On the Finite Subgroups of Connected Lie Groups

William M. Boothby and Hsien-Chung Wang[1])

1. Introduction. According to a beautiful theorem of C. Jordan [7], a finite linear group F of degree n, that is, a finite subgroup of $GL(n, C)$, has a normal abelian subgroup A whose index in F is bounded by a number $k(n)$ depending only on n and not at all on F. Thus, roughly speaking, the nonabelian part of a finite group of $n \times n$ complex matrices has only a finite number of possible forms. More precisely, every such group is an extension of an abelian group A by a group F/A of bounded order; A must, up to a similarity, be just a group of diagonal matrices.

A remarkable analytic proof of this theorem was given by Bieberbach [1], and was subsequently much simplified by Frobenius [4], [5], who obtained estimates for $k(n)$ later improved by Speiser [9]. These proofs make use of the fact that $GL(n, C)$ is a linear Lie group, employing, for example, the matrix algebra. The authors' main purpose in this note is to establish Frobenius' result by using intrinsic properties of compact Lie groups so that it will give Jordan's Theorem for the finite subgroups F of an arbitrary connected Lie group G. Moreover, with the help of an integration formula of Weyl, an integration formula for the bound of the index is given.

To describe our results, let G be a compact Lie group and $\mathfrak{G}$ its Lie algebra. Denote by Q the totality of elements X in $\mathfrak{G}$ such that the absolute values of the characteristic roots of $\operatorname{Ad} X$[2]) are all less than $\frac{\pi}{6}$. Define $U = \exp Q$ and $k(G) = \mu(G)/\mu(U)$ where μ is a Haar measure of G. The main results can be stated as follows:

I. *Let M be a connected Lie group, and G a maximal compact subgroup of M. Each finite subgroup F of M has a normal abelian subgroup A whose index in F is not greater than $k(G)$.*

II. *Let G be a local direct product $G_1 \cdot G_2 \cdot \ldots \cdot G_a \cdot T$ of compact connected simple Lie groups $G_1, \ldots, G_a$ and a toral group T. Then $k(G) = s \cdot k(\operatorname{Ad} G_1) \cdot k(\operatorname{Ad} G_2) \cdot \ldots \cdot k(\operatorname{Ad} G_a)$ where s is the number of connected components of the center of G.*

III. *Let G be a compact connected simple Lie group of rank r, $\alpha_1, \alpha_2, \ldots, \alpha_r$ a system of simple roots, and $\beta = m_1\alpha_1 + m_2\alpha_2 + \ldots + m_r\alpha_r$ the maximum*

1) This work was supported in part by the National Science Foundation under contracts GP-89 and G-24154.

2) Throughout this paper, Ad denotes the adjoint representation of the group while ad denotes that of the Lie algebra.

root. Denote by s the order of the center of G, and denote by D the domain $\{\alpha_1 > 0, \alpha_2 > 0, \ldots, \alpha_r > 0, \beta < 1/12\}$ and regard every root α as a function of $\alpha_1, \alpha_2, \ldots, \alpha_r$. Then

$$\frac{s}{k(G)} = 2^{(\dim G - r)} \int\int \ldots \int_D (\Pi \sin^2 \pi\alpha) d\alpha_1 \ldots d\alpha_r$$

where the product $\Pi \sin^2\pi\alpha$ extends to all the positive roots α of G.

We also, as an example, carry out the integration for the case $G = \mathrm{Sp}(r)$ and express $k(\mathrm{Sp}(r))$ in terms of a determinant.

2. Some results of FROBENIUS. For the sake of completeness, we shall, in this section, first use geometrical language to redefine some concepts of FROBENIUS and then re-establish some of his results in an intrinsic manner.

Let V be a complex vector space of dimension n with positive definite hermitian form $h(\xi, \eta), \xi, \eta \in V$. Considering V as a $2n$-dimensional real euclidean space, we can define the angle $\sphericalangle(\xi, \eta)$ between two non-zero vectors ξ, η in V. This angle is always assumed to lie between 0 and π, and is given by the formula

$$\cos \sphericalangle(\xi, \eta) = Re\, h(\xi, \eta) \,/\, (|\xi|\ |\eta|)$$

where Re denotes the real part, and $|\xi| = \sqrt{h(\xi, \xi)}$, $|\eta| = \sqrt{h(\eta, \eta)}$ denote the lengths of ξ, η respectively.

Let S be the unit sphere in V and $U(n)$ the group of all unitary transformations. The group $U(n)$ acts on S effectively. Since the angle $\sphericalangle(\xi, \eta)$ is a metric on S (in fact the ordinary spherical metric), the real-valued function d over $U(n) \times U(n)$ defined by

$$d(X, Y) = \sup_{\xi \in S} \sphericalangle(X\xi, Y\xi), X, Y \in U(n)$$

gives a two-sided invariant metric on $U(n)$. The right-invariance follows from the definition while the left-invariance from the fact that $\sphericalangle(X\xi, X\eta) = \sphericalangle(\xi, \eta), X \in U(n), \xi, \eta \in S$. Therefore

$$d(E, XY) = d(X^{-1}, Y) \leqq d(E, X^{-1}) + d(E, Y) = d(E, X) + d(E, Y), \tag{2.1}$$

where E is the identity transformation.

Let $X \in U(n)$ and $\{e^{i\vartheta_1}, e^{i\vartheta_2}, \ldots, e^{i\vartheta_s}\}$ be the set of distinct characteristic roots of X. Choose the ϑ's so that $-\pi < \vartheta_j \leqq \pi$ and define

$$\vartheta(X) = \sup \{|\vartheta_j| : j = 1, 2, \ldots, s\}.$$

The angles $\vartheta_1, \ldots, \vartheta_s$ will be called the phase angles of X. We shall verify that $d(E, X) = \vartheta(X)$. For this purpose, let us denote by V_j the eigenspace of X with eigenvalue $e^{i\vartheta_j}$ (note that $\vartheta_j \neq \vartheta_k$ for $j \neq k$). Then

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_s,\ h(V_j, V_k) = 0, j \neq k.$$

For any vector $\xi \varepsilon S$, we write $\xi = \xi_1 + \xi_2 + \ldots + \xi_s$ with $\xi_j \varepsilon V_j$. Then

$$h(\xi, X\xi) = \Sigma_{j=1}^{s} h(\xi_j, X\xi_j) = \Sigma_{j=1}^{s} |\xi_j|^2 e^{i\vartheta_j}$$

and then

$$\cos \sphericalangle(\xi, X\xi) = \Sigma_{j=1}^{s} |\xi_j|^2 \cos\vartheta_j \geqq \Sigma_{j=1}^{s} |\xi_j|^2 \cos\vartheta(X)$$
$$= \cos\vartheta(X)$$

where $|\xi_j|$ denotes the length of ξ_j. It follows that $\sphericalangle(\xi, X\xi) \leqq \vartheta(X)$, $\xi \varepsilon S$, whence $d(E, X) \leqq \vartheta(X)$. On the other hand, $\vartheta(X) = |\vartheta_m|$ for a certain m, and when $\xi \varepsilon S \cap V_m$ we then have $\sphericalangle(\xi, X\xi) = |\vartheta_m| = \vartheta(X)$. This implies $d(E, X) \geqq \vartheta(X)$ and therefore

$$d(E, X) = \vartheta(X). \tag{2.2}$$

We see that although a particular hermitian metric on V was used above, $\vartheta(X)$ and hence $d(E, X)$ do not depend on this choice. We also note that the above statements remain valid for a closed subgroup of $U(n)$ which will be used below.

FROBENIUS **Lemma.** *Let* $X, Y \varepsilon U(n)$ *and* $[X, [X, Y]] = E$ *where* $[X, Y] = XYX^{-1}Y^{-1}$ *denotes the multiplicative commutator operation. If* $\vartheta(Y) < \frac{\pi}{2}$, *then* X *commutes with* Y.

Proof. Let $V_1, \ldots, V_s$ be the eigenspaces of X with eigenvalues $\varrho_1, \ldots, \varrho_s$. Then $W_1 = Y(V_1), \ldots, W_s = Y(V_s)$ must be the eigenspaces of the transformation $T = YXY^{-1}$. Denoting by V'_j the orthogonal complement of V_j, we have $V = V_j \oplus V'_j$. From the hypothesis, X commutes with T and so $T(V_j) = V_j, T(V'_j) = V'_j$. It follows then $W_j = (W_j \cap V_j) \oplus (W_j \cap V'_j)$. But $\vartheta(Y) < \frac{\pi}{2}$, $\sphericalangle(\xi, Y\xi) < \frac{\pi}{2}$, for all vectors ξ. Therefore $W_j \cap V'_j = Y(V_j) \cap V'_j = 0$, and $W_j = W_j \cap V_j$, whence $W_j = V_j$ because they have the same dimension. Now X and T have the same set of eigenspaces corresponding to the same eigenvalues. Thus $X = T$, or what is the same, $XY = YX$.

Remark. FROBENIUS proved this result under the weaker assumption that the phase angles $\vartheta_1, \vartheta_2, \ldots, \vartheta_s$ differ from one another by an angle less than π. However, this follows as an immediate consequence of the above version. In fact, if $|\vartheta_j - \vartheta_k| < \pi$ for all j and k, there exists then a central element Z of $U(n)$ (i.e., scalar unitary transformation) such that $\vartheta(YZ) < \frac{\pi}{2}$. Applying our proposition with YZ taking the place of Y, we obtain $X(YZ) = (YZ)X$ whence $XY = YX$.

3. Two Lemmas. Let G be a compact LIE group and $\mathfrak{G}$ its LIE algebra. The linear adjoint group Ad G, being compact, may be regarded as a subgroup of the orthogonal group $0(n)$ acting on the real LIE algebra $\mathfrak{G}$, i.e., it leaves invariant a euclidean (symmetric, positive definite, bilinear) inner product on $\mathfrak{G}$. Let V be the vector space over the complex numbers C obtained by extending the field of scalars of $\mathfrak{G}$, and let h be any positive definite hermitian form on V whose restriction to $\mathfrak{G}$ is the inner product left invariant by Ad G. Then if $U(n)$ is the group of linear transformations of V leaving h invariant, $0(n)$ and its subgroup Ad G are imbedded as closed subgroups of $U(n)$ and the results of the previous paragraph (in particular, (2.2) and the remarks following it) apply to Ad G acting on V. Thus the function

$$d(\mathrm{Ad}\, x, \mathrm{Ad}\, y) = \vartheta(\mathrm{Ad}\, xy^{-1}),\ x, y \varepsilon G$$

is a two-sided invariant metric on Ad G. Now define $\vartheta(x) = \vartheta(\mathrm{Ad}\, x)$, $x \varepsilon G$. Then ϑ is a real continuous function over G with $0 \leqq \vartheta(x) \leqq \pi$ and

$$\vartheta(x) = \vartheta(x^{-1}),\ \vartheta(yxy^{-1}) = \vartheta(x),\ \vartheta(xz) = \vartheta(x),\ \vartheta(xy) \leqq \vartheta(x) + \vartheta(y)$$

for all elements x, y of G and all z belonging to the center of G. The last inequality follows from (2.1) and (2.2).

Lemma 1. *Let G be a compact* LIE *group and $x, y \varepsilon G, \vartheta(y) < \frac{\pi}{2}$. If $[x,[x,y]]$ belongs to the center Z of G, then $[x,y] \varepsilon Z$ where $[x,y] = xyx^{-1}y^{-1}$ denotes the commutator operation in the group.*

Proof. Let $X = \mathrm{Ad}\, x$, $Y = \mathrm{Ad}\, y$. Then $\vartheta(Y) < \frac{\pi}{2}$ and $[X, [X, Y]] = E$. From the FROBENIUS Lemma in §2, we have $X Y = Y X$, or what is the same, $[x, y] \varepsilon Z$.

This Lemma can be sharpened. We are, however, content with it since any improvement will not help us in the study of finite subgroups of G.

Lemma 2. *Let G be a compact connected LIE group, and ϱ the global metric induced by a two-sided invariant RIEMANNian metric on G. Then for elements x, y and e (= identity in G),*

$$\varrho(e, [x, y]) \leqq 2(\sin \tfrac{1}{2}\vartheta(y))\, \varrho(e, x) .$$

Proof. We will denote by h the invariant scalar product in $\mathfrak{G}$ which gives rise to the metric ϱ on G, and we denote by $\|X\|$ the length of a vector X with respect to h. Now choose $X, Y \varepsilon \mathfrak{G}$ such that

$$x = \exp X,\ y = \exp Y,\ \|X\| = \varrho(e, x),\ \|Y\| = \varrho(e, y) .$$

We use here and below the fact that the geodesics through e are exactly the one-parameter subgroups. Due to compactness any point can be joined to e by a geodesic whose length is equal to the distance from e. If we take

$$x(t) = \exp tX,\ U(t) = (\mathrm{Ad}\, x(t))\, Y,\ u(t) = \exp U(t),$$
$$0 \leqq t \leqq 1,$$

then $u(t)$ is a curve in G joining y to xyx^{-1}. Let l be the arc length of this curve. From the definition of ϱ, we have

$$\varrho(e, [x, y]) = \varrho(y, xyx^{-1}) \leqq l.$$

Now let us give an estimate of l. For this purpose, we write $x(t) = \exp tX$ and then $u(t) = x(t) y x(-t)$. The formula

$$dx/dt = L_{x(t)} X = R_{x(t)} X$$

implies that

$$du/dt = L_{x(t)} R_{yx(-t)} X - L_{x(t)y} R_{x(-t)} X = -\mathrm{Ad}\, x(t) \cdot L_y \cdot (E - \mathrm{Ad}\, y^{-1}) \cdot X .$$

Combining this with the fact that the scalar product in $\mathfrak{G}$ is left unaltered by both left and right translations, we have[3])

$$\left\| \frac{du}{dt} \right\| = \left\| (E - \operatorname{Ad} y^{-1}) X \right\| ,$$

and then

$$l = \int_0^1 \left\| \frac{du}{dt} \right\| dt = \int_0^1 \left\| (E - \operatorname{Ad} y^{-1}) X \right\| dt$$
$$= \left\| (E - \operatorname{Ad} y^{-1}) X \right\| .$$

Now Ad y^{-1} is a rotation in $\mathfrak{G}$, and by using the normal form of a rotation, it is easy to check that

$$\| (E - \operatorname{Ad} y^{-1}) X \| \leqq 2 \sin \tfrac{1}{2} \vartheta(y) \| X \| .$$

Since $\| X \| = \varrho(e, x)$, this gives

$$\varrho(e, [x, y]) \leqq l \leqq 2 (\sin \tfrac{1}{2} \vartheta(y)) \varrho(e, x).$$

The Lemma is thus proved.

Remark. With y fixed and x varied, $2 \sin \frac{1}{2} \vartheta(y)$ is the supremum of the ratio $\varrho(e,[x,y]) / \varrho(e,x)$, and therefore Lemma 2 cannot be further improved. To see this, we choose $X \varepsilon G$ such that the angle between X and $(\operatorname{Ad} y) X$ is exactly $\vartheta(y)$, and put $x(s) = \exp sX$. Then

$$\lim_{s \to 0} \frac{\varrho(e, [x(s), y])}{\varrho(e, x(s))} = 2 \sin \tfrac{1}{2} \vartheta(y).$$

4. A Theorem. Let G be a compact, connected LIE group with LIE algebra $\mathfrak{G}$. For each $0 < c < \pi$, we denote by Q_c the totality of $X \varepsilon \mathfrak{G}$ such that all the characteristic roots of ad X have absolute value less than c. Set

$$U_c = \exp Q_c, \; W_c = \{ x \varepsilon G : \vartheta(x) < c \} .$$

Evidently

$$U_c = U_c^{-1}, W_c = W_c^{-1}, g U_c g^{-1} = U_c, g W_c g^{-1} = W_c,$$
$$(\operatorname{Ad} g)(Q_c) = Q_c, \; U_c \subset W_c, \; g \varepsilon G .$$

---

[3]) We wish to thank the referee for a shorter proof of this formula.

Both U_c and W_c are open in G (since exp is an open mapping), and U_c is connected. Since the closure $\bar{Q}_c$ of Q_c is compact, exp $\bar{Q}_c$ is compact. From the fact that $U_c = (\exp\ \bar{Q}_c) \cap W_c$, it follows that U_c is not only open but closed in W_c. Therefore U_c is the connected component of W_c which contains the identity.

Let us consider $U_c U_{c'}$, $c > 0$, $c' > 0$, $c + c' < \pi$. Suppose $x = \exp X$, $y = \exp Y$, $X \varepsilon Q_c$, $Y \varepsilon Q_{c'}$. Then

$$\vartheta(\exp tX \cdot \exp tY) \leqq \vartheta(\exp tX) + \vartheta(exp\ tY) \leqq c + c'$$
$$0 \leqq t \leqq 1$$

and then $\exp tX \cdot \exp tY \varepsilon W_{c+c'}$ for all t with $0 \leqq t \leqq 1$. This implies that $xy = \exp X \cdot \exp Y$ lies in the connected component of $W_{c+c'}$ which contains the identity, or what is the same $xy \varepsilon U_{c+c'}$. Hence we have shown

$$U_c U_{c'} \subset U_{c+c'}. \tag{4.1}$$

In what follows, we need the following simple property of commutators.

(4.2) *Suppose* $0 < c < \pi/2$ *and* $y \varepsilon U_c$. *If the commutator* $[x, y]$ *belongs to the center* Z *of* G, *then* $[x, y] = e$.

Proof. Let us decompose G into the local direct product $G_s \cdot Z_0$ of its semi-simple part G_s and the identity component of the center Z. Then we have a corresponding LIE algebra decomposition $\mathfrak{G} = \mathfrak{G}_s + \mathfrak{Z}$. Denote by Q_c^s the totality of $X \varepsilon \mathfrak{G}_s$ such that all the characteristic roots of ad X have absolute values less than c, and $U_c^s = \exp Q_c^s$. Then $Q_c = Q_c^s + \mathfrak{Z}$, and $U_c = U_c^s Z_0$. Writing $y = uz$ with $u \varepsilon U_c^s$, $z \varepsilon Z_0$, we have

$$[x, y] = (xyx^{-1})y^{-1} = (xux^{-1})u^{-1} \varepsilon U_c^s U_c^s \subset U_{2c}^s.$$

Therefore $[x, y]$ belongs to the intersection of U_{2c}^s and the center of G_s. But G_s is semi-simple and $2c < \pi$, and so this intersection contains only the identity. It follows that $[x, y] = e$. (4.2) is thus proved.

For each compact and connected LIE group G, we define $k(G) = \mu(G)/\mu(U_{\pi/6})$ where μ is a HAAR measure on G. Evidently $k(G)$ does not depend on the choice of μ and $k(G_1 \times G_2) = k(G_1)k(G_2)$.

Now we are in a position to prove one of our main results.

Theorem 1. *Let* M *be a connected* LIE *group and* G *a maximal compact subgroup of* M. *Then each finite subgroup* F *of* M *has always a normal abelian subgroup* A *such that the index of* A *in* F *is not greater than* $k(G)$.

Proof. We note then any two maximal compact subgroups of M are conjugate. Therefore, without loss of generality, we can assume that $F \subset G$. Let U_c have the same meaning with respect to G as above. We shall show that $A_0 = F \cap U_{\pi/3}$ is a commutative set. To see this, let $x, y \varepsilon A_0$ and consider the sequence of commutators

$$x_1 = [x, y],\ x_2 = [x, x_1],\ \ldots,\ x_{n+1} = [x, x_n],\ \ldots .$$

From Lemma 2, $\varrho(e, x_{n+1}) \leqq (2 \sin \frac{1}{2}\vartheta(x))\, \varrho(e, x_n)$ and hence $\varrho(e, x_i) \leqq \leqq (2 \sin \frac{1}{2}\vartheta(x))^i \varrho(e, y)$. Since F is finite and $2 \sin \frac{1}{2}\vartheta(x) < 1$, there must be an n such that

$$\varrho(e, x_{n+1}) = 0,\ [x, [x, x_{n-1}]] = [x, x_n] = x_{n+1} = e.$$

Let $u = x_{n-2}x^{-1}x_{n-2}^{-1}$. Then $u = x^{-1}x_{n-1}$ and then

$$[x, [x, u]] = [x, [x, x^{-1}x_{n-1}]] = x^{-1}[x, [x, x_{n-1}]]\, x = e.$$

Since $\vartheta(u) = \vartheta(x^{-1}) < \pi/3$, we have, from Lemma 1 and (4.2), $[x, u] = e$, whence $x_n = e$. By repeating this process, we show successively that $x_{n-1} = = e, \ldots, x_1 = e$. Hence $xy = yx$ and A_0 is a commutative set. The subgroup A generated by A_0 is then abelian. But $U_{\pi/3}$ is invariant under the adjoint group, and so A is a normal abelian subgroup of F.

Let $b_1A, b_2A, \ldots, b_qA$ be the totality of cosets of A in F. When $i \neq j$, $b_j^{-1}b_i \notin A$. For simplicity, we set $U = U_{\pi/6}$. From (4.1), $UU^{-1} \subset U_{\pi/3}$. The q open sets

$$b_1U,\ b_2U,\ \ldots,\ b_qU$$

must be disjoint, for if $b_jU \cap b_kU \neq \varnothing$, $j \neq k$, we would have $b_j^{-1}b_k \varepsilon UU^{-1} \subset U_{\pi/3}$ and then $b_j^{-1}b_k \varepsilon A$, which is impossible.
Therefore

$$\mu(G) \geq \Sigma\mu(b_jU) = q\mu(U)$$

and

$$\text{index } A = q \leqq \mu(G)\,/\,\mu(U) = k(G).$$

The Theorem is thus proved.

5. Some properties of $k(G)$. In this section, we shall establish a Theorem which reduces the problem of $k(G)$ for a general G to that for simple LIE groups G.

Theorem 2. *Let* $G = G_1 \cdot G_2 \cdot \ldots \cdot G_a \cdot T$ *be the local direct product of compact connected simple* Lie *groups* $G_1, \ldots, G_a$ *and a toral group* T. *Then*

$$k(G) = s k(\mathrm{Ad}\, G_1) \cdot k(\mathrm{Ad}\, G_2) \cdot \ldots \cdot k(\mathrm{Ad}\, G_a)$$

where s *denotes the number of connected components in the center* Z *of* G.

Proof. Let $G' = \mathrm{Ad}\, G$ and $p : G \to G'$ the projection. Denote by Z_0 the identity component of the center Z of G. Then s is the order of the quotient Z/Z_0. Suppose $U = U_{\pi/6}$, $Q = Q_{\pi/6}$ have the same meaning as in the first paragraph of § 4. Let U' and Q' be, respectively, the counterparts of U and Q for the group G'. Since $\ker p = Z$, $(dp)(Q) = Q'$ and $p(U) = U'$. It follows then that $UZ = p^{-1}(U')$. Suppose

$$Z = z_1 Z_0 \cup z_2 Z_0 \cup \ldots \cup z_s Z_0, \quad z_i^{-1} z_j \notin Z_0, \; i \neq j.$$

From the fact that $U Z_0 = U$, we have

$$p^{-1}(U') = UZ = z_1 U \cup z_2 U \cup \ldots \cup z_s U.$$

This union is a disjoint union. Suppose $z_i U \cap z_j U \neq \varnothing$. Then $z_i^{-1} z_j \in U U^{-1} \subset U_{\pi/3}$. This means that we can choose $X \in Q_{\pi/3}$ such that $z_i^{-1} z_j = \exp X$, and then $E = \mathrm{Ad}\, z_i^{-1} z_j = \exp(\mathrm{ad}\, X)$. But all the characteristic roots of $\mathrm{ad}\, X$ have absolute values less than $\pi/3$ and so the above equality implies $\mathrm{ad}\, X = 0$. In other words, X belongs to the center $\mathfrak{Z}$ of the Lie algebra $\mathfrak{G}$. Hence $z_i^{-1} z_j \in Z_0$ and so $i = j$. The union $z_1 U \cup \ldots \cup z_s U$ is therefore disjoint. Let μ be a normalized Haar measure of G. Then we have

$$\mu(p^{-1}(U')) = \Sigma_{j=1}^{s} \mu(z_j U) = s\mu(U).$$

The set function $\mu' = \mu \cdot p^{-1}$ is evidently a normalized Haar measure of G' and so

$$k(G') = 1/\mu'(U') = 1/s\mu(U) = k(G)/s.$$

Since $\mathrm{Ad}\, G \cong \mathrm{Ad}\, G_1 \times \mathrm{Ad}\, G_2 \times \ldots \times \mathrm{Ad}\, G_a$, we get

$$k(G) = s \cdot k(G') = s \cdot k(\mathrm{Ad}\, G_1) \cdot \ldots \cdot k(\mathrm{Ad}\, G_a).$$

6. An integral formula for $k(G)$. From Theorem 2, to calculate $k(G)$ for a general G, it suffices to calculate $k(G)$ for simple compact Lie groups. In this section, we shall express it in terms of an ordinary RIEMANN integral. For this purpose let us recall some known results about compact semi-simple LIE groups [8], [10], [11].

Suppose that G is a connected and compact semi-simple LIE group of rank r, and $\mathfrak{G}$ its LIE algebra. Take a maximal toral subgroup H of G, and denote by $\mathfrak{H}$ the LIE algebra of H. Restricted to $\mathfrak{H}$, the exponential map is a homomorphism of the additive group $\mathfrak{H}$ onto the multiplicative group H. The kernel of this homomorphism is a lattice γ in $\mathfrak{H}$. Let $\pm\varphi_1, \pm\varphi_2, \ldots, \pm\varphi_m$, $2m = \dim G - r$ be the roots of $\mathfrak{G}$ with respect to $\mathfrak{H}$. They are linear forms on $\mathfrak{H}$ taking integer values on γ. It follows that $\cos 2\pi\varphi_j$, $e^{2\pi i\varphi_j}$ are functions on $\mathfrak{H}$ mod γ, and hence they can be regarded as functions on H. In fact, for $x \varepsilon H$, the characteristic roots of $\operatorname{Ad} x$ are precisely

$$\underbrace{1, 1, \ldots, 1}_{r}, e^{2\pi i\varphi_1}, e^{-2\pi i\varphi_1}, \ldots, e^{2\pi i\varphi_m}, e^{-2\pi i\varphi_m}.$$

Let

$$\varrho(x) = 4^m \Pi_{j=1}^m \sin^2 \pi\varphi_j(x)$$

and dg, dx be the normalized invariant volume elements of G, H, respectively (i.e., $\int_G dg = \int_H dx = 1$). Then, for any class function f over G, we have

$$\int_G f(g)dg = \int_H f(x)\varrho(x)dx/w$$

where w is the order of the WEYL group [8], [10].

For later application, we find it more convenient to express $\int_G f(g)dg$ in terms of integrals over $\mathfrak{H}$. To do this, we take a fundamental region P of γ in $\mathfrak{H}$, and take an invariant volume element dX of the additive group $\mathfrak{H}$ such that $\int_P dX = 1$. Then

$$\int_G f(g)dg = \int_P \tilde{f}(X)\tilde{\varrho}(X)dX/w$$

where $\tilde{f}$, $\tilde{\varrho}$ are functions on $\mathfrak{H}$ given by

$$\tilde{f}(X) = f(\exp X),\ \tilde{\varrho}(X) = \varrho(\exp X),\ X \varepsilon \mathfrak{H}.$$

Theorem 3. *Let G be a compact, connected simple* LIE *group of rank r. Suppose $\alpha_1, \ldots, \alpha_r$ to be a system of simple roots and $\beta = m_1\alpha_1 + m_2\alpha_2 + \ldots + m_r\alpha_r$ the maximal root. Denote by s the number of elements in the* *center* Z *of* G, *by* D *the domain* $\{\alpha_1 > 0, \ldots, \alpha_r > 0, m_1\alpha_1 + \ldots + m_r\alpha_r < \frac{1}{12}\}$, *and regard each root* α *as a function of* $\alpha_1, \ldots, \alpha_r$. *Then*

$$\frac{1}{k(G)} = \frac{2^{(\dim G - r)}}{s} \iint \ldots \int_D (\Pi \sin^2 \pi\alpha)\, d\alpha_1 \ldots d\alpha_r$$

where, in the product $\Pi \sin^2 \pi\alpha$, α *runs through all positive roots.*

Proof. From Theorem 2, it suffices to prove the formula for the case $G = \text{Ad}\, G$. Thus we can assume that G has a trivial center. Then

$$P = \{X \varepsilon \mathfrak{H} : -\tfrac{1}{2} < \alpha_j(X) \leqq \tfrac{1}{2}, j = 1, 2, \ldots, r\}$$

is a fundamental region of γ in $\mathfrak{H}$. Suppose that dg, dX have the same meaning as before, and $Q = Q_{\pi/6}$ is the totality of X in $\mathfrak{G}$ such that all the characteristic roots of ad X have absolute values less than $\pi/6$. Let $U = U_{\pi/6} = \exp Q$ and μ be the normalized HAAR measure of G. Then by WEYL's integration formula, we have

$$\frac{1}{k(G)} = \mu(U) = \int_{P \cap Q} \tilde{\varrho}(X)\, dX / w.$$

Let $R = \{X \varepsilon \mathfrak{H} : |\varphi(X)| < \frac{1}{12}$ for all roots $\varphi\}$. Then $P \cap Q = R$. We find it convenient to divide R into subdomains. For this purpose, let $C_1, C_2, \ldots, C_w$ be all the WEYL chambers. Up to a change of index, we can assume

$$C_1 = \{X \varepsilon \mathfrak{H} : \alpha_j(X) > 0, j = 1, 2, \ldots, r\}.$$

The union $C_1 \cup C_2 \cup \ldots \cup C_w$ is disjoint and $\mathfrak{H} \equiv C_1 \cup C_2 \cup \ldots \cup C_w$ where, as well as in what follows, the symbol "$\equiv$" means "coincides up to a set of measure zero". It follows then that $R \equiv \cup_{i=1}^{w} (R \cap C_i)$ and

$$\frac{1}{k(G)} = \Sigma_{i=1}^{w} \int_{R \cap C_i} \tilde{\varrho}(X)\, dX / w.$$

The WEYL group Φ leaves invariant both the function $\tilde{\varrho}(X)$ and the set R. Therefore the sets $R \cap C_1, R \cap C_2, \ldots, R \cap C_w$ are equivalent to one another under Φ, and we have then

$$\frac{1}{k(G)} = \int_{R \cap C_1} \tilde{\varrho}(X)\, dX.$$

Using the fact that any positive root φ is of the form $\varphi = \Sigma q_i \alpha_i$ with $q_i \leq m_i$, $i = 1, 2, \ldots, r$, we see that $R \cap C_1 = D$ and $\tilde{\varrho}(X) = 4^m \Pi \sin^2 \pi\alpha$ where $2m = \dim G - r$ and the product $\Pi \sin^2 \pi\alpha$ extends to all positive roots α. Hence

$$\frac{1}{k(G)} = 4^m \int_D \Pi \sin^2 \pi\alpha \, dX .$$

By considering $\alpha_1, \ldots, \alpha_r$ as functions over $\mathfrak{H}$, the differential r-form $d\alpha_1 \ldots d\alpha_r$ (more precisely its absolute value) is an invariant volume element of $\mathfrak{H}$ with $\int_P d\alpha_1 \ldots d\alpha_r = 1$ and so we can take it to be our dX. It follows then

$$\frac{1}{k(G)} = 4^m \int \ldots \int_D \Pi \sin^2 \pi\alpha \, d\alpha_1 \ldots d\alpha_r .$$

Theorem 3 is thus proved.

7. $k(G)$ for the symplectic group. As an example, we shall actually carry out the integration for the case $G = \mathrm{Sp}(r)$. Let us first establish a lemma.

Lemma. *Let $f_1, f_2, \ldots, f_r$ be integrable functions in one variable, and*

$$\Delta = \det \begin{vmatrix} f_1(x_1) & f_2(x_1) & \ldots & f_r(x_1) \\ f_1(x_2) & f_2(x_2) & \ldots & f_r(x_2) \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ f_1(x_r) & f_2(x_r) & \ldots & f_r(x_r) \end{vmatrix}$$

Then

$$\int_0^a \ldots \int_0^a \Delta^2 dx_1 \ldots dx_r = r! \det \begin{vmatrix} A_{11} & A_{12} & \ldots & A_{1r} \\ A_{21} & A_{22} & \ldots & A_{2r} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ A_{r1} & A_{r2} & \ldots & A_{rr} \end{vmatrix}$$

where

$$A_{ij} = \int_0^a f_i(t) f_j(t) \, dt .$$

Proof. For every permutation $(j_1, \ldots, j_r)$ of $(1, 2, \ldots, r)$ let $\varepsilon(j_1, j_2, \ldots, j_r)$ denote 1 or —1 according as the permutation is even or odd. Then

$$\Delta^2 = \Sigma_{j,k}\, \varepsilon(j_1, \ldots, j_r)\, \varepsilon(k_1, \ldots, k_r)\, f_{j_1}(x_1) f_{k_1}(x_1) \ldots f_{j_r}(x_r) f_{k_r}(x_r)$$

where the summation extends to all pairs of permutations $(j_1, j_2, \ldots, j_r)$, $(k_1, k_2, \ldots, k_r)$ of $(1, 2, \ldots, r)$. It follows that

$$I = \int_0^a \cdots \int_0^a \Delta^2 \, dx_1 \ldots dx_r$$
$$= \Sigma_{j,k} \, \varepsilon(j_1, \ldots, j_r) \, \varepsilon(k_1, \ldots, k_r) \, A_{j_1 k_1} A_{j_2 k_2} \ldots A_{j_r k_r}$$
$$= \Sigma_j \, \varepsilon(j_1, \ldots, j_r) \, D(j_1, \ldots, j_r)$$

where

$$D(j_1, \ldots, j_r) = \det \begin{vmatrix} A_{j_1 1} & A_{j_1 2} & \ldots & A_{j_1 r} \\ A_{j_2 1} & A_{j_2 2} & \ldots & A_{j_2 r} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ A_{j_r 1} & A_{j_r 2} & \ldots & A_{j_r r} \end{vmatrix} .$$

Since $D(j_1, \ldots, j_r)$ is skew with respect to $(j_1, \ldots, j_r)$ we have

$$I = r! \, D(1, 2, \ldots, r) = r! \det \begin{vmatrix} A_{11} & A_{12} & \ldots & A_{1r} \\ A_{21} & A_{22} & \ldots & A_{2r} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ A_{r1} & A_{r2} & \ldots & A_{rr} \end{vmatrix} .$$

The Lemma is proved.

Now let G be the symplectic group of rank r. We can choose coordinates $x_1, \ldots, x_r$ in $\mathfrak{H}$ such that $\{\pm 2x_j, \pm x_j \pm x_k : j < k\}$ is the totality of roots. Then [2]

$$\alpha_1 = x_1 - x_2, \alpha_2 = x_2 - x_3, \ldots, \alpha_{n-1} = x_{n-1} - x_n, \alpha_n = 2x_n$$

form a fundamental system of simple roots with

$$\beta = 2x_1 = 2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$$

as the maximal root. In terms of x's the domain D in Theorem 3 becomes

$$0 < x_n < x_{n-1} < \ldots < x_1 < \frac{1}{24} .$$

When $G = \mathrm{Sp}(r)$, we know that $s = 2$ and the product $4^m \Pi \sin^2 \pi\alpha$ $(2m = \dim G - r)$ is the square of the determinant [11, p. 59]

$$\Delta = 2^r \det \begin{vmatrix} \sin 2\pi x_1 & \sin 2\pi x_2 & \ldots & \sin 2\pi x_r \\ \sin 4\pi x_1 & \sin 4\pi x_2 & \ldots & \sin 4\pi x_r \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ \sin 2r\pi x_1 & \sin 2r\pi x_2 & \ldots & \sin 2r\pi x_r \end{vmatrix}$$

Therefore

$$\frac{1}{k(\mathrm{Sp}(r))} = \tfrac{1}{2} \int \ldots \int_D \Delta^2 \, d\alpha_1 \ldots d\alpha_r = \int \ldots \int_D \Delta^2 \, dx_1 \ldots dx_r .$$

Any permutation of the coordinates $x_1, \ldots, x_r$ leaves the integrand unchanged. It follows then

$$\frac{1}{k\,(\mathrm{Sp}\,(r))} = \frac{1}{r!} \int \cdots \int_{D'} \Delta^2\, dx_1 \ldots dx_r$$

where $D' = \{0 < x_j < \frac{1}{24},\ j = 1, 2, \ldots, r\}$. Applying the above Lemma, we have

$$\frac{1}{k\,(\mathrm{Sp}\,(r))} = 4^r \det \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ c_{r1} & c_{r2} & \cdots & c_{rr} \end{vmatrix}$$

with $c_{kk} = (k\pi - 6 \sin (k\pi/6))/48\,k\pi$, and for $j \neq k$,
$c_{jk} = [(j + k) \sin ((j - k)\pi/12) - (j - k) \sin ((j + k)\pi/12)]/4(j^2 - k^2)\pi$.

REFERENCES

[1] L. BIEBERBACH: *Über einen Satz des Herrn* C. JORDAN. S.-B. Preuss. Akad. Wiss. 231–240 (1911).
[2] E. CARTAN: *La Géométrie des groupes simples.* Annali di Mat. *4,* 209–256 (1927).
[3] Séminaire de C. CHEVALLEY, 1956–1958, mimeographed notes.
[4] G. FROBENIUS: *Über den* JORDAN*schen Satz.* S.-B. Preuss. Akad. Wiss. 241–248 (1911).
[5] G. FROBENIUS: *Über unitäre Matrizen.* S.-B. Preuss. Akad. Wiss. 373–378 (1911).
[6] S. HELGASON: *Differential Geometry and Symmetric Spaces.* Academic Press, 1962.
[7] C. JORDAN: *Mémoire sur les équations différentielles linéaires à intégrale algébrique.* J. Reine Angew. Math. *84,* 89–215 (1878) = Oeuvres de C. J. *2,* 13–139 (1961).
[8] Séminaire "SOPHUS LIE", 1955, mimeographed notes.
[9] A. SPEISER: *Die Theorie der Gruppen von endlicher Ordnung.* 4th edition, Birkhäuser Verlag, 1956.
[10] E. STIEFEL: *Kristallographische Bestimmung der Charaktere der geschlossenen* LIE*schen Gruppen.* Comment. Math. Helv. *17,* 165–200 (1944–1945).
[11] H. WEYL: *The Structure and Representations of Continuous Groups.* Inst. Adv. Study, Princeton, 1935.

(Received October 24, 1964)