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Fibering manifolds over a circle

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In [7], STALLINGS proves a theorem about fibering 3-manifolds over a circle. We propose to generalize this theorem to higher dimensions, with further restriction on the fundamental group. There are theorems in the differential and piecewise-linear category, but we restrict ourselves here to the differential category. An important consequence will be the fibering of the complements of certain knots and, in particular, the main result of [4].

1. The Fibering Theorem

1.1. Let M and N be smooth compact manifolds. A smooth map $f: M \to N$ is called a *smooth fiber map* if the differential df is an epimorphism at every point of M-furthermore, at every point of ∂M, df must map the tangent space of ∂M onto the tangent space of N. It the follows easily by the techniques of [8] that f is a locally trivial fiber map. Furthermore, the fiber over every point of N is a smooth submanifold of M, whose boundary lies in ∂M, and the coordinate transformations are diffeomorphisms of that submanifold.

We will be interested in the case where N is the circle C, with a fixed orientation. If we fix a base-point $p \in C$ and consider the unit vector v at p tangent to C in the direction determined by the orientation, then the submanifold $F = f^{-1}(p)$ and the normal vector field ν on F which pulls back from v, via df, form a pair (F, ν) which we call a *framed fiber of* f.

1.2. If $f: M \to C$ is a map and $\omega \in H^1(C)$ orients C, we define $\vartheta(f) = f^*(\omega) \in H^1(M)$. If M is bounded, then $\vartheta(f | \partial M)$ is the restriction of $\vartheta(f)$ to $H^1(\partial M)$.

Suppose (F, ν) is a pair consisting of a submanifold F of M of codimension one, meeting ∂M normally along ∂F, and a normal vector field ν to F. Then there exists a smooth function $f: M \to C$, with p a regular value, such that $F = f^{-1}(p)$ and ν is the pull-back, via df, of v, and, in fact, all such maps are homotopic (see [8]). We then define $\vartheta(F, \nu) = \vartheta(f)$, which is, therefore, an invariant of the *framed cobordism class* of (F, ν) (see [2; 1.4]).

1.3. We may ask the question: for which $\vartheta \in H^1(M)$ is there a smooth fiber map $f: M \to C$ with $\vartheta = \vartheta(f)$? More generally, given a smooth fiber map

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$g : \partial M \to S^1$ and an extension $\vartheta \in H^1(M)$ of $\vartheta(g) \in H^1(\partial M)$, is there an extension of g to a smooth fiber map $f : M \to C$ with $\vartheta(f) = \vartheta$? We will restrict ourselves to the case where the fiber will have 1-connected components. It will then follow from the homotopy sequence of the fibration that $\pi_1(M)$ is infinite cyclic and $\vartheta \neq 0$. Furthermore the higher homotopy groups will be isomorphic to those of the fiber and, consequently, finitely generated, by a theorem of SERRE [5, p. 274].

1.4. The main result of this work will be the following theorem which asserts that these conditions on M, necessary for the existence of a fiber map, are, in fact, also sufficient, for higher dimensions.

Theorem. Let M be a smooth compact manifold of dimension exceeding five. Suppose $g : \partial M \to C$ is a smooth fiber map and $\vartheta \in H^1(M)$ an extension of $\vartheta(g)$. If $\vartheta \neq 0$, $\pi_1(M)$ is infinite cyclic and $\pi_i(M)$ is finitely-generated for all $i \geq 1$, then g extends to a smooth fiber map $f : M \to C$ with $\vartheta(f) = \vartheta$.

1.5. The strength of the restrictions imposed on M may be seen e.g. in the consequence that M must be *irreducible*. For if M is a connected sum $M_1 \# M_2$, then either M_1 or M_2 is 1-connected (say M_1) and the universal covering of M is the connected sum of the universal covering of M_2 and a countable number of copies of M_1. Since the homotopy groups of M are finitely-generated, M_1 must be a homotopy sphere; it follows, since $n > 5$, that M is irreducible.

1.6. Let K be a closed smooth submanifold of S^n homeomorphic to S^{n-2}. Then the normal bundle to K is trivial and K is contained in a submanifold of S^n diffeomorphic to $K \times D^2$. Let M be the closure of its complement; ∂M is diffeomorphic to $K \times C$. Suppose $n > 5$ and the homotopy groups of $S^n - K$ are finitely-generated abelian groups; then M satisfies the hypotheses of Theorem (1.4). Since $H^1(M) \approx H^1(\partial M)$, the projection map $\partial M \to C$ extends to a smooth fiber map $M \to C$. As an easy consequence of this we have:

Corollary. If K is a closed smooth submanifold of S^n, homeomorphic to S^{n-2}, with $n > 5$ and the homotopy groups of $S^n - K$ finitely-generated abelian groups, then there is a smooth fiber map $S^n - K \to C$ such that the closure of each fiber of f is a submanifold bounded by K.

1.7. We point out the particular case of (1.6), when $S^n - K$ is homotopy equivalent to the circle C. Then K bounds a contractible submanifold of S^n, and we easily derive [4, Theorem 1].

Conversely it is not difficult to check that the techniques of [4] generalize, in a straightforward manner, to give an alternate proof of (1.6). But this does not seem to yield an alternate proof of our main result, (1.4), because of troubles in the middle dimensions.

2. Some constructions

2.1. Let (F_0, ν_0) be a framed fiber of g (see (1.1)). We first notice that (F_0, ν_0) "extends" to a pair (F, ν), consisting of a submanifold F of M and normal field ν to F, such that $\partial F = F_0, \nu | F_0 = \nu_0$ and $\vartheta(F, \nu) = \vartheta$. This is seen as follows. By obstruction theory g extends to a map $f' : M \to C$ with $\vartheta(f') = \vartheta$. According to [8] we may approximate f' by a smooth map $f'' : M \to C$, also extending g, and with a regular value at the given point $p \in C$. We now define $F = f''^{-1}(p)$ and ν to be the pull-back of v by df''.

2.2. Let M be a smooth compact manifold and F a smooth submanifold, of codimension one, meeting ∂M transversely along ∂F. We may construct a new manifold, which we denote by M_F, by "cutting" M along F. Then ∂M_F consists of two copies of F—which we denote by F' and F''—and $\partial M_{\partial F}$; there are corners at $\partial F'$ and $\partial F''$.

If (F, ν) is a framed fiber of a smooth fiber map $g : M \to C$, then one constructs a smooth fiber map $\bar{g} : M_F \to I$ (I is the unit interval with the usual orientation) as the pull-back, under the orientation-preserving collapsing map $I \to C$, of g. If ν' and ν'' are the normal fields on F' and F'' lifting from ν, then (F', ν') and (F'', ν'') are the framed fibers over $\dot{I}$. Conversely, if we are given a smooth fiber map $\bar{g} : M_F \to I$ with (F', ν') and (F'', ν'') the framed fibers over $\dot{I}$, then one constructs a smooth fiber map $g : M \to C$ with (F, ν) as a framed fiber.

2.3. Returning to the situation of (2.1), suppose we can establish the following strong property of (F, ν):

(*) The components F are 1-connected and F' and F'' are deformation retracts of M_F.

Then $(M_F, \partial M_{\partial F})$ is a relative h-cobordism between $(F', \partial F')$ and $(F'', \partial F'')$. Let $\bar{g} : \partial M_{\partial F} \to I$ be the smooth fiber map constructed from g, as in (2.2). To fix notation assume $\partial F'$ is the fiber over 0.

Since $n > 5$ and the components of F' and F'' are 1-connected, it is a direct consequence of [6, Corollary 3.2] that $\bar{g}$ extends to a smooth fiber map $\bar{f} : M_F \to$ $\to I$, with (F', ν') and (F'', ν'') the framed fibers over 0 and 1, resp. If F has closed components, there is a choice to be made in defining $\bar{f}$ on the corresponding components of M_F; but this choice is determined by demanding that F' be the fiber over 0.

Now $\bar{f}$ induces a smooth fiber map $f: M \to C$ with (F, ν) a framed fiber, as in (2.2). It is clear that f is an extension of g, since $\bar{f}$ is an extension of $\bar{g}$. Since $\vartheta(f) = \vartheta(F, \nu) = \vartheta$, f satisfies the requirements of Theorem (1.4).

2.4. Our task is now reduced to modifying (F, ν), without changing $(\partial F, \nu | \partial F)$ or the element $\vartheta(F, \nu)$, to achieve (*). The basic modification we will perform upon (F, ν) will now be described.

Let d be a k-disk embedded in the interior of M, meeting F transversely along ∂d, so that $\nu | \partial d$ coincides with either the inward or outward radial field of d. We say that d *protrudes from* (F, ν). If T is a tubular neighborhood of d, we may identify the pair (T, d) diffeomorphically with the pair $(d \times D^{n-k}, d \times 0)$, where D^{n-k} is the unit disk in $(n-k)$-space. We may assume $T \cap F = \partial d \times D^{n-k}$. Now define a new submanifold $G = \overline{M - \partial d \times D^{n-k}} \cup d \times \partial D^{n-k}$ (rounding the corners at $\partial d \times \partial D^{n-k}$). The field $\nu | F \cap G$ extends in a unique (up to homotopy) way to a normal field ξ on G.

It is clear that $(\partial G, \xi | \partial G) = (\partial F, \nu | \partial F)$. The fact that $\vartheta(F, \nu) = \vartheta(G, \xi)$ follows from the arguments of [2; 3.3] since (F, ν) and (G, ξ) are "framed cobordant". We refer to (G, ξ) as a *modification of* (F, ν) *along* d.

2.5. Let X be the (not necessarily connected) covering space of M induced from the usual covering $R \to C$ by a map $M \to C$ representing $\vartheta \in H^1(M)$. Then the number of components of X is the order of the quotient group of $H^1(M)$ by the subgroup generated by ϑ; since $\vartheta \neq 0$, this is finite. Since $\pi_1(M)$ is infinite cyclic, each component of X is a copy of the universal cover of M. By hypothesis, their homotopy groups are finitely generated, and, therefore by the above considerations and [5, p.271] the homology groups of X are finitely generated.

2.6. Let (F, ν) be as in (2.1). If $\varrho: M_F \to M$ is the projection, then ϱ lifts to a map $\bar{\varrho}: M_F \to X$, since $\varrho^*(\vartheta) = 0$; moreover it is easily seen that $\bar{\varrho}$ is an imbedding. By means of the covering translations of X we obtain an infinite sequence M_i, $-\infty < i < \infty$, of copies of M_F imbedded in X, whose union is X. If F'_i, F''_i are those parts of ∂M_i lifted from F', F'' we may assume, by correct numbering, that $F''_i = F'_{i+1}$, for every every i, and these are the only identifications among the M_i. This description of X is due essentially to Neuwirth (see [3; 6]).

Let us denote by M^r_F, for $-\infty < r < \infty$, the union of the M_i, $i \leq r$; notice that the pair (M_F, F') may be obtained, by excision from the pair (M^{r+1}_F, M^r_F). The homology groups $H_k(M^r_F)$, $-\infty < r < \infty$, fixed k, form a direct system, under the homomorphisms induced by inclusions, whose direct limit is isomorphic to $H_k(X)$. Similarly the groups $H_k(X, M_F^r)$, $-\infty < r < \infty$, fixed k, form a direct system, under inclusion homomorphisms, whose direct limit is isomorphic to $H_k(X, X) = 0$. Furthermore, it follows from the MAYER-VIETORIS sequence of the triad $(X; M_F^r, \overline{X - M_F^r})$, that the homology groups of M_F^r, and, therefore, of (X, M_F^r), are finitely-generated.

2.7. Note that for every integer s there is a covering translation of X which maps M_F^r onto M_F^{r+s}, for every r. From this it follows that the triples (X, M_F^{r+1}, M_F^r) are all mutually diffeomorphic. This fact will be used quite often.

3. Modification of (F, ν)

3.1. Let $i' : F' \to M_F$ and $i'' : F'' \to M_F$ be the inclusions. We would like the induced functions $i'_* : \pi_0(F') \to \pi_0(M_F)$ and $i''_* : \pi_0(F'') \to \pi_0(M_F)$ to be bijective.

Suppose one of them is not injective—say i'_*. Then there is a path in M_F connecting two distinct components of F'. We now project this path to M, and use a general position argument ($n > 2$) to replace it with a 1-disk d which protrudes from (F, ν) and connects two distinct components of F. A modification of (F, ν) along d will obviously produce a submanifold with one less component than F. Since F has only a finite number of components, we may eventually assume i'_* and i''_* are injective.

But it now follows that i'_* and i''_* are, in fact, bijective. Consider instead the homomorphisms $i'_* : H_0(F') \to H_0(M_F)$ and $i''_* : H_0(F'') \to H_0(M_F)$; it suffices to show they are bijective. By an argument using excision, the homomorphisms $H_0(M_F^r) \to H_0(M_F^{r+1})$ are injective, and are surjective if i'_* is surjective. But if $H_0(M_F^r) \to H_0(M_F^{r+1})$ were not surjective—which would then be true for all r—then X would have an infinite number of components, which, by (2.5), is not true. A similar argument works for i''_*.

So we may now assume that i'_* and i''_* are bijective. Notice that, as a consequence, the functions $\pi_0(F'_r) \to \pi_0(M_r) \to \pi_0(M_F^r) \to \pi_0(X)$, induced by inclusions, are all bijective.

3.2. Our next task is to make the components of F 1-connected. The modifications used here will be analogous to those of (3.1).

Let A be a component of F' or F'', $i : A \to M_F$ the inclusion, and $i_* : \pi_1(A) \to \pi_1(M_F)$ the induced homomorphism. If α is an element of Kernel i_*, we may represent α by an imbedded 1-sphere in A which bounds an imbedded 2-disk in M_F. This follows from a general position argument, since $n > 4$. The 2-disk in M_F then projects to a disk d in M which protrudes from (F, ν). A modification of (F, ν) along d produces a new submanifold identical to F except that A is replaced by a new component whose fundamental group is isomorphic to the group obtained from $\pi_1(A)$ by adding the relation $\alpha = 1$.

It follows from an argument in [4] that, after, a sequence of such modifications, we may assume i_* is injective for every component A of F' or F''. But now it follows from e.g. [4, lemma 4] that A is 1-connected. To fit our present needs this lemma should be changed to read as follows:

Suppose (i) $i_* : \pi_1(A) \to \pi_1(M_F)$ is a monomorphism, for every component A of F' or F'' and (ii) $\pi_1(A) \to \pi_1(M)$ is zero, for every component A of F. Then the components of F are 1-connected.

The proof is identical. Hypothesis (ii) is satisfied because the composition $\pi_1(A) \to \pi_1(M) \xrightarrow{g_*} \pi_1(C)$ is zero, while g_* is injective.

Notice that, since $n > 2$, the modifications just performed do not disturb the results of (3.1). Furthermore, as a consequence, we may conclude that the components of M_F and M_F^r are 1-connected. For, by (3.1), each component of X is a union of one component of each M_r, intersecting in one component of each F'_r. By the van KAMPEN theorem, the 1-connectivity of the components of X and F'_r imply the 1-connectivity of the components of M_r, and, therefore, M_F. Now we can express the components of M_F^r as a union of one component of each M_i, $i \leq r$, intersecting in one component of each F'_i, $i \leq r$; by the van KAMPEN theorem, the 1-connectivity of the components of M_i and F'_i imply the 1-connectivity of the components of M_F^r.

3.3. It is clear that (M_F, F') and (M_F, F'') are now homology 1-connected —a pair (A, B) is *homology k-connected* if $H_i(A, B) = 0$ for $0 \leq i \leq k$. As a consequence, we will show that (X, M_F^r) is homology 2-connected.

From the homology sequence of the triple (X, M_F^{r+1}, M_F^r), and the homology 1-connectedness of (M_F^{r+1}, M_F^r) it follows that the inclusion $j_* : H_i(X, M_F^r) \to \to H_i(X, M_F^{r+1})$ is an isomorphism for $i \leq 1$ and an epimorphism for $i = 2$. But since these two groups are isomorphic (see (2.7)) and finitely generated, j_* must be an isomorphism for $i = 2$ also. Thus we may conclude that, for $i \leq 2$, $H_i(X, M_F^r)$ is isomorphic to the direct limit of the $H_i(X, M_F^r)$, as $r \to \infty$, which is zero (see (2.6)).

3.4. Let us assume now, as an inductive step, the following conditions:

C_k: (i) The inclusions $\pi_0(F') \to \pi_0(M_F)$, $\pi_0(F'') \to \pi_0(M_F)$ are bijective,

(ii) the components of F are 1-connected, and

(iii) (X, M_F^r) is homology $(k-1)$-connected.

Given that (F, ν) satisfies (C_k), we shall show how to modify (F, ν) to satisfy (C_{k+1}), for any value of k satisfying $3 \leq k \leq n-3$.

As a consequence of (C_k), we point out that (M_F, F') is homology $(k-2)$-connected; this follows from the homology sequence of the triple (X, M_F^{r+1}, M_F^r).

3.5. Let $\alpha \in H_k(M_F, F')$; according to [1, lemma 8], which also holds in the case $r = 2$, by an easy argument using the surjectivity of $\pi_3(X, M) \to H_3(X, M)$ —in the notation of [1]—and results of WHITNEY (see, in fact, the discussion of the paragraph preceding lemma 8), α may be represented by an imbedded k-disk. When projected to M, we obtain a k-disk d protruding from (F, ν). Let (G, ξ) be the new submanifold (and normal field) obtained by modifying (F, ν) along d.

Notice that the disk d lifts to a sequence of disks $d_i \subset M_i$, $-\infty < i < \infty$, with $\partial d_i \subset F'_i$. It is clear that we may construct M_G^r from M_F^r by adjoining, in X, a handle whose "core" is d_{r+1}. Also notice that d_{r+1} represents the element β of $H_k(M_F^{r+1}, M_F^r)$ corresponding to α under the natural excision isomorphism.

It is now an easy exercise, using the homology sequence of the triple (X, M_G^r, M_F^r), to check that (X, M_G^r) is homology $(k-1)$-connected but:

$$H_k(X, M_G^r) \approx H_k(X, M_F^r) \,/\, (i_*\beta),$$

where $i : (M_F^{r+1}, M_F^r) \to (X, M_F^r)$ is the inclusion and $(i_*\beta)$ is the subgroup of $H_k(X, M_F^r)$ generated by $i_*\beta$. Also, since $k \leq n-3$, (G, ξ) again satisfies (i) and (ii) of (C_k).

Since $H_k(X, M_F^r)$ is finitely generated, after a finite sequence of such modifications, the homomorphisms:

$$i_* : H_k(M_F^{r+1}, M_F^r) \to H_k(X, M_F^r)$$

will be zero. Consequently, the homomorphisms $H_k(X, M_F^r) \to H_k(X, M_F^{r+1})$, induced by inclusion, will be monomorphisms. Since the direct limit, as $r \to \infty$, of these groups is zero, we conclude that $H_k(X, M_F^r) = 0$, for every r. Thus (F, ν) now satisfies (C_{k+1}).

3.6. We have now reached the point where (F, ν) satisfies (C_{n-2}) of (3.4). Consequently, as pointed out in (3.4), (M_F, F') is homology $(n-4)$-connected; by the Universal Coefficient Theorem, $H^{n-3}(M_F, F')$ is free. Now, by duality and (C_{n-2}), we can conclude that $H_i(M_F, F'') = 0$, unless $i = 2$ or 3, and $H_3(M_F, F'')$ is free.

Since the definitions of F' and F'' are interchangeable, we will make use of our present notation in the remainder of the proof by, at this point, switching their roles. Therefore, we are now assuming that $H_i(M_F, F') = 0$ unless $i = 2$ or 3, and $H_3(M_F, F')$ is free. Conditions (i) and (ii) of (C_k) in (3.4) are, of course, still satisfied.

We now show that $H_i(X, M_F^r) = 0$ unless $i = 3$. If $j: (X, M_F^r) \to (X, M_F^{r+1})$ is the inclusion, it follows immediately from the homology sequence of (X, M_F^{r+1}, M_F^r), that:

$$j_* : H_i(X, M_F^r) \to H_i(X, M_F^{r+1})$$

is either a monomorphism or an epimorphism, unless $i = 3$. If j_* is an epimorphism, then, since the range and domain are isomorphic finitely generated groups, j_* must be an isomorphism. Thus j_* is a monomorphism, unless $i = 3$. But since the direct limit, as $r \to \infty$, is zero, we conclude that $H_i(X, M_F^r) = 0$.

3.7. We now perform a sequence of modifications, as in (3.5), to kill $H_3(X, M_F^r)$. Choose a non-zero element $\alpha \in H_3(M_F, F')$; since $H_3(M_F, F')$ is free, α has infinite order. Since (F, ν) satisfies (C_3), we may modify (F, ν) to obtain (G, ξ). As mentioned in (3.5), (X, M_G^r) is homology 2-connected and $H_3(X, M_G^r)$ is "smaller" than $H_3(X, M_F^r)$. But we also need to notice here that $H_i(X, M_G^r) = 0$ for $i > 3$. This follows from the homology sequence of (X, M_G^r, M_F^r), again, and the fact that α (thus β) has infinite order.

3.8. By the argument of (3.5) we can now achieve the situation in which all the homology groups of (X, M_F^r) vanish. Consequently all the homology groups of (M_F, F') vanish; by duality this is also true of (M_F, F''). Taking into account conditions (i) and (ii) of (C_k), by a theorem of J. H. C. WHITEHEAD, (F, ν) now has property (*) of (2.3).

This completes the proof of Theorem (1.4).

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