zeta)$, and the area between ζ_0 and ζ is $\mathfrak{A}(\zeta) = 2\pi \int_{\zeta_0}^{\zeta} f\sqrt{1 + f'^2}\,|d\zeta|$. *By Theorem* 2, *if* $\mathfrak{S}$ *is simply connected and complete, and if* $C(\zeta) \to C$ *as* $\zeta \to A_2$, *then* $\nu = \lim_{\zeta \to A_2} \dfrac{\mathfrak{L}^2}{4\pi\mathfrak{A}}$ *exists, and* $C = 2\pi(1 - \nu)$. *If* $\mathfrak{S}$ *is doubly connected and complete, and if* $C(\zeta_1, \zeta_2) \to C$ *as* $\zeta_1, \zeta_2 \to A_1, A_2$, *then the corresponding limits* ν_1, ν_2 *exist, and* $C = -2\pi(\nu_1 + \nu_2)$. *By Theorem* 4, *if* $\nu_j \neq 0$ $(j = 1$ *or* $2)$, *then the corresponding level curves* $\zeta =$ const. *are asymptotically geodesic. This will be the case, in particular, if a cone of non-zero vertex angle can be situated interior to* $\mathfrak{S}$.

One may also consider a piece $\mathfrak{S}_B$ of $\mathfrak{S}$ defined by inequalities $A_1 < B_1 \leq \zeta \leq B_2 < A_2$. If $f(\zeta)$ is smooth, $\mathfrak{S}_B$ will not be complete, *however* $\eta_j = \lim_{\zeta \to B_j} \dfrac{d\mathfrak{L}^2}{d\mathfrak{A}}$ *will exist at* B_1, B_2, *and there will hold* $C(\mathfrak{S}_B) = -2\pi(\eta_1 + \eta_2)$ (cf. Theorem 3).

It should be emphasized that the essential features of all the above calculations are local in character; in particular the asymptotic estimates for length and area (and hence the consequences of completeness) depend only on behavior of the metric near an isolated boundary component. These local estimates are related by the formula $C = 2\pi(\chi - \sum_0^n \nu_j)$, which requires for its verification only that the various neighborhoods be joined together smoothly in the given metric.

3. Normal Metrics

Let $\mathfrak{G}$ be a region consisting of the (x, y) plane with $n + 1$ points $p_0, \ldots, p_n$ deleted. It will be convenient to assume that $p_0 =$ point at infinity. A function

$u(x, y)$ defined over $\mathfrak{G}$ determines a conformal metric $ds^2 = e^{2u}(dx^2 + dy^2)$ in $\mathfrak{G}$. In what follows I shall assume that the positive and negative total curvatures associated with $u(x, y)$ are individually finite, that is,

$$T = \iint |\Delta u| dx dy < \infty .$$

In this case, $u(x, y)$ can be represented as the potential of a mass distribution over $\mathfrak{G}$ with density Δu, plus a harmonic function $h(x, y)$.

In any conformal transformation which carries $\mathfrak{G}$ onto a domain of the same type, the law of transformation of $u(x, y)$ will be determined by the requirement of invariance for ds^2. Thus, after a transformation $z = f(\zeta)$, the new function $\bar{u}(\xi, \eta)$ is given by the relation $\bar{u} = u + \log |f'(\zeta)|$, that is, u is changed by an additive harmonic function. Since the only such transformations which leave p_0 invariant are the linear transformations, it is clear that u can be changed at most by an additive constant, in any transformation which leaves invariant the intrinsic geometry and the conformal character of the metric.

A conformal metric defined over $\mathfrak{G}$ by $u(x, y)$ will be called *normal* whenever the harmonic function $h(x, y)$ has the form[6])

$$h(x, y) \equiv \sum_{j=1}^{n} \beta_j \log |z - p_j| = \text{const.}$$

This definition is evidently invariant with respect to the transformations considered. To make it precise, I shall assume given a measure $\mu(E)$ defined over $\mathfrak{G}$, with the property that[7]) $T = \iint_{\mathfrak{G}} |d\mu| < \infty$. *The conformal metric $ds = e^u |dz|$ defined over $\mathfrak{G}$ will be said to be a normal metric whenever*

$$u(z) = \iint_{\mathfrak{G}} \log \left| 1 - \frac{z}{\zeta} \right| d\mu_\zeta + \sum_1^n \beta_j \log |z - p_j| + \text{const.} \tag{6}$$

Here $\mathfrak{G}$ consists of the complex z-plane with $n + 1$ points $p_0 = \infty, p_1, \ldots, p_n$ deleted.

Except in the particular case $\mu + \sum_1^n \beta_j = -1$, μ = measure of $\mathfrak{G}$, it is possible to remove the constant from (6) by an admissible transformation. *In any event, it is of no importance for any of the considerations of this paper, and in the interest of simplicity I shall neglect it in all that follows.*

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[6]) If none of the deleted points is at infinity, it would be necessary to give a more elaborate definition. This is because the law of transformation of $u(x, y)$ is based on the metric properties associated with the manifold described over $\mathfrak{G}$, for which the point at infinity is distinguished.

[7]) The notation is to be interpreted to mean that the absolute variation of the measure is finite.

For any such metric one has for the total curvature, $C = -2\pi\mu$; the EULER Characteristic of $\mathfrak{G}$ is $\chi = 1 - n$.

It will be important to have also a local definition for normality. *A metric* $e^u|dz|$ *defined in a neighborhood* D *of* $p_0 = \infty$ *will be said to be normal at* p_0 *if, in* D, $u(x, y)$ *admits the representation*

$$u(x,y) = \iint\limits_D \log\left|1 - \frac{z}{\zeta}\right| d\mu_\zeta + \beta \log|z| + h(z) \tag{7}$$

where $h(z)$ *is harmonic in* D *and at infinity.* This definition is clearly invariant with respect to transformations which leave the point at infinity unchanged (cf. footnote 6).

A metric is said to be normal at a finite point if, after transforming the point to ∞, the metric is normal at ∞. The definition is in this case formally similar to (7).

For consistency, it is necessary to know that *a normal metric over a region* $\mathfrak{G}$ *of the type described above is normal at each of the points* $\{p_j\}$. This is seen by transforming these points in turn to infinity $\left(\text{observing the transformation law } \bar{u} = u + \log\left|\frac{dz}{d\zeta}\right|\right)$, and noting that the form of the representation is then normal at infinity in the new coordinates.

Conversely, *if a metric defined over* $\mathfrak{G}$ *is normal at each of the* $\{p_j\}$, *then it is normal in* $\mathfrak{G}$. For, without changing the form of the representation, it can be arranged that $h(z)$ has single valued conjugate. Then on transforming each point in turn to infinity, one sees that $h(z)$ (taken from any given configuration) is harmonic at each of the $\{p_j\}$. Hence $h(z)$ is harmonic on the closed RIEMANN sphere and therefore constant.

4. Measures with Compact Support

A considerable simplification arises whenever the measure $\mu(E)$ has compact support in $\mathfrak{G}$, that is, whenever $\mu(E) \equiv 0$ in a neighborhood of each p_j, and I shall consider this case first, in order to clarify the idea. Let Γ_j, γ_j, $\mathfrak{G}_\gamma$, $\mathfrak{L}_j$, $\mathfrak{A}_j$ be as in Theorem 2. We then have

Theorem 5. *Let* $u(z)$ *determine a normal metric* $ds = e^u|dz|$ *in the form* (6), *corresponding to a measure* $\mu(E)$ *having compact support in* $\mathfrak{G}$. *If the metric is complete, then* $\beta_j \leq -1$, $j = 1, \ldots, n$, *and* $\mu \geq 1 - n = \chi$. *For each* $j = 0, \ldots, n$, *the quantity* $\nu_j = \lim\limits_{\gamma_j \to p_j} \mathfrak{L}_j^2/4\pi\mathfrak{A}_j$ *exists, and* $\nu_0 = 1 + \mu + \sum\limits_1^n \beta_j$, *while* $\nu_j = -(1+\beta_j)$ *if* $j \neq 0$. *There holds in this case*

$$C = 2\pi(\chi - \sum_0^n \nu_j) \tag{8}$$

and the curves γ_j *are asymptotically geodesic in the given metric.*

The proof requires little more than formal computation. Near $p_j, j \neq 0$, there holds $e^{u(z)} = a_j \varrho^{\beta_j} + o(\varrho^{\beta_j})$, $a_j \neq 0$, so that completeness implies $\beta_j \leq -1$. Similarly, $\mu + \sum_1^n \beta_j \geq -1$. Also, $\frac{d\mathfrak{L}_j^2}{d\mathfrak{A}_j} = 4\pi(1+\beta_j) + o(1)$ near p_j, so that whenever $\mathfrak{A}_j \to \infty$ there holds $\lim_{\gamma_j \to p_j} \frac{\mathfrak{L}_j^2}{4\pi\mathfrak{A}_j} = -(1+\beta_j)$. If $\mathfrak{A}_j \to \mathfrak{A} \neq \infty$, the above relation shows that $\mathfrak{L}_j$ tends to a finite limit $\mathfrak{L}$. Since

$$\mathfrak{A}_j = \int_r^\varrho \sigma |d\sigma| \oint e^{2u} d\vartheta \geq \frac{1}{2\pi} \int_r^\varrho \frac{\mathfrak{L}_j^2(\sigma)}{\sigma} |d\sigma|$$

it follows that $\mathfrak{L} = 0$, hence also in this case, $\lim_{\gamma_j \to p_j} \frac{\mathfrak{L}_j^2}{4\pi\mathfrak{A}_j} = -(1+\beta_j) = 0$. The point $p_0 = \infty$ is discussed similarly. The result (8) is now immediate, and the asymptotic property of the $\{\gamma_j\}$ follows directly from (5).

Note that under the hypotheses of the theorem, the curves γ_j are asymptotically geodesic without further assumption (cf. the remark under Theorem 4). Note also that the $\{\beta_j\}$ are precisely the fluxes at the boundary points, $\beta_j = -\Phi_j = -\lim_{\gamma_j \to p_j} \frac{1}{2\pi} \oint_{\gamma_j} \frac{\partial u}{\partial n} ds$ if $j \neq 0$, $\beta_0 = \Phi_0$ if $j = 0$. Also, $\beta_0 = \mu + \sum_1^n \beta_j$.

5. Inequalities for Length and Area near a Boundary Component

I shall derive here estimates of length and area from below, which are valid for an arbitrary conformal metric in a neighborhood $\mathfrak{G}_0$ of a boundary component Σ_0. We may suppose Σ_0 to be a circumference or a point. It will be convenient to place Σ_0 at infinity, so that $\mathfrak{G}_0$ is bounded in part by the interior of a circumference, or by the point at infinity. Let Γ_0 be a concentric circumference of radius R, which together with Σ_0 bounds a region lying entirely in $\mathfrak{G}_0$, and let γ_0 be a concentric circumference of radius $r > R$. Let $\mathfrak{L}_0(r)$ be the length of γ_0 in the given metric, and denote by $\mathfrak{A}_0(R; r)$ the area corresponding to the annular region Δ_0 bounded by Γ_0 and γ_0. Let $e^{u(z)}$ denote the local length ratio in the mapping, and set $u_0 = \frac{1}{2\pi R} \oint_{\Gamma_0} e^{u(z)} ds$. Set $\varphi_0 = \frac{1}{2\pi} \oint_{\Gamma_0} \frac{\partial u}{\partial n} ds$ (exterior directed normal), and let $\mu(R;r) = \frac{1}{2\pi}\iint_{\Delta_0} \Delta u\,dx\,dy$. (Then $-2\pi\mu(R;r)$ is the *curvatura integra* over Δ_0.)

Lemma 6. *Under the above hypotheses, there holds*

$$\mathfrak{L}_0(r) \geq 2\pi\, e^{u_0} r \exp\left\{\int_R^r \frac{\mu(R;\varrho)+\varphi_0}{\varrho}\,d\varrho\right\}$$

$$\mathfrak{A}_0(R;r) \geq 2\pi\, e^{2u_0}\int_R^r \varrho \exp\left\{2\int_R^\varrho \frac{\mu(R;\tau)+\varphi_0}{\tau}\,d\tau\right\}d\varrho .$$

Equality holds if and only if $u(z)$ *is a function only of* r *in* Δ_0.

Proof. We have

$$2\pi\mu(R;r) = \iint_{\Delta_0} \Delta u\, dx\,dy = \oint_{\gamma_0} \frac{\partial u}{\partial n}\,ds - \int_{\Gamma_0} \frac{\partial u}{\partial n}\,ds$$

hence $2\pi(\mu+\varphi_0) = r\frac{\partial}{\partial r}\oint_{\gamma_0} u\,d\vartheta$ and letting r vary in this inequality, there follows

$$\begin{aligned}\int_R^r \frac{\mu(R;\varrho)+\varphi_0}{\varrho}\,d\varrho &= \frac{1}{2\pi r}\oint_{\gamma_0} u\,ds - u_0 = \\ &= \frac{1}{2\pi r}\oint_{\gamma_0} \log e^u\,ds - u_0 \leq \log\frac{1}{2\pi r}\oint_{\gamma_0} e^u\,ds - u_0\end{aligned} \tag{9}$$

in consequence of the inequality between arithmetic and geometric means. From this, the first inequality follows.

The second relation is proved similarly. In fact, we write

$$\int_R^\varrho \frac{\mu(R;\tau)+\varphi_0}{\tau}\,d\tau = \frac{1}{4\pi\varrho}\oint \log e^{2u}\,ds - u_0 \leq \tfrac{1}{2}\log\frac{1}{2\pi\varrho}\oint e^{2u}\,ds - u_0 .$$

Thus,

$$e^{2u_0}\exp\left\{\int_R^\varrho \frac{\mu(R;\tau)+\varphi_0}{\tau}\,d\tau\right\} \leq \frac{1}{2\pi\varrho}\oint e^{2u}\,ds = \frac{1}{2\pi\varrho}\,\frac{d\mathfrak{A}_0}{d\varrho}$$

from which the result follows on a further integration.

Theorem 7. *Suppose Σ_0 consists of the single point $p_0 = \infty$. Let $C(R;r)$ be the total curvature in Δ_0, and suppose $C(R;r) \to C_0 \neq +\infty$ as $r \to \infty$. If $\Phi_0 > -1$, or if $\Phi_0 = -1$ and $C(R;r) \leq C_0$ for sufficiently large r, then the area associated with the metric* (3) *exterior to γ_0 will be infinite.*

Proof. Observe that $\mu(R;r) + \varphi_0 = \frac{1}{2\pi} \oint\limits_{\gamma_0} \frac{\partial u}{\partial n}\, dr \to \Phi_0$ as $r \to \infty$. Hence, since $C(R;r) = -2\pi\mu(R;r)$, one has by the hypotheses and by Lemma 6,

$$\mathfrak{A}_0(R;r) \geq 2\pi e^{2u_*} R^2 \int\limits_R^r \varrho^{-1} d\varrho \to \infty, \quad \text{Q.E.D.}$$

Remark 1. If the metric is complete at Σ_0, then Σ_0 is a single point and $\Phi_0 \geq -1$. See HUBER [3], Theorems 1 and 15. For the cases considered in this paper, independent demonstrations of these results will be given in later sections.

Remark 2. The hypothesis $C(R;r) \leq C_0$ is satisfied in particular if the region of negative curvature has compact support near Σ_0. Compare HUBER [3], Theorem 14.

Remark 3. There exist complete metrics with finite area, which satisfy all the above hypotheses except the assumption $C(R;r) \leq C_0$. An example is the conformal metric $ds = \dfrac{|dz|}{(|z|+1)\log(|z|+2)}$ over the z-plane.

6. Normal Metrics; General Case

Consider again a metric $ds^2 = e^{2u}(dx^2 + dy^2)$ defined by a relation

$$u(x,y) = \iint\limits_{\mathfrak{G}} \log\left|1 - \frac{z}{\zeta}\right| d\mu_\zeta + \sum_1^n \beta_j \log|z - p_j| \tag{10}$$

where $\mathfrak{G}$ is the (x,y) plane with $n+1$ points $p_1, \ldots, p_n, p_0 = \infty$ deleted, and $\iint\limits_{\mathfrak{G}} |d\mu_\zeta| = T < \infty$. No further assumption will be made, but one may already conclude that the quantities $\Phi_j = \lim\limits_{\gamma_j \to p_j} \frac{1}{2\pi} \oint\limits_{\gamma_j} \frac{\partial u}{\partial n}\, ds$ (outer directed normal) exist and are finite. Let $\mathfrak{L}_j(r)$, as above, be the length of γ_j in the given metric and $\mathfrak{A}_j(R; r)$ the area of the annular region Δ_j between γ_j and a fixed circumference Γ_j centered at p_j. One may always assume that the $\{\Gamma_j\}$, and hence also the $\{\gamma_j\}$, are non-intersecting.

The first results have a local character and depend only on the behavior of the metric near one of the boundary points, which may be chosen to be $p_0 = \infty$. Accordingly I shall assume at first only that the metric is normal at p_0, so that we may write

$$u(z) = \iint_{D_0} \log \left| 1 - \frac{z}{\zeta} \right| d\mu_\zeta + \beta \log |z| + h(z) \tag{11}$$

where D_0 is a neighborhood of infinity, and $h(z)$ is harmonic at infinity. Again set $u_0 = \frac{1}{2\pi R} \int_{\Gamma_0} e^{u(z)} ds$, and define $C(R; r)$, $\mu(R; r)$ as above. Let $Q(R; r) = \int_R^r \frac{\mu(R; \varrho) - \varphi_0}{\varrho} d\varrho$. It is assumed that[7])

$$T_0 = \iint_{D_0} |d\mu_\zeta| < \infty. \tag{12}$$

Note that $Q(R; r) = [\Phi_0 + o(1)] \log r$, as $r \to \infty$.

Theorem 8. *For a conformal metric* $ds^2 = e^{2u} |dz|^2$ *determined by* (11) *and satisfying* (12), *there holds as* $r \to \infty$,

$$\mathfrak{L}_0(r) = 2\pi\, e^{u_0 + o(1)} r\, e^{Q(R; r)}. \tag{13}$$

Also, for R, $r \to \infty$, *there holds*

$$\mathfrak{A}_0(R; r) = 2\pi\, e^{2u_0 + o(1)} \int_R^r \varrho\, e^{2Q(R; \varrho)} d\varrho. \tag{14}$$

Note that if the area is infinite at p_0, it is unnecessary to let $R \to \infty$ in (14).

Proof. Let $|z| = r$ and let D_r be the intersection of D_0 with a disk of radius r about the origin. Let $\mathfrak{E}_r$ be the exterior of D_r and set

$$u(z) = u_1(z) + u_2(z) + \beta \log |z| + h(z) \tag{15}$$

where

$$u_1(z) = \iint_{D_{r/2}} \log \left| \frac{z - \zeta}{\zeta} \right| d\mu_\zeta, \quad u_2(z) = \iint_{\mathfrak{E}_{r/2}} \log \left| \frac{z - \zeta}{\zeta} \right| d\mu_\zeta .$$

In $D_{r/2}$ we have $\log\left|\frac{z-\zeta}{\zeta}\right| = \log\left|\frac{z}{\zeta}\right| - \log\left|1-\frac{\zeta}{z}\right|^{-1}$ and

$$\iint_{D_{r/2}} \log\frac{1}{\left|1-\frac{\zeta}{z}\right|}|d\mu_\zeta| \le T\log\frac{1}{1-\eta} + \iint_{D_{r/2}-D_{\eta r}} \log\frac{1}{\left|1-\frac{\zeta}{z}\right|}|d\mu_\zeta|$$

where $0<\eta<\frac{1}{2}$. In the last term on the right, the integrand is bounded, hence for a suitable constant A,

$$\iint_{D_{r/2}} \log\frac{1}{\left|1-\frac{\zeta}{z}\right|}|d\mu_\zeta| \le T\log\frac{1}{1-\eta} + A\int_{\mathfrak{E}_{\eta r}} |d\mu_\zeta|.$$

If we choose $\eta=\eta(r)$ tending to zero but such that $\eta r\to\infty$, we see that $\iint_{D_{r/2}} \log\frac{1}{\left|1-\frac{\zeta}{z}\right|}|d\mu_\zeta| = o(1)$ as $r\to\infty$. Thus, $u_1(z)$ has the form

$$u_1(z) = f(r) + o(1). \tag{16}$$

Also, since $h(z)$ is harmonic at infinity, $h(z)=h(\infty)+O(r^{-1})$. Hence, for a circumference $\gamma_0(r)$ about the origin, there holds

$$\frac{1}{2\pi r}\oint_{\gamma_0} [u(z)-u_2(z)]\,ds = \log\frac{1}{2\pi r}\oint e^{u-u_2}ds + o(1). \tag{17}$$

Consider now the integral $I_{\gamma_0}(\zeta) = \frac{1}{2\pi r}\int_{\gamma_0}\log|z-\zeta|\,|dz|$. If $|\zeta|>r$, then $I_{\gamma_0}(\zeta)$ is the mean value of a function of z which is harmonic in D_r, hence equals the value of the function at $z=0$, that is, if $|\zeta|>r$ then $I_{\gamma_0}(\zeta) = \log|\zeta|$. If $|\zeta|<r$, then $I_{\gamma_0}(\zeta)$ is a harmonic function of ζ in D_r which by symmetry is constant on any concentric interior circumference. Hence $I_{\gamma_0}\equiv$ const. for $|\zeta|<r$. But $I_{\gamma_0}(0)=\log r$, hence this is its value throughout the interior of γ_0. We conclude that if $|\zeta|\ge\frac{r}{2}$, then $\left|\frac{1}{\log|\zeta|}I_{\gamma_0}(\zeta)\right| \le \frac{\log r}{\log r-\log 2}$, and hence, in particular,

$$\frac{1}{2\pi r}\oint_{\gamma_0} u_2(z)ds = o(1) \quad \text{as } r\to\infty. \tag{18}$$

Finally, consider $\int_0^{2\pi}[e^{u_2(z)}-1]\,d\vartheta$ for $|z|=r$ and $\vartheta=\arg z$. Let $\alpha(M)$ be the measure of the set E_M of ϑ on which $|u_2(z)|>M$. Then

$$M\alpha(M) < \int_{E_M} |u_2(z)|\,d\vartheta < \int_{\mathfrak{C}_{r/2}} H(\zeta)\,|d\mu_\zeta| \tag{19}$$

where $H(\zeta) = \left|\int_{E_M} \log\left|1 - \frac{z}{\zeta}\right| d\vartheta\right|$.

For fixed $|\zeta| > \frac{r}{2}$ and z on E_M and outside the circle $|z - \zeta| = \zeta$, the integrand in $H(\zeta)$ is uniformly bounded. But for z inside this circle, the integrand increases in magnitude as $|z - \zeta|$ decreases. Hence $H(\zeta)$ is maximized when the part of E_M interior to the circle is an interval with its midpoint at arg ζ. For this configuration we compute $H(\zeta) < A(1 + |\log \alpha(M)|)\alpha(M)$ for some constant A, and hence from (19), $M < (1 + |\log\alpha(M)|)\,\varepsilon(r)$ where $\varepsilon(r) \to 0$ as $r \to \infty$. We conclude $\alpha(M) < A\mathrm{e}^{-M/\varepsilon(r)}$ for a suitable A. But

$$\int_0^{2\pi} |\mathrm{e}^{u_2(z)} - 1|\,d\vartheta \leq \int_{M>0} |\mathrm{e}^{M} - 1|\,d\alpha(M) + \int_{M>0} |\mathrm{e}^{-M} - 1|\,d\alpha(M)$$

from which there follows easily

$$\int_0^{2\pi} |\mathrm{e}^{u_2(z)} - 1|\,d\vartheta = o(1) \tag{20}$$

as $r \to \infty$.

We are now prepared to estimate $\mathfrak{L}_0(r)$. We have by (17) and (18)

$$\frac{1}{2\pi r}\oint u\,ds = \log\frac{1}{2\pi r}\oint \mathrm{e}^{u-u_2}ds + o(1).$$

Also, $\mathfrak{L}_0(r) = \oint \mathrm{e}^{u}ds = \oint \mathrm{e}^{u-u_2}ds + \oint \mathrm{e}^{u-u_2}(\mathrm{e}^{u_2} - 1)ds = \oint \mathrm{e}^{u-u_2}ds\,\mathrm{e}^{o(1)}$ by (15), (16) and (20). Hence $\frac{1}{2\pi r}\oint u\,ds = \log\frac{1}{2\pi r}\mathfrak{L}_0(r) + o(1)$.

Referring back to (9), we find immediately the stated relation (13).

To prove (14), observe that the method of proof of (20) yields also

$$\int_0^{2\pi} |\mathrm{e}^{2u_2(z)} - 1|\,d\vartheta = o(1).$$

Thus, as above,

$$\begin{aligned}\frac{1}{2\pi r}\oint u\,ds &= \log\frac{1}{4\pi r}\oint \mathrm{e}^{2u}ds + o(1)\\ &= \log\frac{1}{4\pi r}\frac{d\mathfrak{A}_0}{dr} + o(1)\end{aligned} \tag{21}$$

from which the result follows again from (9) by an integration.

Theorem 9. *Suppose* $\Phi_0 > -1$. *Then*

$$\mathfrak{A}_0(R;r) = \frac{\pi}{1+\Phi_0} e^{2u_0+o(1)} r^2 e^{2Q(R;r)} \tag{22}$$

as $r \to \infty$.

Proof. By Theorem 7, the area associated with the metric is infinite at p_0. Hence by Theorem 8,

$$\mathfrak{A}_0(R;r) = 2\pi e^{2u_0+o(1)} \int_R^r \varrho e^{2Q(R;\varrho)} d\varrho$$

for any fixed R. We have

$$\int_R^r \varrho e^{2Q} d\varrho = \left[\frac{\varrho^2}{2} e^{2Q(R;\varrho)}\right]_R^r - \int_R^r \varrho e^{2Q(R;\varrho)} [\mu(\gamma_0) - \sum_1^n \Phi_j] d\varrho$$

$$= \left[\frac{\varrho^2}{2} e^{2Q(R;\varrho)}\right]_R^r - \Phi_0 \int_R^r \varrho e^{2Q(R;\varrho)} d\varrho + \int_R^r \varrho e^{2Q(R;\varrho)} \varepsilon(\varrho) d\varrho$$

where $\varepsilon(\varrho) \to 0$ as $\varrho \to \infty$. Again using the fact that the area is infinite at p_0, we find

$$(1 + \Phi_0 + o(1)) \int_R^r \varrho e^{2Q(R;\varrho)} d\varrho = \frac{r^2}{2} e^{2Q(R;r)} \tag{23}$$

and from this the result follows.

Theorem 10. *Suppose* $\Phi_0 \geq -1$. *Then for any fixed sufficiently large* R *there holds*

$$\nu_0 = \lim_{r\to\infty} \frac{\mathfrak{L}_0^2(r)}{4\pi \mathfrak{A}_0(R;r)} = 1 + \Phi_0.$$

The assertion implies, in particular, that the indicated limit exists.

Proof. If $\Phi_0 > -1$, the result is immediate from Theorems 8 and 9. If $\Phi_0 = -1$ and the area is infinite at p_0, the result may still be obtained from (23). Suppose $\lim_{r\to\infty} \mathfrak{A}_0(R;r) \neq \infty$. Since the integrand in (23) is positive, the integral is increasing in r, hence $\lim_{r\to\infty} \int_R^r \varrho e^{2Q(R;\varrho)} d\varrho$ exists and is finite. There holds

$$\int_R^r \varrho e^{2Q(R;\varrho)} d\varrho = \left[\frac{\varrho^2}{2} e^{2Q}\right]_R^r + \int_R^r (1 + \varepsilon(\varrho)) \varrho e^{2Q(R;\varrho)} d\varrho \tag{24}$$

where $\varepsilon(\varrho) \to 0$ as $\varrho \to \infty$. By the above remarks, both integrals in (24) tend to finite limits as $r \to \infty$. We conclude therefore from (24) that $r^2 e^{2Q(R;r)}$ tends to a finite limit. Hence, by Theorem 8, $\mathfrak{L}_0(r) \to \mathfrak{L}_0^* \neq \infty$ as $r \to \infty$. Again using Theorem 8, one sees that

$$\mathfrak{A}_0(R;r) = \frac{1}{2\pi} e^{o(1)} \int_R^r \frac{\mathfrak{L}_0^2(\varrho)}{\varrho} d\varrho = \frac{1}{2\pi} e^{o(1)} \int_R^r \frac{\mathfrak{L}_0^{*2} + o(1)}{\varrho} d\varrho .$$

Since $\lim_{r\to\infty} \mathfrak{A}_0(R;r) < \infty$, there must hold $\mathfrak{L}_0^* = 0$, and from this the result follows.

Theorem 11. *Let* $e^{u(z)}|dz|$ *be a metric* (3) *which is normal at* $p_0 = \infty$ *and which satisfies* (12). *Then for any* $\delta > 0$ *there holds asymptotically for the length* $L(r)$ *of the image of a radial segment of length* r *from the origin in the z-plane,*

$$r^{1+\Phi_0-\delta} < L(r) < r^{1+\Phi_0+\delta} + \text{const.} \tag{25}$$

Proof. We follow, essentially, the proof of Theorem 8. Using the decomposition (15), we find again

$$u_1(z) = \iint_{D_{r/2}} \log\left|\frac{z}{\zeta}\right| d\mu_\zeta + o(1)$$

whence, choosing D_0 to be the exterior of a circumference $\Gamma_0(R)$,

$$u_1(z) = \mu(R;r)\log r + o(\log r). \tag{26}$$

Thus

$$\frac{1}{2\pi r}\oint_{\gamma_0(r)} u(z)|dz| = \mu(R;r)\log r + \beta \log r + \frac{1}{2\pi r}\oint u_2(z)|dz| + o(\log r).$$

By (18) the integral on the right is $o(1)$ as $r \to \infty$. By (9), since $\mu(R;r) + \varphi_0 = = \Phi_0 + o(1)$,

$$\frac{1}{2\pi r}\oint_{\gamma_0(r)} u(z)|dz| = u_0 + \Phi_0 \log r + o(\log r).$$

Hence, $\mu(R;r) + \beta = \Phi_0 + o(1)$, as $r \to \infty$, and we find

$$u(z) = (\Phi_0 + o(1))\log r + u_2(z). \tag{27}$$

Choose $|z| = r$ in the range $r_0 \leq |z| \leq 2r_0$. In the given metric, the length $L(r)$ of a radial line is $L(r) = \int_0^r e^{u(z)}|dz|$, and for any $\delta > 0$ we have, for large r_0,

$$\int_{r_0}^{r} \varrho^{\Phi_0-\delta} e^{u_2(z)} d\varrho \leq \int_{r_0}^{r} e^{u(z)} d\varrho \leq \int_{r_0}^{r} \varrho^{\Phi_0+\delta} e^{u_2(z)} d\varrho . \tag{28}$$

For any β we have

$$\int_{r_0}^{r} \varrho^{\beta} e^{u_2(z)} d\varrho = \int_{r_0}^{r} \varrho^{\beta} d\varrho + \int_{r_0}^{r} \varrho^{\beta} [e^{u_2(z)} - 1] d\varrho \tag{29}$$

and for $r \leq 2r_0$ we may write

$$\left| \int_{r_0}^{r} \varrho^{\beta} [e^{u_2(z)} - 1] d\varrho \right| < r^{\beta} \int_{r_0}^{r} | e^{u_2(z)} - 1 | \, d\varrho . \tag{30}$$

Let E_M be the set on $[r_0, r]$ where $|u_2(z)| > M$ and let $\alpha(M)$ be its measure. Then

$$\alpha(M) \cdot M < \int_{E_M} |u_2(z)| ds < \int_{\mathfrak{C}_{r_0/2}} |d\mu_\zeta| \int_{E_M} \left| \log \left| 1 - \frac{z}{\zeta} \right| \right| ds_z . \tag{31}$$

We may clearly assume that the given radial line is the positive x-axis. Then for ζ to the left of the line $|\zeta - 2r_0| = |\zeta|$ and outside the circumference $|\zeta| = r_0$, the integrand on the right in (31) will be bounded. For ζ to the right of $|\zeta - 2r_0| = |\zeta|$ but interior to $|\zeta| = r \leq 2r_0$, the integrand is increased if ζ is replaced by a point of the same magnitude, but on the given radial line. The integral will then be maximized if that part of E_M is replaced by a segment of length $\alpha(M)$ and containing ζ. Setting $\tau = \left|\frac{z}{\zeta}\right|$ and letting E_M^* be the image of the modified E_M, we find $\int_{E_M} \left|\log\left|1 - \frac{z}{\zeta}\right|\right| ds_z \leq |\zeta| \int_{E_M^*} |\log|1 - \tau|| \, d\tau$ and $\int_{E_M^*} |\log|1 - \tau|| \, d\tau \leq A\alpha^*(M)\left[1 + \log\frac{1}{\alpha^*(M)}\right] = A\frac{1}{|\zeta|}\alpha(M)\left[1 + \log\frac{|\zeta|}{\alpha(M)}\right]$.

Thus, $\alpha(M) \cdot M < \varepsilon(r) \cdot \alpha(M)\left[1 + \log\frac{2r_0}{\alpha(M)}\right]$ where $\varepsilon(r) \to 0$ as $r \to \infty$, from which $\alpha(M) < Cre^{-\frac{1}{\varepsilon(r)}M}$. Hence, for $r \leq 2r_0$,

$$\left|\int_{r_0}^{r} [e^{u_2(z)} - 1] ds\right| \leq \left|\int_{M>0} [e^{M} - 1] d\alpha(M)\right| + \left|\int_{M>0} [e^{-M} - 1] d\alpha(M)\right| \leq C \cdot \varepsilon(r) \cdot r$$

for some constant C. Hence in this case (30) becomes

$$\left|\int_{r_0}^{r} \varrho^{\beta} [e^{u_2(z)} - 1] d\varrho\right| < C \cdot \varepsilon(r_0) \cdot r_0^{1+\beta} . \tag{32}$$

Consider now an arbitrary $r > r_0$ and let n be the smallest integer, for which $2^n r_0 > r$. Then $r < 2^n r_0 < 2r$, and we find from (32)

$$\left|\int_{r_0}^{r} \varrho^{\beta}[e^{u_2(z)} - 1]d\varrho\right| < C \cdot \varepsilon(r_0) \cdot r_0^{1+\beta}[1 + 2^{1+\beta} + \ldots + 2^{n(1+\beta)}]$$

$$= C \cdot \varepsilon(r_0) \cdot r_0^{1+\beta} \frac{2^{(n+1)(1+\beta)} - 1}{2^{1+\beta} - 1} \tag{33}$$

$$< C \cdot \varepsilon(r_0) \begin{cases} r^{1+\beta} \text{ if } \beta > -1 \\ r_0^{1+\beta} \text{ if } \beta < -1 \,. \end{cases}$$

Using this inequality, the theorem follows easily from (28) and (29).

Remark 1. The example of the complete conformal metric $ds = \log(2 + |z|)|dz|$ spread over the z-plane, shows that the constant δ cannot in general be removed from the exponent in (25). In this case $\Phi_0 = 0$ and $L(r) = r \log r + O(r)$.

Remark 2. A particular consequence is that if $\Phi_0 > -1$, the circumference γ_0 is essentially a geodesic circle, that is, it is a locus of points approximately equidistant in the metric from a fixed point.

Corollary 11. *Suppose the given metric is complete at* p_0, *that is, every path tending to infinity has, in the given metric, infinite length. Then*[8]) $\Phi_0 \geq -1$.

We are now prepared to discuss the situation described at the beginning of this section, of a normal metric defined over a region $\mathfrak{G}$ consisting of the complex plane with $n + 1$ points deleted. The function $u(x, y)$ is then given by (10), and it is supposed that (12) holds. The fluxes $\{\Phi_j\}$ are then related to the total measure $\mu = \mu(\mathfrak{G})$ by GREEN's formula, and one has

$$\mu = \sum_0^n \Phi_j .$$

On the other hand, the *curvatura integra* is

$$C = -2\pi\mu .$$

By Corollary 11, if the metric is complete at p_0, then $\Phi_0 \geq -1$. Similarly, as one sees by transforming p_j to ∞, completeness at p_j implies $\Phi_j \geq 1$, $j = 1, \ldots, n$. By Theorem 10, $\nu_0 = \lim\limits_{\gamma_0 \to p_0} \dfrac{\mathfrak{L}_0(r)}{4\pi\mathfrak{A}_0(R; r)}$ exists, and $\nu_0 = \Phi_0 + 1$. Similarly, $\nu_j = \Phi_j - 1$. Collecting these results, we obtain:

[8]) This result is also a consequence of the more general Theorem 1 of HUBER [3].

Theorem 12. *Suppose the metric determined by* (10) *is complete and that* (12) *holds. Then for each* j, $\nu_j = \lim\limits_{\gamma_j \to p_j} \dfrac{\mathfrak{L}_j(r)}{4\pi \mathfrak{A}_j(R; r)}$ *exists and is independent of* R, *and*

$$C = 2\pi(\chi - \sum_0^n \nu_j). \tag{34}$$

In the case $n = 0$ there is an evident formal connection of Theorem 12 with a theorem of A. Huber [4], who proved that for any simple closed curve γ the inequality

$$\frac{\mathfrak{L}^2}{4\pi \mathfrak{A}} \geq 1 - \frac{C^+(\gamma)}{2\pi} \tag{35}$$

holds for the length $\mathfrak{L}$ of γ and area $\mathfrak{A}$ bounded by γ in the given metric. Here C^+ is the *curvatura integra*, evaluated over that part of the region bounded by γ, in which the curvature is non-negative. The inequality (35) is in general not sharp, and it is incorrect if C^+ is replaced by C. However, if γ is chosen to be a large circumference, then by Theorem (12), (35) becomes

$$\lim_{r \to \infty} \frac{\mathfrak{L}_0^2(r)}{4\pi \mathfrak{A}_0(R; r)} = 1 - \frac{C}{2\pi}$$

that is, for the selected curves $\gamma_0(r)$ a result which is stronger than (35) holds asymptotically with equality sign.

The following estimate is again local and refers to the behavior of the metric near an isolated boundary component.

Theorem 13. *Under the assumptions of Theorem 11, let* σ_0 *be a divergent path tending to* $p_0 = \infty$. *Let* $L_r(\sigma_0)$ *be the length of that part of* σ_0 *which lies interior to a circumference* $\gamma_0(r)$ *of radius* r *about the origin. Then for any* $\delta > 0$ *there holds asymptotically* $L_r(\sigma_0) \geq r^{1+\Phi_0-\delta}$ *as* $r \to \infty$.

Comparing this result with Theorem 11, we see that the images of the radial lines behave asymptotically as approximations to geodesics in the given metric.

Proof. Setting $|\zeta| = \varrho$, we have

$$L_r(\sigma_0) = \int_{\sigma_0 \cap \gamma_0} e^{u(\zeta)} |d\zeta| \geq \int_{\varrho \leq r} e^{u(\zeta)} |d\varrho|.$$

This inequality will not be weakened if we omit all arcs of σ_0 on which values of ϱ are repeated; that is, if the maximum value of ϱ attained on σ_0 for all arc lengths $s \leq s_a$ is ϱ_a, all arcs on σ_0 for which $s > s_a$, $\varrho < \varrho_a$ are to be omitted in the integration. In this case the integration is monotonic in ϱ and the estimates in the proof of Theorem 11 are easily seen to apply, so that for the length $L_r(\sigma_0)$ of that part of σ_0 for which $\varrho < r$ we obtain $L_r(\sigma_0) \geq r^{1+\Phi_0-\delta}$ by (25) for any $\delta > 0$, the stated result.

7. A Geometrical Assumption; Sharpening of the above Estimates

The asymptotic estimates for length and area derived above can be improved under a suitable assumption on the decay of the curvature at the singular points $\{p_j\}$. Such an assumption, if it is to be meaningful, should involve only quantities which can be determined a priori in terms of the intrinsic geometry of the surface and should not depend on properties of the representation over the z-plane (although it will still be assumed that the metric is normal). The simplest hypothesis available to us involves the rate of decay of curvature as the point of evaluation moves along a divergent path. To make this concept precise, select a fixed point P and define the distance $d(Q)$ from P to Q as the greatest lower bound of lengths (in the given metric) of paths which join P to Q. I shall assume in this section[9]) that *there are fixed constants* C *and* $\delta > 0$ *such that uniformly for all* Q *near* $p_0 = \infty$, *there holds* $|K| < Cd^{-2-\delta}$, *where* K *is the* GAUSSIAN *curvature associated with the metric.*

Under this hypothesis we find:

Theorem 14. *Under the hypotheses of Theorem 11 and the additional hypothesis* $|K| < Cd^{-2-\delta}$, *there holds, for any* $\varepsilon < \min[\delta, 1]$,

$$L(r) = A r^{1+\Phi_0}[1 + O(r^{-\varepsilon})]$$

for some positive constant A, *whenever* $\Phi_0 > -1$.

Remark. The assumption $|K| < Cd^{-2-\delta}$ cannot be deleted, and even an assumption $|K| < C(d \log d)^{-2}$ is not sufficient. This can be seen from the

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[9]) This assumption assures a sufficient rate of decay so that the curvature is absolutely integrable. An assumption $|K| < d^{-2}$ would not suffice.

example (which we have already considered in another context) of the conformal metric $ds = \log(2 + |z|)|dz|$ spread over the z-plane. For this metric there holds $T < \infty$, $\mu = 0$, $K = (r\log^2 r)^{-2} \sim (d\log d)^{-2}$, $L(r) = r\log r + O(r)$.

Proof of Theorem 14. We may clearly assume that P is the origin in the z-plane. Consider a radial segment from P and let Q be a point on this segment such that $|z_Q| = r$. Consider a smooth path joining P to Q, whose length approximates the distance $d(Q)$. Applying Theorem 11 to this path, we obtain, for given $\delta > 0$ and large r, $d(Q) + \varepsilon \geq r^{1+\Phi_0-\delta}$ for any $\varepsilon > 0$. Hence $d(Q) \geq r^{1+\Phi_0-\delta}$ as $r \to \infty$.

By assumption, $|K(Q)| < d^{-2-\delta}$. Hence $|K(Q)| < r^{-2(1+\Phi_0)-\delta}$ (δ not the same in all contexts), as $r \to \infty$. In the notation of the proof of Theorem 8,

$$\iint_{\mathfrak{E}_r} |d\mu| = \iint_{\mathfrak{E}_r} |K| e^{2u} \varrho\, d\varrho\, d\vartheta$$

where $\mathfrak{E}_r$ is the exterior of the disk D_r, while from the above estimates

$$\iint_{\mathfrak{E}_r} |K| e^{2u} \varrho\, d\varrho\, d\vartheta \leq \int_r^\infty \varrho^{-2\Phi_0-1-\delta} d\varrho \oint e^{2u(z)} d\vartheta.$$

The circuit integral on the right equals $\varrho^{-1}\dfrac{d\mathfrak{A}_0}{d\varrho}$. Returning to the proof of Theorem 8, we find from (21) and from (9)

$$\oint e^{2u(z)} d\vartheta = 2\pi e^{2u_0+o(1)} \varrho^{2\Phi_0+o(1)}.$$

Thus,

$$\iint_{\mathfrak{E}_r} |K|\, e^{2u(z)} \varrho\, d\varrho\, d\vartheta \leq 2\pi e^{2u_0} \int_r^\infty \varrho^{-1-\delta} d\varrho = \frac{2\pi}{\delta} e^{2u_0} r^{-\delta}$$

for some $\delta > 0$. We have proved:

$$\iint_{\mathfrak{E}_r} |d\mu| = \iint_{\mathfrak{E}_r} |K| e^{2u} \varrho\, d\varrho\, d\vartheta = O(r^{-\delta}) \text{ as } r \to \infty. \tag{36}$$

Consider now the definition (15) of $u_1(z)$, the region D_0 being chosen as the exterior of $\Gamma_0(R)$. We have

$$u_1(z) = \iint_{D_{r/2}} \log\left|\frac{z-\zeta}{\zeta}\right| d\mu_\zeta = \mu(R;r)\log|z| + \iint_{D_{r/2}} \log\left|\frac{z-\zeta}{z}\right| d\mu_\zeta - \iint_{D_{r/2}} \log|\zeta|\, d\mu_\zeta. \tag{37}$$

Set $|\zeta| = \varrho$, $|z| = r$. Then $\left|\log\left|\frac{z-\zeta}{z}\right|\right| \leq \left|\log\left(1 - \frac{\varrho}{r}\right)\right|$ for $\varrho < r$. Thus, setting $\mu(R) = \lim_{r\to\infty} \mu(R; r)$, integrating by parts and using (36),

$$\left|\iint_{D_{r/2}} \log\left|\frac{z-\zeta}{z}\right| d\mu_\zeta\right| \leq \left|\log\left(1 - \frac{\varrho}{r}\right) \cdot (\mu(R) - \mu(R; r))\right|_R^{r/2} +$$

$$+ \int_R^{r/2} |\mu(R) - \mu(R; r)| \frac{1}{r-\varrho} d\varrho \leq A r^{-\delta}$$

for some constant A, provided $\delta < 1$. Similarly, we estimate

$$\iint_{D_{r/2}} \log|\zeta|\, |d\mu_\zeta| \leq A r^{-\delta} \log r.$$

Thus, we have from (37) $u_1(z) = \mu(R) \log|z| + O(r^{-\delta} \log r)$. The reasoning which led to (27) shows that

$$\mu(R) + \beta = \Phi_0. \tag{38}$$

Thus $u(z) = u_1(z) + u_2(z) + \beta \log|z| + h(z) = \Phi_0 \log|z| + u_2(z) + O(r^{-\delta})$ for some $\delta > 0$.

An examination of the reasoning which led to (32) shows that the quantity $\varepsilon(r_0)$ in (32) can be chosen in the form

$$\varepsilon(r_0) < C\eta(r_0) \log \frac{1}{\eta(r_0)}$$

where $\eta(r_0) = \iint_{\mathfrak{C}_{r_0/2}} |d\mu|$. By (36) we have in the present case $\eta(r) = O(r^{-\delta})$. Hence we will have an estimate of the form (33) with β replaced by $\beta - \bar{\delta}$, for any $\bar{\delta} < \delta$. Placing this result in (29) and using (38) yields

$$L(r) = e^A r^{1+\Phi_0} [1 + O(r^{-\varepsilon})]$$

which was to be proved. Such an estimate holds for any $\varepsilon < \min(\delta, 1)$, as one sees by retracing the steps in the derivation.

Similarly one may prove:

Theorem 15. *Under the assumptions of Theorem 14 there holds (cf. Theorem 8 and 9)*

$$\mathfrak{L}_0(r) = 2\pi e^{u_0} r^{1+\Phi_0}[1 + O(r^{-\varepsilon})]$$

$$\mathfrak{A}_0(R; r) = \frac{\pi}{1 + \Phi_0} e^{2u_0} r^{2(1+\Phi_0)}[1 + O(r^{-\varepsilon})]$$

$$\frac{\mathfrak{L}_0^2(r)}{4\pi \mathfrak{A}_0(R; r)} = (1 + \Phi_0) + O(r^{-\varepsilon})$$

for any $\varepsilon < \min[\delta, 1]$.

We omit details.

8. Asymptotic Estimates for the Length Ratio

I shall denote the local length ratio by $\lambda(z) = e^{u(z)}$.

Theorem 16. *For a conformal metric defined by* (11) *for which* (12) *is satisfied, suppose the region in which* $K > 0$ (*that is, the region in which* $\mu < 0$) *has compact support. Then there is a constant* A *such that* $\lambda(z) \leq A r^{\Phi_0}$ *as* $r \to \infty$. *If the region on which* $K < 0$ *is compact, then* $\lambda(z) \geq A r^{\Phi_0}$ *as* $r \to \infty$, *for some* A.

Proof. Let us again use the decomposition (15). Suppose $K \leq 0$ outside the circumference $\gamma(r_0)$. By the material leading to (16),

$$u_1(z) = \iint_{D_{r/2}} \log\left|\frac{z-\zeta}{\zeta}\right| d\mu_\zeta = \iint_{D_{r/2}} \log\left|\frac{z}{\zeta}\right| d\mu_\zeta + o(1) = \iint_{D_r} \log\left|\frac{z}{\zeta}\right| d\mu_\zeta + o(1)$$

since $\log\left|\frac{z}{\zeta}\right|$ is bounded when $1 \leq \left|\frac{z}{\zeta}\right| \leq 2$. Integrating by parts yields

$$u_1(z) = \int_R^r \frac{\mu(R; \varrho)}{\varrho} d\varrho + o(1)$$

$$= \int_R^{r_0} \frac{\mu(R; \varrho)}{\varrho} d\varrho + \int_{r_0}^r \frac{\mu(R)}{\varrho} d\varrho - \int_{r_0}^r \frac{[\mu(R) - \mu(R; \varrho)]}{\varrho} d\varrho + o(1)$$

for any (fixed) r_0 in the range $R < r_0 < r$.

By assumption, the last integral on the right is non-positive, hence $u_1(z) \leq A + \mu(R) \log r$ as $r \to \infty$. But, repeating the derivation of (27), one sees that $\mu(R) + \beta = \Phi_0$. Thus, $u(z) \leq A + \Phi_0 \log r + o(1)$ from which the first assertion follows. The corresponding inequality, when $K \geq 0$ outside D_{r_0}, is proved similarly.

Corollary 16. *If in addition* $\Phi_0 > -1$, *then* $L(r) \leq A r^{\Phi_0}$, $L(r) \geq A r^{\Phi_0}$, *respectively, in the two cases considered. If* $\Phi_0 = -1$, *then* $L(r) \leq A \log r$, $L(r) \geq A \log r$, *respectively.*

Note that by Corollary 11, $\Phi_0 \geq -1$ whenever the metric is complete at p_0.

Remark. If the curvature has compact support, then one obtains $\lambda = A r^{\Phi_0}[1 + O(r^{-1})]$ (cf. Theorem 16). Estimates of this type cannot be expected, however, in a general case, even under assumptions of the type introduced in § 7. One may imagine, for example, a situation in which the measure μ is concentrated at a sequence of points tending to infinity. Such a measure can be constructed such that $\iint_{\mathfrak{E}_r} |d\mu|$ tends to zero as rapidly as desired, but λ will nevertheless be singular at each point of the sequence. This situation may occur, for example, when the measure μ arises from the conformal representation of a polyhedral surface. In order to obtain asymptotic estimates for $\lambda(z)$ in a general case, it would be necessary to introduce a new postulate on the *local* smoothness of the curvature with respect to the given metric.

9. Applications to Differential Geometry in the Large

The significance of the preceding developments for the general theory of abstract surfaces consists in the fact that for an important class of such surfaces, the associated metrics, when represented in terms of conformal parameters over a plane domain, turn out to be *normal* in the sense of § 3. It seems likely that this result is true for arbitrary complete open surfaces of finite connectivity, over which the curvature is absolutely integrable. I am, however, presently able to prove it only by invoking an additional supposition.

Hypothesis S. *The region of positive curvature has compact support on the surface.*

Under this assumption I shall show first that a neighborhood of each boundary component can be mapped conformally onto the (open) exterior of a disk in the complex z-plane. This result follows alternatively from more general results of HUBER [3] (esp. Theorem 15); however, in the case which I consider it is possible to provide a somewhat simpler demonstration, and it seems desirable to do so. Thus it is possible to speak of normal metrics in the sense of § 3, and the remainder of this section will then be devoted to proving that under any such mapping the metric becomes a normal metric at $p_0 = \infty$ in the form (11). Thus all results derived in §§ 6–8 will apply.

By an *abstract surface* $\mathfrak{S}$ I shall mean a finitely connected, open RIEMANN surface on which a conformal metric $e^{u(z)}|dz|$ is defined. Every such surface is homeomorphic to a closed surface from which a finite number of points $p_0, \ldots, p_n$ has been deleted (KERÉKJÁRTÓ [6], Chapter 5). A doubly-connected annular region surrounding p_0 can be mapped conformally onto a plane annulus bounded by inner and outer circumferences $\Gamma_0(R)$, $\gamma_0(r)$, such that $\gamma_0(r)$ corresponds to p_0. In terms of conformal parameters there holds $K = -e^{-2u}\Delta u$, $\iint K dA = -\iint \Delta u dx dy$, and we may introduce, as before, a measure μ corresponding to $u(x, y)$.

Theorem 17. *Suppose $\mathfrak{S}$ is complete at p_0 and that the curvature is absolutely integrable over $\mathfrak{S}$ in a neighborhood of this point. Assume also Hypothesis S. Then γ_0 consists of the single point at infinity.*

Proof. Suppose the theorem were false, so that γ_0 is an entire outer circumference. In the annular region D we have

$$u(z) = \iint_D \log|\zeta - z|\, d\mu_\zeta + \beta \log z + h(z) \tag{39}$$

where $h(z)$ is harmonic in D. We may choose β so that $h(z)$ has single-valued conjugate $h^*(z)$. Because of Hypothesis S, the integral over the measure is bounded above near γ_0 (cf. the proof of Theorem 16). Evidently, $\beta \log z$ is bounded at γ_0. We may write, because of the choice of β,

$$h(z) = h_0(z) + h_1(z)$$

where $h_0(z)$ is harmonic interior to γ_0, $h_1(z)$ is harmonic exterior to Γ_0. Consider the mapping[10])

$$w(z) = \int_0^z e^{h_0 + i h_0^*}\, dz.$$

[10]) The underlying idea in the ensuing discussion is due to HUBER [3, p. 53].

The function $w(z)$ carries the interior of γ_0 onto an unbranched RIEMANN surface over the w-plane. Let Σ denote a disk centered at the origin in the w-plane, whose radius is the least upper bound of all values for which Σ lies interior to a sheet of the surface. The inverse mapping $z(w)$ is by the monodromy theorem analytic and single valued in Σ. Under this mapping the image of Σ cannot be compact in the interior of γ_0, for each point of the boundary image would then lie interior to a circle of analyticity, and one then could conclude that Σ could be enlarged. Hence there is a sequence of points in Σ, tending to a boundary point q_0, whose inverse images tend to γ_0.

Consider a radius σ_0 joining q_0 to $w = 0$. If its inverse image were compact interior to γ_0, one could conclude that q_0 would be interior to a circle of analyticity, which we have just shown cannot happen. Hence the inverse image of σ_0 corresponds to a divergent path on $\mathfrak{S}$. We have

$$\infty > \int_{\sigma_0(w)} |dw| = \int_{\sigma_0(z)} \left|\frac{dw}{dz}\right| |dz| = \int_{\sigma_0(z)} e^{h_0} |dz| > A \int_{\sigma_0(z)} e^{u(z)} |dz| = \infty$$

by the above estimates on the terms in (39). From this contradiction it follows that the radius of Σ is infinite, that is, one sheet of the RIEMANN surface must cover the entire w-plane. But the inverse function is 1-valued on this sheet and achieves only values interior to γ_0. Hence γ_0 has infinite radius, which was to be shown.

Theorem 18. *Under the hypotheses (and hence also the conclusion) of Theorem 17, $u(z)$ admits near $p_0 = \infty$ a representation of the form*

$$u(z) = \iint_D \log \left|1 - \frac{z}{\zeta}\right| d\mu_\zeta + \beta \log |z| + h(z) \tag{40}$$

where $h(z)$ is harmonic at infinity, that is, the metric defined by $u(z)$ is normal at p_0.

Proof. We need only establish that for suitable choice of β, $h(z)$ is harmonic at infinity. We may write $h(z) = h_0(z) + h_1(z)$, where $h_0(z)$ is entire and $h_1(z)$ is harmonic at infinity. Because of Hypothesis S (cf. again, the proof of Theorem 16), there is a positive integer N such that, near $p_0 = \infty$,

$$\iint_D \log \left|1 - \frac{z}{\zeta}\right| d\mu_\zeta + \beta \log z \leq N \log |z|.$$

Consider the mapping defined for all finite z,

$$w(z) = \int_0^z z^N e^{h_0 + i h_0^*} dz. \tag{41}$$

This mapping is unbranched except at the origin, where the simply covered z-plane is taken to an $(N+1)$ sheeted surface over the w-plane. As in the proof of Theorem 17 above, one finds as a consequence of completeness that there are no finite boundary points. Hence the $(N+1)$ sheeted w-plane corresponds $1-1$ with the simply covered z-plane, and $z = 0 \leftrightarrow w = 0$. Hence $w \equiv A z^{N+1}$, from which it follows from (41) that $h_0(z) \equiv$ const., Q. E. D.

The main results

The material of §§ 6–8 implies the following general properties of abstract surfaces.

Theorem 19. *Let $\mathfrak{S}$ be an abstract surface which is complete and has finite total curvature in a doubly-connected neighborhood $\mathfrak{N}$ of one of its ideal boundary components, and suppose $\mathfrak{S}$ satisfies Hypothesis S in $\mathfrak{N}$. Then $\mathfrak{N}$ can be mapped conformally onto a neighborhood D of $p_0 = \infty$ in the complex z-plane. Let $\Gamma_0(R)$, $\gamma_0(r)$ be concentric circumferences of radii R, $r > R$ in D. Let $\mathfrak{L}_0(r)$ denote the length on $\mathfrak{S}$ corresponding to $\gamma_0(r)$, let $\mathfrak{A}_0(R; r)$ be the area corresponding to the annulus between Γ_0, γ_0. Then relations* (13), (14) *hold for $\mathfrak{L}_0$, $\mathfrak{A}_0$, and $\mathfrak{L}_0(r)$ tends to a limit as $r \to \infty$, which is infinite whenever $\Phi_0 = \lim_{r\to\infty} \frac{1}{2\pi} \int_{\gamma_0(r)} \frac{\partial u}{\partial n} ds > -1$. The quantity Φ_0 exists and is finite, and $\Phi_0 \geq -1$. If $\Phi_0 > -1$, then* (22) *holds. Also, $\lim_{r\to\infty} \frac{\mathfrak{L}_0^2(r)}{4\pi \mathfrak{A}_0(R; r)} = \nu_0$ exists, and $\nu_0 = 1 + \Phi_0$. The radial lines through γ_0 are asymptotically geodesic in the sense of Theorem 13, and their lengths can be estimated by* (25). *The curves $\gamma_0(r)$ are, in the corresponding sense, asymptotically geodesic circles on $\mathfrak{S}$. Under the additional assumption of Theorem 14 at p_0, correspondingly improved estimates hold.*

Theorem 20. *Let $\mathfrak{S}$ be a complete abstract surface satisfying Hypothesis S and which has finite total curvature C. Then (cf. Theorem 12) $\nu_j = \lim_{\gamma_j \to p_j} \frac{\mathfrak{L}_j(r)}{4\pi \mathfrak{A}_j(R; r)}$*

exists at each boundary component p_j, *there holds* $\nu_j = 1 + \Phi_j$, *and*

$$C = 2\pi(\chi - \sum_0^n \nu_j).$$

Hypothesis S can be deleted for any case in which it is known that the metric is normal at each boundary component.

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