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Homology and fibrations I

Coalgebras, cotensor product and its derived functors

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1. Introduction

The study of the relations between the homology structure of the base space, the total space and the fiber of a fibration offers ample opportunity for application of homological algebra. This series of papers develops some of this algebra and derives its relations with the geometric situations.

In this paper the basic notion is that of a (graded differential) coalgebra A over a commutative ring K . Left and right (graded differential) A -comodules are defined as well as a cotensor product $A \square_A B$ of a right A -comodule A and a left A -comodule B . Using a suitable relative notion of an injective resolution the derived functor $\text{Cotor}^A(A, B)$ (which is a graded K -module) is defined. This functor is the target of a spectral sequence $\{E^r(A, A, B), d^r\}$ and under some flatness conditions (which are always satisfied if K is a field) the term $E^2(A, A, B)$ is isomorphic with $\text{Cotor}^{H(A)}(H(A), H(B))$. This algebraic apparatus is developed in §§ 2–10.

The contact with geometry is established in the following way. We consider a commuting diagram

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ \pi' \downarrow & & \downarrow \pi \\ B' & \xrightarrow{f} & B \end{array}$$

of topological spaces and continuous maps. The normalized singular chains of B with coefficients in K yield a coalgebra $(B; K)$. We regard $(B'; K)$ as a left $(B; K)$ -comodule and for any coefficient K -module C we may regard $(E; C)$ as a right $(B; K)$ -comodule. The diagram above then yields a natural transformation

$$\tau : H(E'; C) \rightarrow \text{Cotor}^{(B; K)}((E; C), (B'; K)) .$$

The main result (Theorem 12.1) asserts that if the space B is pathwise connected and simply connected, π is a fibration and π' is the induced fibration by f then τ yields an isomorphism

¹⁾ Supported by Contract NONR 266 (57).

$$H(E'; C) \approx \text{Cotor}^{(B; K)}((E; C), (B'; K)). \quad (1.1)$$

A very important special case is when B' is a single point. Then E' is the fiber F corresponding to the point $f(B')$ of B and (1.1) becomes

$$H(F; C) \approx \text{Cotor}^{(B; K)}((E; C), K). \quad (1.2)$$

In § 15 we give a more elaborate relative version of (1.1). In § 18 we show that if $C = K$ and suitable flatness conditions hold then both sides of (1.1) and (1.2) acquire the structure of graded K -coalgebras and (1.1) and (1.2) are isomorphisms of K -coalgebras. For general C (1.1) and (1.2) are isomorphisms of right comodules over these coalgebras. Similar considerations apply to the terms of the spectral sequences approximating the right sides of (1.1) and (1.2).

A very important special case has to be attributed to ADAMS [1]. If E is the space of paths in B with fixed origin b_0 and $\pi: E \rightarrow B$ is the evaluation at the end point of each path, the fiber F is the loop space $\Omega(B)$. Since E is contractible, the isomorphism (1.2) may be reduced to

$$H(\Omega(B); C) \approx \text{Cotor}^{(B; K)}(C; K) \quad (1.3)$$

and in particular

$$H(\Omega(B); K) \approx \text{Cotor}^{(B; K)}(K, K). \quad (1.4)$$

Starting with the coalgebra $(B; K)$, ADAMS has constructed a complex X (the co-bar construction) and has shown that $H(X) \approx H(\Omega(B); K)$ when K is a principal ideal integral domain. In our theory X appears as an injective resolution which allows to compute $\text{Cotor}^{(B; K)}(K, K)$.

The definition of Cotor properly belongs to the domain of relative homological algebra that will be treated by us in a forthcoming publication. However, it has been possible without serious loss of space to give here an entirely self-contained account.

2. Coalgebras and comodules

Let K denote a commutative ring (with unit). We shall consider complexes C

$$\dots \rightarrow C_n \xrightarrow{d_n} C_{n-1} \rightarrow \dots \rightarrow C_0 \rightarrow 0$$

of K -modules. A *morphism* $f: C \rightarrow D$ will be a family of K -morphisms

$f_n: C_n \rightarrow D_n$ commuting with the differentiations. The resulting category is denoted by DGK . Any K -module C is regarded as an object of DGK by taking $C_0 = C$ and $C_n = 0$ for $n \neq 0$. In particular K itself is an object of DGK . The tensor product $C \otimes D$ for $C, D \in DGK$ is defined in the usual way and is again in DGK . We adopt the usual identifications $K \otimes C = C = C \otimes K$.

A coalgebra A over K (or a K -coalgebra) consists of an object A of DGK together with a pair of morphisms

$$\varepsilon: A \rightarrow K, \quad \delta: A \rightarrow A \otimes A$$

satisfying the identities

$$(\varepsilon \otimes A)\delta = 1_A = (A \otimes \varepsilon)\delta, \quad (\delta \otimes A)\delta = (A \otimes \delta)\delta.$$

A right A -comodule is an object A of DGK together with a “structure morphism”

$$\nabla: A \rightarrow A \otimes A$$

satisfying

$$(\varepsilon \otimes A)\nabla = 1_A, \quad (\nabla \otimes A)\nabla = (A \otimes \delta)\nabla.$$

With the obvious morphisms, these right comodules form a category DGA (the category of graded differential right comodules over A). Analogously we define the left comodules and obtain the category ADG . The ring K itself is a coalgebra with $\varepsilon = \delta = 1_K$ and every complex over K is also a comodule. Thus the notation DGK is unambiguous.

All the above can be repeated by requiring that all differential operators be zero. We then obtain the categories GK , GA and AG where A is a K -coalgebra with zero differentiation. We shall regard GA as a subcategory of DGA .

The category DGA is an additive category with cokernels. If A is K -flat (i. e., if each A_n is a flat K -module) then DGA is an abelian category. The category DGA is equipped with a natural functor $DGA \rightarrow DGK$ obtained by neglecting the structure morphism. There is an adjoint functor in the opposite direction which to each object $C \in DGK$ assigns the *extended* A -comodule $C \otimes A$ with the structure morphism $\nabla = C \otimes \delta$. More generally, if A is a right A -comodule and C is in DGK then $C \otimes A$ is a right A -comodule with the structure morphism

$$C \otimes \nabla: C \otimes A \rightarrow C \otimes A \otimes A.$$

Let A be a right Λ -comodule and B a left Λ -comodule. The *cotensor* product $A \square_{\Lambda} B$ is defined as the object in DGK which is the kernel of

$$i: A \otimes B \rightarrow A \otimes \Lambda \otimes B, \quad i: \nabla_A \otimes B - A \otimes \nabla_B.$$

Proposition 2.1. If $A = C \otimes \Lambda$ is an extended comodule then the morphism

$$j: C \otimes \nabla_B: C \otimes B \rightarrow C \otimes \Lambda \otimes B = A \otimes B$$

establishes an isomorphism

$$C \otimes B \approx (C \otimes \Lambda) \square_{\Lambda} B.$$

Proof. Let $k = C \otimes \varepsilon \otimes B: C \otimes \Lambda \otimes B \rightarrow C \otimes B$. We verify by computation that

$$ij = 0, \quad kj = 1_{C \otimes B}, \quad jkf = f \quad \text{if} \quad if = 0.$$

Thus j is a kernel for i .

Corollary 2.2. $A \square_{\Lambda} \Lambda = A$.

3. Resolutions

The functor Cotor^{Λ} will be defined as the right derived functor of the cotensor product, in a suitable relative sense. This relativity is indicated by defining the terms "injective" and "exact". For the purposes of this paper we choose the *injective* comodules to be the direct summands of extended comodules.

Proposition 3.1. A right Λ -comodule A is injective if and only if there exists a morphism $f: A \otimes \Lambda \rightarrow A$ of right Λ -comodules such that the composition

$$A \xrightarrow{\nabla} A \otimes \Lambda \xrightarrow{f} A$$

is the identity.

Proof. If A is a direct summand of an extended Λ -comodule $C \otimes \Lambda$ then we have morphisms

$$A \xrightarrow{g} C \otimes \Lambda \xrightarrow{h} A$$

in DGA whose composition is the identity. Then define f as the composition

$$A \otimes A \xrightarrow{g \otimes A} C \otimes A \otimes A \xrightarrow{C \otimes A \otimes \varepsilon} C \otimes A \xrightarrow{h} A.$$

The *exact sequences* in DGA are defined to be the sequences in DGA which are split exact when viewed in DGK .

A complex X in DGA will always be assumed to be negative

$$X^0 \rightarrow X^1 \rightarrow \dots \rightarrow X^n \rightarrow \dots$$

with the usual convention $X^n = X_{-n}$. The complex X is said to be *injective* if each X^n is injective in the tense defined above. A complex X is augmented if it is accompanied by a morphism $\varepsilon: A \rightarrow X^0$ such that the composition $A \rightarrow X^0 \rightarrow X^1$ is zero. We write $\varepsilon: A \rightarrow X$, regarding A as a complex concentrated in degree zero and ε as a morphism of complexes. The augmented complex $\varepsilon: A \rightarrow X$ is said to be *acyclic* if the sequence

$$0 \rightarrow A \rightarrow X^0 \rightarrow X^1 \rightarrow \dots \rightarrow X^n \rightarrow \dots$$

is exact, i.e. is split exact in the category DGK . If $\varepsilon: A \rightarrow X$ is both injective and acyclic then we say that $\varepsilon: A \rightarrow X$ is an *injective resolution* of A . The existence of injective resolutions in the above sense will be shown in § 6. The only other fact needed here is

Proposition 3.2. Consider a diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & X^0 & \rightarrow & X^1 \rightarrow \dots \rightarrow X^n \rightarrow \dots \\ & & f \downarrow & & & & \\ 0 & \rightarrow & A' & \rightarrow & X'^0 & \rightarrow & X'^1 \rightarrow \dots \rightarrow X'^n \rightarrow \dots \end{array}$$

in DGA in which the upper row is exact and in the lower row X'^n are injective for $n = 0, 1, \dots$. Then there exists a family of morphisms $f^n: X^n \rightarrow X'^n$ which render the diagram commutative.

As usual the sequence $\{f^n\}$ is defined inductively using

Proposition 3.3. Given an exact sequence

$$\dots \rightarrow A_n \rightarrow A_{n-1} \rightarrow \dots \quad -\infty < n < \infty \quad (3.1)$$

in DGA and given an injective comodule B in DGA the sequence

$$\dots \leftarrow \Lambda(A_n, B) \leftarrow \Lambda(A_{n-1}, B) \leftarrow \dots$$

is exact. Here $\Lambda(A, B)$ denotes the K -module of all morphisms $A \rightarrow B$ in DGA .

Proof. Without loss of generality we may assume that $B = C \otimes_A \Lambda$ is an extended comodule. We then have the natural isomorphism $\Lambda(A, C \otimes \Lambda) \approx K(A, C)$ where $f: A \rightarrow C \otimes \Lambda$ and $g: A \rightarrow C$ determine each other as follows

$$g = (C \otimes \varepsilon)f, \quad f = (g \otimes \Lambda) \nabla_A.$$

Since in the category DGK the sequence (3.1) is split exact, the conclusion follows.

4. Complexes and filtrations

Let X be a complex in DG and Y a complex in DG . We then have the K -modules

$$T_{q,s,t}(X, \Lambda, Y) = (X_s \square_\Lambda Y_t)_q$$

with commuting differential operators

$$d' : T_{q,s,t} \rightarrow T_{q-1,s,t}, \quad d'' : T_{q,s,t} \rightarrow T_{q,s-1,t}, \quad d''' : T_{q,s,t} \rightarrow T_{q,s,t-1}.$$

We convert this "triple" complex into a "single" complex $T(X, \Lambda, Y)$ using the direct product as follows:

$$T_n(X, \Lambda, Y) = \prod T_{q,s,t}(X, \Lambda, Y), \quad q + s + t = n$$

and considering the total differential operator which on each $T_{q,s,t}$ is

$$d = d' + (-1)^q d'' + (-1)^{q+s} d'''.$$

Note that $T(X, \Lambda, Y)$ is a complex of K -modules ranging in general from $-\infty$ to $+\infty$.

Similarly we define

$$S_{q,s,t}(X, \Lambda, Y) = H_q(X_s \square_\Lambda Y_t)$$

$$S_{q,p}(X, \Lambda, Y) = \prod S_{q,s,t}(X, \Lambda, Y), \quad s + t = p$$

$$S_n(X, \Lambda, Y) = \prod S_{q,p}(X, \Lambda, Y), \quad p + q = n.$$

The differential operators on the complexes $S_{q,*}$ and S are given by $(-1)^q d'' + (-1)^{q+s} d'''$. Thus in a sense S is the homology of T with respect to the partial differential operator d' . Clearly

$$H_n(S(X, \Lambda, Y)) = \coprod H_r(S_{q,*}(X, \Lambda, Y)), \quad q + r = n.$$

We define the *main filtration* of $T(A, \Lambda, B)$ by setting

$$F_{r,n} = \coprod T_{q,s,t}, \quad n = q + s + t, \quad s + t \leq r.$$

We then have

$$T = F_0 \supset F_{-1} \supset \dots \supset F_p \supset F_{p-1} \supset \dots$$

The filtration is complete [5] because of the direct product used in the definition of T . Further we find that

$$F_{p,n}/F_{p-1,n} = \coprod T_{q,s,t}, \quad p = s + t, \quad n = p + q.$$

Therefore

$$E_{p,q}^1 = S_{q,p}$$

and

$$E_{p,q}^2 = H_p(S_{q,*}).$$

Proposition 4.1. Let X be an injective complex in DGA and let $\eta: B \rightarrow Y$ be an acyclic augmented complex in ADG . Then

$$HT(X, \Lambda, \eta): HT(X, \Lambda, B) \rightarrow HT(X, \Lambda, Y)$$

$$HS(X, \Lambda, \eta): HS(X, \Lambda, B) \rightarrow HS(X, \Lambda, Y)$$

are isomorphisms.

Proof. In view of the main filtration, the statement for S implies that for T . We now filter the complexes $S_{q,*}(X, \Lambda, B)$ and $S_q(X, \Lambda, Y)$ by the resolution degree of X . The associated graded objects are then $S_{q,*}(X_p, \Lambda, B)$ and $S_{q,*}(X_p, \Lambda, Y)$. Thus it suffices to show that

$$HS_{q,*}(A, \Lambda, \eta): HS_{q,*}(A, \Lambda, B) \rightarrow HS_{q,*}(A, \Lambda, Y)$$

is an isomorphism for every injective A . Without loss of generality we may assume that A is an extended comodule $A = C \otimes \Lambda$. From 2.1 we deduce

that $S_{q,*}(A, \Lambda, B) = H_q(C \otimes B)$ while $S_{q,*}(A, \Lambda, Y)$ is the complex

$$H_q(C \otimes Y^0) \rightarrow H_q(C \otimes Y^1) \rightarrow \dots$$

Since the sequence

$$0 \rightarrow B \rightarrow Y^0 \rightarrow Y^1 \rightarrow \dots$$

is split exact in the category DGK , it remains exact after the application of the functor $H_q(C \otimes -)$. This implies that $HS_{q,*}(A, \Lambda, \eta)$ is an isomorphism as required.

5. The functor Cotor

Consider $A \in DGA$, $B \in \Lambda DG$ and let

$$\varepsilon: A \rightarrow X, \quad \eta: B \rightarrow Y$$

be injective resolutions. It follows from 4.1 that we have isomorphisms

$$HT(X, \Lambda, B) \xrightarrow{HT(X, \Lambda, \eta)} HT(X, \Lambda, Y) \xleftarrow{HT(\varepsilon, \Lambda, Y)} HT(A, \Lambda, Y)$$

of graded K -modules. Any of these three graded K -modules is denoted by $\text{Cotor}^A(A, B)$. The independence of $\text{Cotor}^A(A, B)$ of the choice of resolutions and the functorial properties of Cotor will be established below.

By 4.1 we also have isomorphisms

$$H_q S_{p,*}(X, \Lambda, B) \rightarrow H_q S_{p,*}(X, \Lambda, Y) \leftarrow H_q S_{p,*}(A, \Lambda, Y)$$

and the common value of these K -modules is denoted by

$$E_{p,q}^2(A, \Lambda, B).$$

These are the terms $E_{p,q}^2$ of the spectral sequences of the main filtration applied to the complexes $T(X, \Lambda, B)$ or $T(X, \Lambda, Y)$ or $T(A, \Lambda, Y)$. The terms $E_{p,q}^r$, $r \geq 2$ of these spectral sequences will be denoted by $E_{p,q}^r(A, \Lambda, B)$. We note that $E_{p,q}^r = 0$ unless $p \leq 0$ and $q \geq 0$.

The augmentations ε and η induce morphisms

$$e: H(A \square_{\Lambda} B) \rightarrow \text{Cotor}^A(A, B)$$

$$e': H_q(A \square_{\Lambda} B) \rightarrow E_{0,q}^2(A, \Lambda, B).$$

If A is injective then we may choose $A = X$ and e and e' are then isomorphisms. Similarly if B is injective.

Suppose now that Λ , A and B have zero differentiation. Then (see § 6) the injective resolutions X and Y may be chosen so that each X_s and Y_t has zero differentiation. Then in the complex $T(X, \Lambda, Y)$ the differential operator d' is identically zero so that $T(X, \Lambda, Y)$ coincides with $S(X, \Lambda, Y)$. Thus the spectral sequence collapses and $\text{Cotor}^A(A, B)$ may be identified with $E^2(A, \Lambda, B)$. Thus $\text{Cotor}^A(A, B)$ is bigraded in the sense that

$$\text{Cotor}_n^A(A, B) = \prod \text{Cotor}_{p,q}(A, B), \quad p + q = n$$

where

$$\text{Cotor}_{p,q}(A, B) = E_{p,q}^2(A, B) = H_q(S_{p,*}(X, \Lambda, Y)).$$

Let $\varphi: \Lambda \rightarrow \Lambda'$ be a morphism of K -coalgebras. Every Λ -comodule A may then be regarded as a Λ' -comodule with the structure morphism

$$A \xrightarrow{\nabla} A \otimes \Lambda \xrightarrow{A \otimes \varphi} A \otimes \Lambda'.$$

A φ -morphism $f: A \rightarrow A'$ for $A \in DGA$, $A' \in DGA'$ is defined as a morphism in DGA' of A regarded as a Λ' -comodule. If further $g: B \rightarrow B'$ is a φ -morphism with $B \in \Lambda DG$, $B' \in \Lambda' DG$ then we readily define the induced morphism $f \square_{\varphi} g: A \square_{\Lambda} B \rightarrow A' \square_{\Lambda'} B'$ in DGK .

Now let

$$\varepsilon': A' \rightarrow X' \quad \eta': B' \rightarrow Y'$$

be a Λ' -injective resolution of $A' \in DGA'$, $B' \in \Lambda' DG$. From 3.1 we deduce the existence of φ -morphisms $F: X \rightarrow X'$ and $G: Y \rightarrow Y'$ such that $F\varepsilon = \varepsilon'f$, $G\eta = \eta'g$. We then have the commutative diagram

$$\begin{array}{ccccc} HT(X, \Lambda, B) & \longrightarrow & HT(X, \Lambda, Y) & \longleftarrow & HT(A, \Lambda, Y) \\ HT(F, \varphi, g) \downarrow & & GT(F, \varphi, G) \downarrow & & HT(f, \varphi, G) \downarrow \\ HT(X', \Lambda', B') & \longrightarrow & HT(X', \Lambda', Y') & \longleftarrow & HT(A', \Lambda', Y') \end{array}$$

where all the horizontal morphisms are isomorphisms. It follows that the vertical morphisms are independent of the "liftings" F and G of f and g . There results a morphism

$$\text{Cotor}^{\varphi}(f, g): \text{Cotor}^A(A, B) \rightarrow \text{Cotor}^{A'}(A', B')$$

defined for a morphism $\varphi: \Lambda \rightarrow \Lambda'$ and φ -morphisms $f: A \rightarrow A'$, $g: B \rightarrow B'$.

If $A = A'$ and $\varphi = 1_A$ then we write $\text{Cotor}^A(f, g)$ for $\text{Cotor}^\varphi(f, g)$, and this defines the structure of Cotor^A as a functor. Incidentally, taking $\varphi = 1_A$, $f = 1_A$, $g = 1_B$, the argument above shows the independence of $\text{Cotor}^A(A, B)$ of the choice of the resolutions.

The same procedure applied to the complexes S instead of T yields the morphism

$$E_{p,q}^2(f, \varphi, g) : E_{p,q}^2(A, A, B) \rightarrow E_{p,q}^2(A', A', B').$$

These are of course the morphisms of the terms E^2 of the spectral sequences induced by $T(F, \varphi, G)$.

6. Canonical and tapered resolutions

Given $A \in DG\mathcal{A}$ consider the sequence

$$0 \longrightarrow A \xrightarrow{\nabla} A \otimes A \xrightarrow{l} A^1 \longrightarrow 0$$

where $A \otimes A$ has $A \otimes \delta$ as structural morphism. Then ∇ is a morphism in $DG\mathcal{A}$ and we define l as the cokernel of ∇ . Since $(A \otimes \varepsilon)\nabla = 1_A$ the sequence is a split exact sequence in DGK and thus is an exact sequence in the sense required here.

Iterating this procedure, we obtain exact sequences

$$0 \longrightarrow A^p \xrightarrow{k^p} A^p \otimes A \xrightarrow{l^p} A^{p+1} \longrightarrow 0 \quad (6.1)$$

where $A = A^0$, $p = 0, 1, \dots$. Thus setting

$$X^p = A^p \otimes A, \quad d^p = k^{p+1} l^p$$

we obtain an injective resolution $\nabla : A \rightarrow X$. This is the *canonical* injective resolution of A . This resolution has many useful properties. In particular, it inherits many properties from A and A . For instance, a fact that we used in § 5, if A and A have zero differentiation then so does each X^p .

The coalgebra A will be called *connected* if the morphism $\varepsilon : A \rightarrow K$ induces an isomorphism $A_0 \approx K$. In this case we usually identify A_0 and K . This imbedding of K into A is a morphism of K -coalgebras and thus permits us to regard K as a left or right A -comodule.

If further $A_1 = 0$, then A is called *simply connected*. More generally A is *k-connected* ($k \geq 0$) if it is connected and if $A_i = 0$ for $0 < i \leq k$. Thus the

terms “0-connected” and “1-connected” coincide with “connected” and “simply connected”.

Proposition 6.1. If the coalgebra A is k -connected then in the canonical injective resolution $\nabla : A \rightarrow X$ we have

$$(X^p)_i = 0 \quad \text{if } i < (k+1)p.$$

Indeed, if A is k -connected and if $A_i = 0$ for $i < s$, then $A_i = (A \otimes A)_i$ for $i < k+1+s$, so that $A_i^1 = 0$ for $i < k+1+s$. Thus by induction it follows that $A_i^p = 0$ for $i < (k+1)p$.

Corollary 6.2. If A is connected then

$$\text{Cotor}_n^A(A, B) = 0 \quad \text{if } n < 0$$

$$E_{p,q}^2(A, B) = 0 \quad \text{if } p+q < 0.$$

Indeed, if X is the canonical resolution of A then for $n < 0$

$$T_n(X, A, B) = 0, \quad S_n(X, A, B) = 0.$$

We shall be particularly interested in the case when A is simply connected. In this case the canonical resolution X satisfies

$$(X^p)_i = 0 \quad \text{if } i < 2p.$$

A complex X with the above property will be called *tapered*. If X and Y are tapered complexes then we have

$$T_{q,s,t}(X, A, Y) = 0 \quad \text{except when } -n \leq s+t \leq 0$$

where $n = q+s+t$. This implies that the product used to define $T_n(X, A, Y)$ is finite. For the main filtration $\{F_r\}$ of $T(X, A, Y)$ we have

$$T_n = F_{0,n} \supset F_{-1,n} \supset \dots \supset F_{-n,n} \supset F_{-n-1,n} = 0.$$

Thus the main filtration is finite in each degree. Consequently, the spectral sequence converges in the naive sense.

For a tapered complex X in DGA we define a right A -comodule \tilde{X} as follows

$$(\tilde{X})_n = \Sigma (X^p)_i, \quad n = i - p.$$

We note that the direct sum is finite. The differential operator is $d = d' + (-1)^i d''$ where

$$d' : (X^p)_i \rightarrow (X^p)_{i-1}, \quad d'' : (X^p)_i \rightarrow (X^{p+1})_i.$$

Now observe that for any K -module B , the complex $X \otimes B$ given by

$$0 \rightarrow X^0 \otimes B \rightarrow X^1 \otimes B \rightarrow \dots \rightarrow X^n \otimes B \rightarrow \dots$$

is again tapered and so we have $(X \otimes B)^\sim$. Since the tensor product commutes with direct sums, this may be identified with $\tilde{X} \otimes B$. If we now take $B = A$ then the morphisms $\nabla^p : X^p \rightarrow X^p \otimes A$ induce a morphism $X \rightarrow X \otimes A$ of complexes, which in turn induce $\tilde{\nabla} : \tilde{X} \rightarrow \tilde{X} \otimes A$. This converts \tilde{X} into a right A -comodule.

For any complex Y in ADG , we now observe that

$$T(X, A, Y) = T(\tilde{X}, A, Y).$$

If Y also is tapered, then

$$T(X, A, Y) = T(\tilde{X}, A, \tilde{Y}) = \tilde{X} \square_A \tilde{Y}.$$

If X is a tapered injective resolution of A then we have $H(A) = \text{Cotor}^A(A, A) = H(\tilde{X} \square_A A) = H(\tilde{X})$. This yields

Proposition 6.3. If $\varepsilon : A \rightarrow X$ is a tapered injective resolution of A then $H(\tilde{\varepsilon}) : H(A) \rightarrow H(\tilde{X})$ is an isomorphism.

This fact can of course also be verified directly by filtering \tilde{X} by the resolution degree of X .

7. Flatness conditions

A DGK -module B will be called K -flat if B_n is K -flat for each n . If B is K -flat and A is any DGK -module then there is a spectral sequence converging to $H(A \otimes B)$ and with $\text{Tor}^K(H(A), H(B))$ as term E^2 . This will be called the KÜNNETH spectral sequence [3, Ch. XVII].

Theorem 7.1. Let $\varphi : A \rightarrow A'$ be a morphism of K -coalgebras and let $f : A \rightarrow A'$, $g : B \rightarrow B'$ be φ -morphisms with $A \in DGA$, $A' \in DGA'$,

$B \in \Lambda DG$, $B' \in \Lambda' DG$. Assume further that Λ, Λ', B, B' are K -flat. If $H(\varphi): H(\Lambda) \rightarrow H(\Lambda')$, $H(f): H(A) \rightarrow H(A')$ and $H(g): H(B) \rightarrow H(B')$ are isomorphisms then

$$\text{Cotor}^\varphi(f, g): \text{Cotor}^\Lambda(A, B) \rightarrow \text{Cotor}^{\Lambda'}(A', B')$$

$$E^2(f, \varphi, g): E^2(A, \Lambda, B) \rightarrow E^2(A', \Lambda', B')$$

are isomorphisms.

Proof. Clearly, the conclusion concerning E^2 implies that for Cotor .

Let X and X' be the canonical resolutions of A and A' . Since the canonical resolutions are functorial, the φ -morphism $f: A \rightarrow A'$ yields a φ -morphism $F: X \rightarrow X'$. More precisely, we have the commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^p & \longrightarrow & A^p \otimes \Lambda & \longrightarrow & A^{p+1} \longrightarrow 0 \\ & & f^p \downarrow & & f^p \otimes \varphi \downarrow & & f^{p+1} \downarrow \\ 0 & \longrightarrow & A'^p & \longrightarrow & A'^p \otimes \Lambda' & \longrightarrow & A'^{p+1} \longrightarrow 0 \end{array}$$

with f^p defined inductively starting with $f: A \rightarrow A'$ and with $F^p = f^p \otimes \varphi$. If $H(f^p)$ is an isomorphism then so is $H(f^p \otimes \varphi)$, by the KÜNNETH spectral sequence. Therefore by the “five lemma”, $H(f^{p+1})$ also is an isomorphism. Hence by induction it follows that $H(f^p)$ is an isomorphism for every p .

To show that $E^2(f, \varphi, g)$ is an isomorphism it suffices to show that

$$S(F, \varphi, g): S(X, \Lambda, B) \rightarrow S(X', \Lambda', B')$$

is an isomorphism. For this it suffices to show that

$$H(F^p \square_\varphi g): H(X^p \square_\Lambda B) \rightarrow H(X'^p \square_{\Lambda'} B')$$

is an isomorphism. Since $F^p = f^p \otimes \varphi$. This reduces to

$$H((f^p \otimes \varphi) \square_\varphi g): H((A^p \otimes \Lambda) \square_\Lambda B) \rightarrow H((A'^p \otimes \Lambda') \square_{\Lambda'} B').$$

By 2.1 this reduces to

$$H(f^p \otimes g): H(A^p \otimes B) \rightarrow H(A'^p \otimes B').$$

The fact that this is an isomorphism follows again from the KÜNNETH spectral sequence.

8. Finiteness conditions

A DGK -module A will be said to be of finite type if A_n is finitely K -generated for each n .

Proposition 8.1. Assume that the ring K is noetherian, A is simply connected, A and B are K -flat and $H(A)$, $H(A)$ and $H(B)$ are of finite type. Then $\text{Cotor}^A(A, B)$ is of finite type.

Proof. Let C be any DGK -module with $H(C)$ of finite type. Then $\text{Tor}_p^K(H_n(C), H_m(B))$ is finitely generated for each p, n, m and therefore by the KÜNNETH spectral sequence $H(C \otimes B)$ is of finite type.

We now consider the construction of the canonical resolution X of A . From the exact sequence (6.1) we deduce the exact triangle

$$\begin{array}{ccc} H(A^p) & \rightarrow & H(A^p \otimes A) \\ & \nwarrow & \swarrow \\ & H(A^{p+1}) & \end{array}$$

Thus if $H(A^p)$ is of finite type, then so is $H(A^p \otimes A)$ and consequently also $H(A^{p+1})$. Thus by induction, $H(A^p)$ is of finite type. Since $X^n \square_A B = (A^n \otimes A) \square_A B \approx A^n \otimes B$, it follows that $H(X^n \square_A B)$ is of finite type. As a consequence $E_{p,q}^2(A, A, B)$ is finitely generated for every p, q . Since A is simply connected the convergence of the spectral sequence yields that $\text{Cotor}^A(A, B)$ is of finite type.

9. Calculation of $E_{p,q}^2$

If A and B are DGK -modules, then we have the morphism

$$H(A) \otimes H(B) \rightarrow H(A \otimes B). \quad (9.1)$$

If B and $H(B)$ (or A and $H(A)$) are K -flat, then it follows from the KÜNNETH spectral sequence that (9.1) is an isomorphism. Under these conditions we shall regard (9.1) as an identification.

If A is a K -coalgebra and if A and $H(A)$ are K -flat, then it follows readily that the mappings

$$H(\varepsilon): H(A) \rightarrow K, \quad H(\delta): H(A) \rightarrow H(A \otimes A) = H(A) \otimes H(A)$$

convert $H(\Lambda)$ into a K -coalgebra. Similarly, if A is a right Λ -comodule, then

$$H(\nabla): H(A) \rightarrow H(A \otimes \Lambda) = H(A) \otimes H(\Lambda)$$

converts $H(A)$ into a right $H(\Lambda)$ -comodule. Similarly, for left Λ -comodules.

Now let A be in $DG\Lambda$ and B in ΛDG . Assume that Λ , $H(\Lambda)$, B and $H(B)$ are K -flat. We then have the exact sequences

$$\begin{aligned} 0 \longrightarrow A \square_{\Lambda} B \longrightarrow A \otimes B \xrightarrow{i} A \otimes \Lambda \otimes B \\ 0 \longrightarrow H(A) \square_{H(\Lambda)} H(B) \longrightarrow H(A) \otimes H(B) \xrightarrow{i'} H(A) \otimes H(\Lambda) \otimes H(B) \end{aligned}$$

and we may identify i' with $H(i)$. There results a natural morphism

$$H(A \square_{\Lambda} B) \rightarrow H(A) \square_{H(\Lambda)} H(B). \quad (9.2)$$

Proposition 9.1. If under the conditions above, A is Λ -injective then $H(A)$ is $H(\Lambda)$ -injective and (9.2) is an isomorphism. Similarly, if B is Λ -injective, then $H(B)$ is $H(\Lambda)$ -injective and (9.2) is an isomorphism.

Proof. If A is Λ -injective then by 3.1 there exists a morphism $f: A \otimes \Lambda \rightarrow A$ such that the composition

$$A \xrightarrow{\nabla} A \otimes \Lambda \xrightarrow{f} A$$

is the identity. It follows that the composition

$$\begin{aligned} H(A) \xrightarrow{H(\nabla)} H(A \otimes \Lambda) \xrightarrow{H(f)} A \\ \parallel \\ H(A) \otimes H(\Lambda) \end{aligned}$$

is the identity, so that $H(A)$ is $H(\Lambda)$ -injective. To prove that (9.2) is an isomorphism, we may replace A by $A \otimes \Lambda$. Then by 2.1 both sides of (9.2) become $H(A) \otimes H(B)$. The case when B is Λ -injective is entirely similar.

Now assume that Λ , $H(\Lambda)$, B and $H(B)$ are K -flat and let $\varepsilon: A \rightarrow X$ be an injective resolution of A . Then $E_{p,*}^2(A, \Lambda, B)$ is the homology of

$$H(X_{p+1} \square_{\Lambda} B) \rightarrow H(X_p \square_{\Lambda} B) \rightarrow H(X_{p-1} \square_{\Lambda} B)$$

which by 9.1 is the homology of

$$H(X_{p+1}) \square_{H(\Lambda)} H(B) \rightarrow H(X_p) \square_{H(\Lambda)} H(B) \rightarrow H(X_{p-1}) \square_{H(\Lambda)} H(B) . \quad (9.3)$$

Since the sequence $0 \rightarrow A \rightarrow X^0 \rightarrow X^1 \rightarrow \dots$ when regarded as a sequence in DGK is split exact, it follows that the sequence

$$H(A) \rightarrow H(X^0) \rightarrow H(X^1) \rightarrow \dots \quad (9.4)$$

is a split exact sequence of graded K -modules and therefore is an exact sequence in the category $DGH(\Lambda)$. Since $H(X^n)$ is $H(\Lambda)$ -injective, it follows that (9.4) is an injective resolution of $H(A)$ in the category $DGH(\Lambda)$. Thus the homology of (9.3) is $\text{Cotor}_{p,*}^{H(\Lambda)}(H(A), H(B))$. This yields

Theorem 9.2. If Λ , $H(\Lambda)$, B and $H(B)$ are K -flat, then we have a natural isomorphism

$$E_{p,q}^2(A, \Lambda, B) \approx \text{Cotor}_{p,q}^{H(\Lambda)}(H(A), H(B)) .$$

The same holds if the hypotheses that B and $H(B)$ are K -flat are replaced by the assumption that A and $H(A)$ are flat. In the proof we then use a resolution of B .

10. Properly filtered comodules

In this section we shall assume that the coalgebra Λ is K -flat, i.e. that for each p , the functor $\otimes \Lambda_p$ is exact.

Define the subcomodules $S_p \Lambda$ by setting

$$(S_p \Lambda)_q = \begin{cases} \Lambda_q & \text{if } q \leq p, \\ 0 & \text{if } q > p. \end{cases}$$

Then for any right Λ -comodule A set

$$S_p A = \nabla^{-1}(A \otimes S_p \Lambda) .$$

We then have

$$\begin{aligned} 0 &= S_{-1} A \subset S_0 A \subset \dots \subset S_p A \subset S_{p+1} A \subset \dots \\ &\cup S_p A = A \\ \nabla S_p A &\subset \Sigma S_u A \otimes S_v \Lambda, \quad u + v = p. \end{aligned}$$

Generalizing this we define a *proper filtration* T of A to be a sequence of subcomodules $T_p A$ of A such that

$$\begin{aligned}
0 = T_{-1} A &\subset T_0 A \subset \dots T_p A \subset T_{p+1} A \subset \dots \\
&\cup T_p A = A \\
\bigtriangledown T_p A &\subset T_u A \times S_v A, \quad u + v = p.
\end{aligned}$$

The filtration S described above is therefore a proper filtration of A and is called the filtration by *coskeletons*.

Since A is K -flat we find that \bigtriangledown induces morphisms

$$T_p/T_{p-1} \rightarrow T_u/T_{u-1} \otimes \Lambda_v, \quad u + v = p$$

from which we deduce, by projection, morphisms

$$\gamma_p^0: T_p/T_{p-1} \rightarrow T_0 \otimes \Lambda_p.$$

These morphisms are compatible with differentiation provided on the right side we use the differentiation $d \otimes \Lambda_p$. Thus, again using K -flatness of Λ_p , we obtain morphisms

$$\gamma_{p,q}^1: E_{p,q}^1(A) \rightarrow H_q(T_0 A) \otimes \Lambda_p.$$

We shall say that the proper filtration T of A is *perfect* if the following two conditions hold

(10.1) The diagrams

$$\begin{array}{ccc}
E_{p,q}^1(A) & \xrightarrow{\gamma_{p,q}^1} & H_q(T_0 A) \otimes \Lambda_p \\
d_{p,q}^1 \downarrow & & \downarrow (-1)^q 1 \otimes d_p \\
E_{p-1,q}^1(A) & \xrightarrow{\gamma_{p-1,q}^1} & H_q(T_0 A) \otimes \Lambda_{p-1}
\end{array}$$

are commutative.

(10.2) The induced morphisms

$$\gamma_{p,q}^2: E_{p,q}^2(A) \rightarrow H_q(H_p(T_0 A) \otimes \Lambda)$$

are isomorphisms.

Proposition 10.1. Let A, A' be right Λ -comodules with perfect filtrations T, T' and let $f: A \rightarrow A'$ be a morphism compatible with the filtrations. Then $H(f): H(A) \rightarrow H(A')$ is an isomorphism if and only if f induces an isomorphism $H(T_0 A) \approx H(T'_0 A')$.

This follows directly from the comparison theorem for spectral sequences (Seminaire Cartan 1954/55, Exposé 3).

Proposition 10.2. Let A be simply connected and let X be a tapered injective complex in DGA . Then the filtration of \tilde{X} by coskeletons is perfect.

Proof. Since each X^p is injective it is a direct factor of a comodule $C^p \otimes A$ where C^p is in DGK . Thus adding a direct summand to X^p , we may assume that $X^p = C^p \otimes A$ for every p . As a consequence we may assume that as a graded K -module we have $\tilde{X} = C \otimes A$.

This reduces the proof of 10.2 to the following

Proposition 10.3. Let A be simply connected, let $C \in GK$ and let d be a differentiation in $A = C \otimes A$ such that A is a right A -comodule with the structure morphism $\nabla = C \otimes \delta$. Then the filtration of A by coskeletons is perfect.

Proof. First observe that

$$S_p A = C \otimes S_p A.$$

Therefore

$$S_0 A = C \otimes A_0 = C$$

and thus C is a subcomodule of A with a differential operator d_C . Consider the differential operator

$$\tilde{d} = d_C \otimes 1 + 1 \otimes d_A.$$

For $c \in C_q$, $\lambda \in A_p$ we have

$$\nabla d(c \otimes \lambda) = d \nabla(c \otimes \lambda) = d(c \otimes \delta \lambda).$$

Since $A_1 = 0$ we have

$$\delta \lambda \equiv 1 \otimes \lambda \pmod{S_{p-2} A}.$$

Therefore $\text{mod } S_{p-2}(A \otimes A)$ we have

$$\nabla d(c \otimes \lambda) \equiv d(c \otimes 1 \otimes \lambda) \equiv d_C c \otimes 1 \otimes 1 + (-1)^q c \otimes 1 \otimes d_A \lambda.$$

Since the result depends only upon d_C and d_A , we have the same with d replaced by \tilde{d} . Thus we have shown that

$$\nabla d y \equiv \nabla \tilde{d} y \pmod{S_{p-2}(A \otimes A)} \quad \text{for } y \in S_p A.$$

Since $(C \otimes \varepsilon \otimes A) \nabla = (C \otimes \varepsilon \otimes A)(C \otimes \delta) = C \otimes 1_A = 1_A$ it follows that

$$d y = \tilde{d} y \pmod{S_{p-2} A} \quad \text{for } y \in S_p A.$$

Therefore $(E^0(A), d^0)$ and $(E^1(A), d^1)$ and $E^2(A)$ (but not necessarily d^2) will be the same for the differential operator d as for \tilde{d} . Thus we may assume that $d = \tilde{d}$, i.e., that $A = C \otimes A$ is an extended comodule. In this case $S_p A = C \otimes S_p A$ and the verification that the filtration is perfect is entirely obvious.

We now note that if A is connected then for any right comodule A , the subcomodule $S_0 A$ consists of all elements $a \in A$ such that $\nabla a = a \otimes 1$. From the definition of the cotensor product we then see that

$$S_0 A = A \square_A K.$$

If T is any proper filtration of A then $T_0 A \subset S_0 A$ and therefore we have a morphism

$$H(T_0 A) \rightarrow H(A \square_A K).$$

Composing this with the morphism

$$e_{A,K}: H(A \square_A K) \rightarrow \text{Cotor}^A(A, K)$$

we obtain

$$\beta: H(T_0 A) \rightarrow \text{Cotor}^A(A, K).$$

Theorem 10.4. Let A be a simply connected and K -flat coalgebra over K , let A be a right A -comodule with a perfect filtration T . Then β is an isomorphism.

Proof. Let $\varepsilon: A \rightarrow X$ be a tapered injective resolution of A and consider $\tilde{\varepsilon}: A \rightarrow \tilde{X}$. By 10.2, the filtration of \tilde{X} by coskeletons is perfect. Since $T_p A \subset S_p A$ we have that $\tilde{\varepsilon}(T_p A) \subset S_p \tilde{X}$. Further, by 6.3, $H(\tilde{\varepsilon})$ is an isomorphism. Thus 10.1 implies that $\tilde{\varepsilon}$ induces an isomorphism

$$H(T_0 A) \approx H(S_0 \tilde{X}).$$

However $H(S_0 \tilde{X}) = H(\tilde{X} \square_A K) = \text{Cotor}^A(A, K)$ and the proof is complete.

11. The geometric filtration

Let A be a simplicial set (i.e. a complete semi-simplicial complex) and let K be a commutative ring fixed once and for all. We shall denote by the same symbol A the DGK -module of normalized chains in A with coefficients in K

and write $H(A)$ for the homology GK -module. If C is any DGK -module then we write $H(A; C)$ for $H(A \otimes C)$.

If X is a topological space and $S(X)$ is its total singular simplicial set, then we shall frequently write X for $S(X)$. Thus X will denote a space, a simplicial set or a DGK -module depending upon the context.

Given a simplex s in A of dimension n , we denote by s_p^q ($0 \leq p \leq q \leq n$) the simplex of dimension $q - p$ obtained from s by applying the face operator ε_{q+1} ($n - q$)-times and then the face operator ε_0 p -times. We now define

$$\delta: A \rightarrow A \otimes A$$

by the usual ALEXANDER-WHITNEY formula

$$\delta s = \sum s_0^q \otimes s_q^n, \quad 0 \leq q \leq n.$$

We also define $\varepsilon: A \rightarrow K$ by setting $\varepsilon s = 0$ if $n > 0$ and $\varepsilon s = 1$ if $n = 0$. The mappings ε, δ convert A into a K -coalgebra and $A \otimes C$ is a left A -comodule.

A morphism $f: A \rightarrow B$ of simplicial sets induces a morphism of K -coalgebras and an f -morphism $f \otimes C: A \otimes C \rightarrow B \otimes C$ of comodules.

The *geometric filtration* G of $A \otimes C$ induced by f is defined by setting G_p to be the submodule of $A \otimes C$ generated by simplexes s such that $f s$ is an iterated degeneracy of a simplex of dimension $\leq p$. This is equivalent with the condition that $f(s')$ be degenerate for any face s' of s of dimension $> q$. To verify that G is a proper filtration of $A \otimes C$ as a left B -comodule, consider an n -simplex s in A . Then

$$\nabla(s \otimes c) = \sum s_0^q \otimes f(s_q^n) \otimes c, \quad 0 \leq q \leq n.$$

Thus if $s \otimes c \in G_p$ and $f(s_p^n)$ is non degenerate then $s_0^q \otimes c \in G_{p-q}$. Thus

$$\nabla G_p \subset \sum_q S_q B \otimes G_{p-q}$$

as required. The morphism

$$\gamma_p^0: G_p/G_{p-1} \rightarrow B \otimes G_0$$

takes then the form

$$\gamma_p^0(s \otimes c) = s_0^{n-p} \otimes f(s_{n-p}^n) \otimes c.$$

Now consider a fibration $\pi: E \rightarrow B$ of topological spaces, let M be a subcomplex of $S(B)$ with a single 0-simplex $b_0 \in B$, let N be the subcomplex

$\pi^{-1}(M)$ of $S(E)$ and let $\varrho: N \rightarrow M$ be induced by π . Then for the geometric filtration G of $N \otimes C$ induced by ϱ we have

$$G_0 = F \otimes C, \quad \text{where } F = \pi^{-1}(b_0)$$

and

$$\gamma^1: E^1(G) \rightarrow M \otimes H(F; C).$$

We shall use the following facts, which constitute the basic facts of the computation of the Serre spectral sequence of a fibration; [10], [9], [2].

(11.1) γ^1 is an isomorphism.

(11.2) Each 1-simplex of M defines an automorphism of $H(F; C)$ and there results a local coefficient system on M . If in $M \otimes H(F; C)$ this local system is used, then γ^1 commutes with the differentiation.

It follows from the above that the filtration G of $N \otimes C$ is perfect if the local system on M is constant. In particular, this is the case when M has only one 1-simplex, i.e. when M is a simply connected coalgebra.

12. The main results

Consider a commuting diagram

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ \pi' \downarrow & & \downarrow \pi \\ B' & \xrightarrow{f} & B \end{array}$$

of simplicial sets. The diagram is then also a commuting diagram of K -coalgebras. In particular, B' may be regarded as a right B -comodule and $E \otimes C$ (where $C \in DGK$) as a left B -comodule. There results a commuting diagram

$$\begin{array}{ccc} & H(E'; C) & \\ & \swarrow \approx & \searrow \approx \\ H(E' \square_{E'} (E' \otimes C)) & \xrightarrow{\approx} & \text{Cotor}^{E'}(E', E' \otimes C) \\ \downarrow H(\pi' \square_{\varphi} (g \otimes C)) & & \downarrow \text{Cotor}^{\varphi}(\pi', g \otimes C) \\ H(B' \square_B (E \otimes C)) & \longrightarrow & \text{Cotor}^B(B', E \otimes C) \end{array}$$

where $\varphi = \pi g = f\pi'$. There results a morphism

$$\tau: H(E'; C) \rightarrow \text{Cotor}^B(B', E \otimes C)$$

which is the prime objective of our investigation.

Theorem 12.1. Consider a commuting diagram of topological spaces

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ \pi' \downarrow & & \downarrow \pi \\ B' & \xrightarrow{f} & B \end{array}$$

where B is (pathwise) connected and simply connected, π is a fibration (in the sense of Serre) and π' is the fibration induced by f . Then for any commutative ring K and any DGK -module C we have

$$H(E'; C) \approx \text{Cotor}^B(B', E \otimes C).$$

In the special case when B' is the space consisting of a single point, E' is the fiber F of π corresponding to $b_0 = f(B')$ as base point. This yields

Theorem 12.2. If $\pi: E \rightarrow B$ is a fibration with fiber $F = \pi^{-1}(b_0)$ and if the space B is connected and simply connected, then

$$H(F; C) \approx \text{Cotor}^B(K, E \otimes C).$$

In both theorems the isomorphisms are instances of the morphism τ .

It should be observed that we could equally well regard $C \otimes E$ as a right B -comodule and B' as a left B -comodule, thereby interchanging the two variables in the functor Cotor^B .

13. Proof of Theorem 12.1

By considering the pathwise connected components of B' separately, we may assume without any loss of generality that B' is pathwise connected. We select base points $b'_0 \in B'$, $b_0 \in B$ such that $fb'_0 = b_0$. Then the fibers $F = \pi^{-1}(b_0)$ and $F' = \pi'^{-1}(b'_0)$ may be identified under g .

Let M be a minimal subcomplex of $S(B)$ relative to b_0 as base point. Let $N = \pi^{-1}(M)$ be the subcomplex of $S(E)$ consisting of all singular simplexes s with $\pi(s)$ in M . We note (without proof) that we have a commuting diagram

$$\begin{array}{ccccc}
N & \xrightarrow{k} & S(E) & \xrightarrow{l} & N \\
\varrho \downarrow & & S(\pi) \downarrow & & \varrho \downarrow \\
M & \xrightarrow{j} & S(B) & \xrightarrow{h} & M
\end{array}$$

of simplicial maps with the following properties.

(13.1) j and k are inclusions.

(13.2) hj and lk are identities.

(13.3) j, k, h and l induce isomorphisms of homology.

Similarly we choose a minimal complex in $S(B')$ relative to b'_0 and obtain a diagram as above with "primes".

Now consider the commuting diagram

$$\begin{array}{ccccccc}
N' & \xrightarrow{k'} & S(E') & \xrightarrow{S(g)} & S(E) & \xrightarrow{l} & N \\
\varrho' \downarrow & & S(\pi') \downarrow & & S(\pi) \downarrow & & \varrho \downarrow \\
M' & \xrightarrow{j'} & S(B') & \xrightarrow{S(f)} & S(B) & \xrightarrow{h} & M
\end{array} .$$

There results a commuting diagram

$$\begin{array}{ccc}
H(N'; C) & \xrightarrow{H(k'; C)} & H(E'; C) \xrightarrow{\tau} \text{Cotor}^B(B', E \otimes C) \\
\tau' \downarrow & & \text{Cotor}^h(B', l \otimes C) \downarrow \\
\text{Cotor}^M(M', N \otimes C) & \xrightarrow{\text{Cotor}^M(j, N \otimes C)} & \text{Cotor}^M(B', N \otimes C)
\end{array} \quad (13.4)$$

where τ' is the morphism resulting from the diagram

$$\begin{array}{ccc}
N' & \xrightarrow{v} & N \\
\varrho' \downarrow & & \downarrow \varrho \\
M' & \xrightarrow{u} & M
\end{array} \quad (13.5)$$

with $u = hS(f)j'$, $v = ls(g)k'$.

By the isomorphism theorem 7.1, the morphisms in (13.4) except τ and τ' are isomorphisms. Thus τ is an isomorphism if and only if τ' is an isomorphism.

We note that in diagram (13.5), M' has one 0-simplex, M has one 0-simplex and one 1-simplex and the fibers of ϱ and ϱ' are both $S(F)$, and are identified under v .

Since the coalgebra M is simply connected we may choose a tapered injective resolution of $N \times C$ as a left M -comodule

$$\varepsilon: N \otimes C \rightarrow X$$

from which we derive the morphism

$$\tilde{\varepsilon}: N \times C \rightarrow \tilde{X}$$

of left M -comodules.

The mapping τ' is then induced by

$$\xi: N' \otimes C \rightarrow M' \square_M \tilde{X}$$

defined for any n -simplex s in N' by

$$\xi(s \otimes c) = \sum \varrho'(s_0^q) \otimes \tilde{\varepsilon}(vs_q^n \otimes c), \quad 0 \leq q \leq n$$

where $M' \square_M \tilde{X}$ is regarded as a K -submodule of $M' \otimes \tilde{X}$.

In $N' \otimes C$ we consider the geometric filtration G given by ϱ' while in $M' \square_M \tilde{X}$ we consider the filtration R given by the degrees in M' . Then, if $s \otimes c \in G_p$ then $\varrho'(s_0^q)$ is degenerate for $q > p$ so that $\xi(s \otimes c) \in R_p$. Thus ξ is compatible with the filtrations G and R . By (11.1) we have

$$E^1 G = M' \otimes H(F; C).$$

The terms $E_{p,*}^0$ for R are

$$M'_p \square_M \tilde{X}$$

where M'_p has the trivial structure morphism $s \rightarrow s \otimes 1$. Thus M'_p as a right M -comodule is a direct sum of copies of K . Consequently we have the identification

$$M'_p \square_M \tilde{X} = M'_p \otimes (K \square_M \tilde{X}).$$

Therefore

$$E^1 R = M'_p \otimes H(K \square_M \tilde{X}) = M'_p \otimes \text{Cotor}^M(K, N \otimes C).$$

Consequently,

$$E^1 \xi = M' \otimes \beta$$

where

$$\beta: H(F; C) \rightarrow \text{Cotor}^M(K, N \otimes C)$$

is induced by

$$F \otimes C \longrightarrow N \otimes C \xrightarrow{\tilde{\varepsilon}} \tilde{X}.$$

Since the geometric filtration G of $N \otimes C$ given by ϱ is perfect and since $G_0 = F \otimes C$, the fact that β is an isomorphism follows from 10.4.

14. Connecting morphisms

Let

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

be a sequence of right Λ -comodules which is exact in the ordinary sense, i.e. such that

$$0 \rightarrow A'_n \rightarrow A_n \rightarrow A''_n \rightarrow 0 \quad (14.1)$$

is exact for every n . Assume that Λ is K -flat and that B is a K -flat left Λ -comodule. Then in the canonical resolution of B , the comodules B^n are K -flat. Since $A \square_{\Lambda} X^n = A \otimes B^n$ it follows that the sequences

$$0 \rightarrow A' \square_{\Lambda} X^n \rightarrow A \square_{\Lambda} X^n \rightarrow A'' \square_{\Lambda} X^n \rightarrow 0$$

are all exact. Consequently we obtain an exact sequence of DGK -modules

$$0 \rightarrow T(A', \Lambda, X) \rightarrow T(A, \Lambda, X) \rightarrow T(A'', \Lambda, X) \rightarrow 0$$

and passing to homology we obtain the exact triangle

$$\begin{array}{ccc} \text{Cotor}^{\Lambda}(A'', B) & \xrightarrow{\partial} & \text{Cotor}^{\Lambda}(A', B) \\ & \nwarrow \quad \swarrow & \\ & \text{Cotor}^{\Lambda}(A, B) & \end{array} \quad (14.2)$$

with the connecting morphism ∂ of degree -1 .

If

$$0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$$

is an exact sequence as above of right Λ -comodules then it follows that in the canonical resolutions of B' , B , B'' we have the exact sequences

$$0 \rightarrow B'^n \rightarrow B^n \rightarrow B''^n \rightarrow 0$$

$$0 \rightarrow X'^n \rightarrow X^n \rightarrow X''^n \rightarrow 0.$$

If, therefore Λ and B'' are K -flat, then B''^n are K -flat and it follows as above that the sequence

$$0 \rightarrow T(A, \Lambda, X') \rightarrow T(A, \Lambda, X) \rightarrow T(A, \Lambda, X'') \rightarrow 0$$

is exact. This yields the exact triangle

$$\begin{array}{ccc}
 \text{Cotor}^A(A, B'') & \xrightarrow{\partial} & \text{Cotor}^A(A, B') \\
 & \swarrow \quad \searrow & \\
 & \text{Cotor}^A(A, B) &
 \end{array}$$

The usual formal rules for the two connecting morphisms apply and will not be stated here. In particular, the connecting morphisms commute with the morphisms τ of § 12. Thus, in particular under the conditions of 12.1, if

$$0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$$

is an exact sequence of DGK -modules, then the triangles

$$\begin{array}{ccc}
 H(E'; C'') & \xrightarrow{\partial} & H(E'; C') \\
 & \swarrow \quad \searrow & \\
 & H(E'; C) &
 \end{array}$$

and

$$\begin{array}{ccc}
 \text{Cotor}^B(B', E \times C'') & \xrightarrow{\partial} & \text{Cotor}^B(B', E \times C) \\
 & \swarrow \quad \searrow & \\
 & \text{Cotor}^B(B', E \times C) &
 \end{array}$$

are isomorphic.

Remark. To obtain the exact triangle (14.2), the condition that A and B are K -flat may be replaced by the condition that the sequence (14.1) be split exact.

15. The relative theorem

Let A be a simplicial set, A_1, A_2 simplicial subsets of A and let $A_{12} = A_1 \cap A_2$. For any DGK -module C we have the commuting diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & A_{12} \otimes C & \longrightarrow & A_1 \otimes C & \longrightarrow & (A_1/A_{12}) \otimes C & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & A_2 \otimes C & \longrightarrow & A \otimes C & \longrightarrow & (A/A_2) \otimes C & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & (A_2/A_{12}) \otimes C & \longrightarrow & (A/A_1) \otimes C & \longrightarrow & (A/A_1 \cup A_2) \otimes C & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

with exact rows and columns. The module

$$H(A, A_1; C) = H((A/A_1) \otimes C)$$

is the *relative* homology module, while

$$H(A, A_1, A_2; C) = H(A, A_2, A_1; C) = H((A/A_1 \cup A_2) \otimes C)$$

is the *triad* homology module. Note that $H(A, 0; C) = H(A; C)$ and $H(A, A_1, A_2; C) = H(A, A_1; C)$ if $A_2 \subset A_1$.

Note that if X is a topological space and X_1, X_2 are subspaces of X then $S(X_1) \cup S(X_2)$ is a simplicial subset of $S(X_1 \cup X_2)$. If the inclusion $S(X_1) \cup S(X_2) \rightarrow S(X_1 \cup X_2)$ induces isomorphisms of homology groups then the triad (X, X_1, X_2) is called *proper* and we then have $H(X, X_1, X_2; C) = H(X, X_1 \cup X_2; C)$. This is always the case when $X_2 \subset X_1$. Since in our notation X and $S(X)$ frequently are denoted by the same symbol, we shall write $X_1 \vee X_2$ for $S(X_1) \cup S(X_2)$ so as not to confuse it with $S(X_1 \cup X_2)$.

Consider the commuting diagram

$$\begin{array}{ccccc} & & D' & \xrightarrow{h} & D \\ & & \downarrow i' & & \downarrow i \\ E'' & \xrightarrow{k} & E' & \xrightarrow{g} & E \\ \pi'' \downarrow & & \downarrow \pi' & & \downarrow \pi \\ B'' & \xrightarrow{j} & B' & \xrightarrow{f} & B \end{array} \quad (15.1)$$

of simplicial sets in which i, i', j, k , are inclusions. We then have the commuting diagram

$$\begin{array}{ccc} & H(E', E'', D'; C) & \\ \approx \swarrow & & \searrow \approx \\ H(E' \square_{E'} (E'/E'' \cup D'); C) & \xrightarrow{\approx} & \text{Cotor}^{E'}(E', (E'/E'' \cup D') \otimes C) \\ \downarrow H(\alpha \square_{\varphi} (\beta \otimes C)) & & \downarrow \text{Cotor}^{\varphi}(\alpha, \beta \otimes C) \\ H((B'/B'') \square_B (E/D \otimes C)) & \longrightarrow & \text{Cotor}^B(B'/B'', E/D \otimes C) \end{array}$$

where $\varphi = \pi g = f\pi'$ and

$$\alpha: E' \rightarrow B'/B'', \quad \beta: E'/E'' \cup D' \rightarrow E/D$$

are induced by π' and g .

There results a morphism

$$\tau : H(E', E'', D'; C) \rightarrow \text{Cotor}^B(B'/B'', E/D \otimes C). \quad (15.2)$$

Theorem 15.1. Assume that (15.1) is a diagram of topological spaces and that

- (i) i, i', j, k , are inclusions,
- (ii) π and πi are fibrations,
- (iii) π' and $\pi' i'$ are fibrations induced by f ,
- (iv) π'' is the fibration induced by fj ,
- (v) B is pathwise connected and simply connected.

Then for every DGK -module C , (15.2) is an isomorphism.

Proof. From the exact sequences

$$0 \rightarrow B'' \rightarrow B' \rightarrow B'/B'' \rightarrow 0$$

$$0 \rightarrow E'' \rightarrow E' \rightarrow E'/E'' \rightarrow 0$$

we deduce the commuting diagram with $T = \text{Cotor}^B(-, E \otimes C)$

$$\begin{array}{ccccccc} \dots & \longrightarrow & H(E''; C) & \longrightarrow & H(E'; C) & \longrightarrow & H(E', E''; C) \longrightarrow H(E''; C) \longrightarrow \dots \\ & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\ \dots & \longrightarrow & T(B'') & \longrightarrow & T(B') & \longrightarrow & T(B'/B'') \longrightarrow T(B'') \longrightarrow \dots \end{array}$$

Thus 12.1 and the "5 lemma" imply that

$$\tau : H(E', E''; C) \rightarrow \text{Cotor}^B(B'/B'', E \otimes C)$$

is an isomorphism.

We now consider the exact sequences with $D'' = E'' \cap D'$

$$0 \rightarrow E'' \rightarrow E' \rightarrow E'/E'' \rightarrow 0$$

$$0 \rightarrow E''/D'' \rightarrow E'/D' \rightarrow E'/E'' \cup D' \rightarrow 0$$

and have the commuting diagram with $T = \text{Cotor}^B(B'/B', - \otimes C)$

$$\begin{array}{ccccccc} \dots & H(E'', D''; C) & \rightarrow & H(E', D'; C) & \rightarrow & H(E', E'', D'; C) & \rightarrow H(E'', D''; C) \rightarrow \dots \\ & \downarrow \tau & & \downarrow \tau & & \downarrow \tau & \\ \dots & T(E'') & \longrightarrow & T(E') & \longrightarrow & T(E'/E'') & \longrightarrow T(E'') \longrightarrow \dots \end{array}$$

Applying the “5 lemma” again it follows that $H(E', E'', D'; C) \rightarrow T(E'/E'')$ is an isomorphism as required.

16. External products

Given A and B in DGK we define the permutation morphism

$$\sigma: A \otimes B \rightarrow B \otimes A$$

by the usual formula

$$\sigma(a \otimes b) = (-1)^{p_a} b \otimes a, \quad a \in A_p, \quad b \in B_q.$$

Let Λ and Γ be K -coalgebras, let A be a right Λ -comodule and B a right Γ -comodule. The morphisms

$$\Lambda \otimes \Gamma \xrightarrow{\varepsilon_\Lambda \otimes \varepsilon_\Gamma} K \otimes K = K$$

$$\Lambda \otimes \Gamma \xrightarrow{\delta_\Lambda \otimes \delta_\Gamma} \Lambda \otimes \Lambda \otimes \Gamma \otimes \Gamma \xrightarrow{\Lambda \otimes \sigma \otimes \Lambda} \Lambda \otimes \Gamma \otimes \Lambda \otimes \Gamma$$

$$A \otimes B \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes \Lambda \otimes B \otimes \Gamma \xrightarrow{A \otimes \sigma \otimes \Gamma} A \otimes B \otimes \Lambda \otimes \Gamma$$

convert $\Lambda \otimes \Gamma$ into a K -coalgebra and $A \otimes B$ into a right $\Lambda \otimes \Gamma$ -comodule. Similarly if C is a left Λ -comodule and D is a left Γ -comodule then $C \otimes D$ is a left $\Lambda \otimes \Gamma$ -comodule. Further we have the commutative diagram

$$\begin{array}{ccc} A \otimes B \otimes C \otimes D & \xrightarrow{\nabla \otimes 1} & A \otimes B \otimes \Lambda \otimes \Gamma \otimes C \otimes D \\ 1 \otimes \sigma \otimes 1 & & \downarrow \sigma' \\ A \otimes C \otimes B \otimes D & \xrightarrow{\nabla \otimes 1 \otimes \nabla \times 1} & A \otimes \Lambda \otimes C \otimes B \otimes \Gamma \otimes D \end{array}$$

where the vertical maps are suitable permutation morphisms. The same diagram holds with the horizontal arrows replaced by $1 \otimes \nabla$ and $1 \otimes \nabla \otimes 1 \otimes \nabla$. This implies a natural transformation

$$\xi: (A \square_\Lambda C) \otimes (B \square_\Gamma D) \rightarrow (A \otimes B) \square_{\Lambda \otimes \Gamma} (C \otimes D).$$

Proposition 16.1. If A is Λ -injective and B is Γ -injective then $A \otimes B$ is $\Lambda \otimes \Gamma$ -injective and ξ is an isomorphism.

Proof. Without any loss of generality we may assume that A and B are extended, i.e. that $A = A' \otimes \Lambda$, $B = B' \otimes \Gamma$ where A' and B' are DGK -modules. Then $A \otimes B = A' \otimes \Lambda \otimes B' \otimes \Gamma \approx A' \otimes B' \otimes \Lambda \otimes \Gamma$ so that $A \otimes B$ is $\Lambda \otimes \Gamma$ -injective. Further, using 2.1, ξ reduces to

$$\xi: (A' \otimes C) \otimes (B' \otimes D) \rightarrow (A' \otimes B') \otimes (C \otimes D)$$

which upon inspection turns out to be a switching isomorphism.

Proposition 16.2. If $\varepsilon: A \rightarrow X$ is an injective resolution in DGA and $\eta: B \rightarrow Y$ is an injective resolution in $DG\Gamma$ then $\varepsilon \otimes \eta: A \otimes B \rightarrow X \otimes Y$ is an injective resolution in $DG(\Lambda \otimes \Gamma)$.

Proof. We recall that $X \otimes Y$ is defined as the complex

$$(X \otimes Y)^n = \Sigma X^p \otimes Y^q, \quad p + q = n$$

and with the usual derivation operator. It is clear from 16.1 that each $(X \otimes Y)^n$ is $\Lambda \otimes \Gamma$ -injective. To prove that $\varepsilon \otimes \eta$ is acyclic, we observe that the condition that $\varepsilon: A \rightarrow X$ is acyclic is easily seen to be equivalent with the condition that $\varepsilon: A \rightarrow X$ regarded as a morphism in the category of complexes in DGK is a chain homotopy equivalence. It now follows readily that

$$A \otimes B \xrightarrow{\varepsilon \otimes B} X \otimes B \xrightarrow{X \otimes \eta} X \otimes Y$$

are chain homotopy equivalences and thus $\varepsilon \otimes \eta$ also is a chain homotopy equivalence.

We consider the complexes

$$T = T(X \otimes Y, \Lambda \otimes \Gamma, C \otimes D)$$

$$T' = T(X, \Lambda, C), \quad T'' = T(Y, \Gamma, D).$$

We have by 16.1

$$\begin{aligned} T_{*,s,0} &= (X \otimes Y)_s \square_{\Lambda \otimes \Gamma} (C \otimes D) \\ &= \Sigma (X_{s'} \otimes Y_{s''}) \square_{\Lambda \otimes \Gamma} (C \otimes D) \\ &= \Sigma (X_{s'} \square_{\Lambda} C) \otimes (Y_{s''} \square_{\Gamma} D) \\ &= \Sigma T'_{*,s',0} \otimes T''_{*,s'',0} \end{aligned}$$

where $s' + s'' = s$. There results a natural morphism

$$T' \otimes T'' \rightarrow T. \quad (16.1)$$

Passing to homology we have

$$H(T') \otimes H(T'') \rightarrow H(T' \otimes T'') \rightarrow H(T). \quad (16.2)$$

Consequently, we obtain a natural morphism

$$\text{Cotor}^A(A, C) \otimes \text{Cotor}^T(B, D) \rightarrow \text{Cotor}^{A \otimes T}(A \otimes B, C \otimes D). \quad (16.3)$$

Proposition 16.3. If A and T are simply connected and if A, C and $\text{Cotor}^A(A, C)$ are K -flat then (16.3) is an isomorphism.

Proof. Since A and T are simply connected, the resolutions X and Y may be chosen tapered. Then $X \otimes Y$ is a tapered resolution of $A \otimes B$. It follows that the various products involved in T, T' and T'' are finite. Therefore (16.1) is an isomorphism. It therefore suffices to show that $H(T') \otimes H(T'') \rightarrow H(T' \otimes T'')$ is an isomorphism. For this it suffices to establish that T' and $H(T')$ are K -flat. The assumption that $\text{Cotor}^A(A, C)$ is K -flat yields the K -flatness of $H(T')$. If we choose X to be the canonical resolution of A , then since A and A are K -flat, it follows that A^n and X^n are K -flat. Then $X^n \square_A C \approx A^n \otimes C$ and therefore $X^n \square_A C$ also is K -flat. Therefore, with this choice of X, T' is K -flat.

It should be noted that the morphism (16.3) is associative in the following sense. If Σ is a third K -coalgebra, then the diagram

$$\begin{array}{ccc} \text{Cotor}^A \times \text{Cotor}^T \times \text{Cotor}^\Sigma & \longrightarrow & \text{Cotor}^{A \otimes T} \otimes \text{Cotor}^\Sigma \\ \downarrow & & \downarrow \\ \text{Cotor}^A \otimes \text{Cotor}^{T \otimes \Sigma} & \longrightarrow & \text{Cotor}^{A \otimes T \otimes \Sigma} \end{array}$$

commutes. For convenience we omitted the variables. The proof of this associative law is omitted.

We now return to the complexes T, T', T'' above. The main filtrations of T' and T'' induce a filtration of $T' \otimes T''$ using the usual rule

$$F_r(T' \otimes T'') = \sum F_u T' \otimes F_v T'', \quad u + v = r$$

and the morphism (16.1) is compatible with the filtration. There result morphisms

$$\xi^r : E'^r \otimes E''^r \rightarrow E^r$$

which are compatible with the differentiations. Further the diagram

$$\begin{array}{ccc} H(E'^r \otimes E''^r) & \xrightarrow{H(\xi^r)} & H(E^r) \\ \uparrow & & \parallel \\ E'^{r+1} \otimes E''^{r+1} & \xrightarrow{\xi^{r+1}} & E^{r+1} \end{array}$$

commutes. If

$$\Lambda, \Gamma, A, B, H(\Lambda), H(\Gamma), H(A), H(B) \quad (16.4)$$

are K -flat then by 9.2 E'^2 may be replaced by $\text{Cotor}^{H(\Lambda)}$, etc. The morphism ξ^2 then becomes

$$\begin{aligned} & \text{Cotor}^{H(\Lambda)}(H(A), H(C)) \otimes \text{Cotor}^{H(\Gamma)}(H(B), H(D)) \rightarrow \\ & \rightarrow \text{Cotor}^{H(\Lambda) \otimes H(\Gamma)}(H(A) \otimes H(B), H(C) \otimes H(D)). \end{aligned} \quad (16.5)$$

From 16.3 we know that (16.5) is an isomorphism if

$$\begin{aligned} & H(\Lambda) \text{ and } H(\Gamma) \text{ are simply connected and } H(\Lambda), H(A) \\ & H(C) \text{ and } \text{Cotor}^{H(\Lambda)}(H(A), H(C)) \text{ are } K\text{-flat.} \end{aligned} \quad (16.6)$$

Thus if both (16.4) and (16.6) hold then ξ^2 is an isomorphism. This implies that ξ^3 is an isomorphism if we know that $E'^3 = H(E'^2)$ is K -flat. By induction we thus obtain

Proposition 16.4. If $H(\Lambda)$ and $H(\Gamma)$ are simply connected and if $\Lambda, \Gamma, A, B, C, H(\Lambda), H(\Gamma), H(A), H(B), H(C)$ and $E''^r (r \geq 2)$ are K -flat then $\xi^r : E'^r \otimes E''^r \rightarrow E^r$ is an isomorphism.

Of course, all the flatness conditions are automatically fulfilled if K is a field.

17. The EILENBERG-ZILBER Theorem

Given simplicial sets A and B the product $A \times B$ is a simplicial set with $(A \times B)_n = A_n \times B_n$. The EILENBERG-ZILBER [8], [4] theorem establishes morphisms

$$A \times B \begin{array}{c} \xrightarrow{\zeta} \\ \xleftarrow{\eta} \end{array} A \otimes B \quad (17.1)$$

of DGK -modules which have the following properties

(17.2) ζ and η are natural in A , B (and K).

(17.3) $\zeta\eta = 1$.

(17.4) $\eta\zeta$ is homotopic to 1 (in a natural fashion). It follows that

(17.5) $H(\eta)$ is an isomorphism with $H(\zeta)$ as inverse.

Since both sides of (17.1) are K -coalgebras it is natural to ask whether ζ and η are compatible with the coalgebra structures. The mapping ζ is defined by

$$\zeta(a, b) = \sum a_0^p \otimes b_p^n, \quad 0 \leq p \leq 0, \quad a \in A_n, \quad b \in B_n$$

and is not compatible with the coalgebra structures. The definition of η is much more elaborate. Given integers $0 \leq m \leq n$ we denote by (m, n) the set of all integers q such that $m \leq q \leq n$. Consider diagrams

$$\begin{array}{ccc} & & (0, m) \\ & \nearrow \pi_1 & \\ (0, m+n) & \xrightarrow{\omega} & (0, m) \times (0, n) \\ & \searrow \pi_2 & \\ & & (0, n) \end{array}$$

where π_1 and π_2 are projections, $\pi_1\omega$ and $\pi_2\omega$ are weakly monotone and $\pi_1\omega + \pi_2\omega$ is the identity. If ω has this property then the points $\omega(t)$, $0 \leq t \leq p+q$ are the vertices of a path in the rectangle with corners $(0, 0)$, (m, n) leading from $(0, 0)$ to (m, n) and composed of $m+n$ intervals of length 1 parallel to one of the axes. The area of the rectangle under the path is then an integer k and we denote $\iota(\omega) = (-1)^k$. Then η is given for $a \in A_m$, $b \in B_n$ by

$$\eta(a \otimes b) = \sum \iota(\omega) (\pi_1\omega a, \pi_2\omega b)$$

the summation extending over all paths ω from $(0, 0)$ to (m, n) as described above. We assert that

$$\eta \text{ is a morphism of } K\text{-coalgebras.} \quad (17.6)$$

The commutativity of the diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\eta} & A \otimes B \\ \varepsilon \otimes \varepsilon \searrow & & \swarrow \varepsilon \\ & K & \end{array}$$

is evident. We must show that the diagram

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{\eta} & A \times B \\
\delta \otimes \delta \downarrow & & \downarrow \delta \\
A \otimes A \otimes B \otimes B & & \\
1 \otimes \sigma \otimes 1 \downarrow & & \\
A \otimes B \otimes A \otimes B & \xrightarrow{\eta \otimes \eta} & (A \times B) \otimes (A \times B)
\end{array}$$

commutes. Applying $\delta \otimes \delta$ to $a \otimes b$, $a \in A_m$, $b \in B_n$ yields

$$\Sigma a_0^p \otimes a_p^m \otimes b_0^q \otimes b_q^n, \quad p \in (0, m), \quad q \in (0, n).$$

Applying $1 \otimes \sigma \otimes 1$ to each term yields

$$\Sigma_{p,q} (-1)^{(m-p)q} a_0^p \otimes b_0^q \otimes a_p^m \otimes b_q^n.$$

Applying $\eta \otimes \eta$ yields

$$\Sigma_{p,q} \Sigma_{\omega', \omega''} (-1)^{(m-p)q} \iota(\omega') \iota(\omega'') (\pi_1' \omega' a_0^p, \pi_2' \omega' b_0^q) \otimes (\pi_1'' \omega'' a_p^m, \pi_2'' \omega'' b_q^n) \quad (17.7)$$

where ω' ranges through all paths from $(0, 0)$ to (p, q) while ω'' ranges through all paths from $(0, 0)$ to $(m-p, n-q)$. By translation, the path ω'' may be regarded as running from (p, q) to (m, n) and thus we obtain a path $\omega = \omega' + \omega''$ from $(0, 0)$ to (m, n) . A calculation of areas shows that

$$(-1)^{(m-p)q} \iota(\omega') \iota(\omega'') = \iota(\omega).$$

Further

$$(\pi_1' \omega' a_0^p, \pi_2' \omega' b_0^q) = (\pi_1 \omega a, \pi_2 \omega b)_0^{p+q}$$

$$(\pi_1'' \omega'' a_p^m, \pi_2'' \omega'' b_q^n) = (\pi_1 \omega a, \pi_2 \omega b)_{p+q}^{m+n}.$$

Thus (17.7) becomes

$$\Sigma_{p,q} \Sigma_{\omega} \iota(\omega) (\pi_1 \omega a, \pi_2 \omega b)_0^{p+q} \otimes (\pi_1 \omega a, \pi_2 \omega b)_{p+q}^{m+n}$$

where ω ranges through all the paths from $(0, 0)$ to (m, n) passing through (p, q) . Fixing $r = p + q$ we obtain this way all paths ω . Thus the summation above is

$$\Sigma_r \Sigma_{\omega} (\pi_1 \omega a, \pi_2 \omega b)_0^r \otimes (\pi_1 \omega a, \pi_2 \omega b)_r^{m+n}, \quad r \in (0, m+n)$$

which is precisely $\delta\eta(a \otimes b)$. This concludes the proof.

Let B' be a simplicial subset of B . The morphism $\eta: A \otimes B \rightarrow A \times B$ then induces a morphism

$$\eta': A \otimes B/B' \rightarrow A \times B/A \times B'.$$

The left side is a left $A \otimes B$ -comodule while the right one is a left $A \times B$ -comodule. It is trivial to see that η' is an η -morphism of comodules. Further $H(\eta')$ is an isomorphism.

18. Cotor as a coalgebra

Let A be a simplicial set. We then have a commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{d} & A \times A \\ & \searrow \delta & \downarrow \zeta \\ & & A \otimes A \end{array}$$

where d is the diagonal map. Since $H(\zeta)$ and $H(\eta)$ are inverses of each other, we obtain the commuting diagram

$$\begin{array}{ccc} H(A) & \xrightarrow{H(d)} & H(A \times A) \\ & \searrow H(\delta) & \uparrow H(\eta) \\ & & H(A \otimes A). \end{array}$$

Assume now that $H(A)$ is K -flat. Then by § 9, $H(A)$ is a K -coalgebra with a structure morphism δ_H and we have the commuting diagram

$$\begin{array}{ccc} H(A) & \xrightarrow{H(d)} & H(A \times A) \\ \delta_H \downarrow & & \uparrow H(\eta) \\ H(A) \otimes H(A) & = & H(A \otimes A). \end{array}$$

Since d and η are morphisms of coalgebras it follows that $H(d)$ and $H(\eta)$ are also morphisms of K -coalgebras and thus δ_H is a morphism of K -coalgebras. This is equivalent to the well known fact that the K -coalgebra $H(A)$ is commutative.

If A_1, A_2 are simplicial subsets of A and C is in DGK then $H(A, A_1, A_2; C)$ is a left (or right) $H(A)$ -comodule and we have the commuting diagram

$$\begin{array}{ccc}
H(A, A_1, A_2; C) & \xrightarrow{H(d \otimes C)} & H(A \times A, A \times A_1, A \times A_2; C) \\
\downarrow \nabla_H & & \uparrow H(\eta \otimes C) \\
H(A) \otimes H(A, A_1, A_2; C) & = & H(A \otimes (A/A_1 \cup A_2) \otimes C)
\end{array} \quad (18.1)$$

where ∇_H is the structure morphism of $H(A, A_1, A_2; C)$. It follows that ∇_H is a morphism of $H(A)$ -comodules.

We return to diagram (15.1) of simplicial sets and assume throughout this section that

(18.2) There exists a simplicial set M with a single 1-simplex and a morphism $\varrho: B \rightarrow M$ of simplicial sets such that $H(\varrho)$ is an isomorphism.

This hypothesis is satisfied if B is the singular simplicial set of a pathwise connected and simply connected topological space.

We have the diagonal maps

$$d: B \rightarrow B \times B, \quad d_1: B'/B'' \rightarrow B' \times B'/B' \times B''$$

$$d_2: E/D \rightarrow E \times E/E \times D$$

and we consider the diagram

$$\begin{array}{ccc}
\text{Cotor}^B(B'/B'', E/D \otimes C) & \xrightarrow{\text{Cotor}^d(d_1, d_2 \otimes C)} & \text{Cotor}^{B \times B}(B' \times B'/B' \times B'', E \times E/E \times D \otimes C) \\
& & \uparrow \text{Cotor}^\eta(\eta_1, \eta_2 \otimes C) \\
(18.3) \quad & & \text{Cotor}^{B \otimes B}(B' \otimes (B'/B''), E \otimes E/D \otimes C) \\
& & \uparrow \xi \\
& & \text{Cotor}^B(B', E) \otimes \text{Cotor}^B(B'/B'', E/D \otimes C).
\end{array}$$

Since $H(\eta)$, $H(\eta_1)$, $H(\eta_2)$ are isomorphisms it follows from the isomorphism theorem 7.1 that $\text{Cotor}^\eta(\eta_1, \eta_2 \otimes C)$ is an isomorphism. From 16.2 we know that ξ is an isomorphism if

$$\text{Cotor}^B(B', E) \text{ is } K\text{-flat.} \quad (18.4)$$

If (18.4) holds, then (18.3) yields a morphism

$$\text{Cotor}^B(B'/B'', E/D \otimes C) \rightarrow \text{Cotor}^B(B', E) \otimes \text{Cotor}^B(B'/B'', E/D \otimes C). \quad (18.5)$$

As a special case of (18.5) we have

$$\text{Cotor}^B(B', E) \rightarrow \text{Cotor}^B(B', E) \otimes \text{Cotor}^B(B', E). \quad (18.6)$$

We also have a morphism

$$\text{Cotor}^B(B', E) \rightarrow K \quad (18.7)$$

induced by mapping B, B' and E into a pointlike simplicial set (which has one simplex in each dimension).

It is a formal matter to verify that (18.6) and (18.7) convert $\text{Cotor}^B(B', E)$ into a graded K -coalgebra and that (18.5) converts $\text{Cotor}^B(B'/B'', E/D \otimes C)$ into a graded left $\text{Cotor}^B(B', E)$ -comodule.

Assume now that

$$H(E') \text{ is } K\text{-flat}. \quad (18.8)$$

Writing down the appropriate fairly large commuting diagram one easily obtains that

$$\tau_0: H(E') \rightarrow \text{Cotor}^B(B', E)$$

is a morphism of graded K -coalgebras and that

$$\tau: H(E', E'', D'; C) \rightarrow \text{Cotor}^B(B'/B'', E/D \otimes C)$$

is a τ_0 -morphism of graded left comodules.

In diagram (18.2) we may replace the functor Cotor by the spectral sequence functors E^r . Then assuming

$$H(B), H(B'), H(B'/B''), H(E) \text{ and } E^r(B', B, E) \text{ } (r \geq 2) \text{ are } K\text{-flat} \quad (18.9)$$

we obtain that $E^r(B', B, E)$ is a (graded differential) K -coalgebra and that $E^r(B'/B'', B, E/D \otimes C)$ is a left $E^r(B', B, E)$ -comodule.

Conditions (18.9) together with 9.2 yield isomorphisms

$$E^2(B', B, E) \approx \text{Cotor}^{H(B)}(H(B'), H(E))$$

$$E^2(B'/B'', B, E/D \otimes C) \approx \text{Cotor}^{H(B)}(H(B', B''), H(E, D; C)).$$

This yields morphisms (18.5) and (18.6) with B, B' , etc. replaced by $H(B), H(B')$, etc. These morphisms can be obtained from diagram (18.3) by re-

placing B, B' , etc. by $H(B), H(B')$, etc. However, in view of (18.1) the following slightly simpler diagram may be used

$$\begin{array}{ccc} \text{Cotor}^{H(B)}(H(B', B''), H(E, D; C)) & \longrightarrow & \text{Cotor}^{H(B) \otimes H(B)}(H(B') \otimes H(B', B''), \\ & & \begin{array}{c} H(E) \otimes H(E, D; C) \\ \uparrow \xi \end{array} \\ \text{Cotor}^{H(B)}(H(B'), H(E)) \otimes \text{Cotor}^{H(B)}(H(B', B''), H(E, D; C)) & & \end{array}$$

The horizontal morphism is induced by the structure morphism

$$H(B) \rightarrow H(B) \otimes H(B), H(B', B'') \rightarrow H(B') \otimes H(B', B'')$$

and

$$H(E, D; C) \rightarrow H(E) \otimes H(E, D; C).$$

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