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Autor(en): **Eckmann, B. / Hilton, P.J.**

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Unions and intersections in Homotopy Theory

by B. Eckmann and P. J. Hilton*)

1. Introduction

Much of the algebra of category theory has been concerned with abelian categories: this is natural in view of the importance of homological algebra and the fact that, for such categories $\mathfrak{C}$, the set of morphisms $M_{\mathfrak{C}}(A, B)$ has the rich algebraic structure of an abelian group with respect to which composition is distributive. However in representing group theory or homotopy theory it is no longer possible to confine attention to abelian categories: the resulting algebraic theory of categories which emerges from a generalization of such mathematical theories has been discussed in [5, 6, 7] and [8]. In both these works one has retained from homological algebra at least the notion of the existence of kernels and cokernels, and unions and intersections. However it was remarked in [6] that, for example, even where there are no kernels in the strict sense one always has a *kernel ideal;* but it is not necessarily the case that this ideal is principal, let alone principal and generated by a monomorphism.

Our interest in this paper is almost exclusively in the homotopy category $\mathfrak{T}_h$ belonging to $\mathfrak{T}$, the category of based spaces of the based homotopy type of CW-complexes and based maps; the morphisms $[f]$ of $\mathfrak{T}_h$ are the based homotopy classes (of based maps f in $\mathfrak{T}$). In this category $\mathfrak{T}_h$ it turns out that the kernel, cokernel, union, intersection and equalizer ideals are all principal but are not furnished with canonical generators. Let us exemplify the situation with regard to kernels and cokernels. Consider first the fibration (in $\mathfrak{T}$)

$$S^1 \xrightarrow{i} SO(3) \xrightarrow{p} S^2. \tag{1.1}$$

Were $X \xrightarrow{[q]} SO(3)$ the kernel of $[p]$ in $\mathfrak{T}_h$, so that $[q]$ is a monomorphism in $\mathfrak{T}_h$, we would have the following commutative diagram

$$\begin{tikzcd}
\pi_2(S^2) \arrow[r] & \pi_1(S^1) \arrow[r] \arrow[d, shift right, "\alpha"'] & \pi_1(SO(3)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_1(X) \arrow[r] \arrow[u, shift right, "\beta"'] & \pi_1(SO(3)) \arrow[r] & 0
\end{tikzcd}$$

This would imply $\pi_1(X) = Z_2$ and $\alpha\beta = 1$, an evident contradiction. Now

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consider the cofibration (in $\mathfrak{T}$), where P^2 is the real projective plane and S^1 an embedded projective line,

$$S^1 \xrightarrow{j} P^2 \xrightarrow{k} S^2. \tag{1.2}$$

Were $P^2 \xrightarrow{[q]} X$ the cokernel of $[j]$ in $\mathfrak{T}_h$, so that $[q]$ is an epimorphism in $\mathfrak{T}_h$, we would have the following commutative diagram (cohomology with integer coefficients)

$$\begin{tikzcd} H^1(S^1) \arrow[r] & H^2(S^2) \arrow[r] \arrow[d, shift right, "\alpha"'] & H^2(P^2) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & H^2(X) \arrow[r] \arrow[u, shift right, "\beta"'] & H^2(P^2) \arrow[r] & 0. \end{tikzcd}$$

This would imply $H^2(X) = Z_2$ and $\alpha\beta = 1$, an evident contradiction.

It is plain that in these examples $[i]$ generates the kernel ideal of $[p]$ and $[k]$ generates the cokernel ideal of $[j]$; clearly these statements generalize to embeddings of fibres in any fibration and projection onto cofibres in any cofibration. These observations are the basis of the study of these and related ideals in $\mathfrak{T}_h$.

We have preferred to work in an arbitrary (pointed) category $\mathfrak{C}$ rather than in the category $\mathfrak{T}$ and its related homotopy category $\mathfrak{T}_h$. Our reason has not been primarily that of increased generality nor even the exploitation of the duality principle; but rather to exhibit how little of the richness of the categories $\mathfrak{T}$ and $\mathfrak{T}_h$ is required to prove the ideals in question principal. Essentially one requires only an equivalence relation compatible with composition to associate a homotopy category $\mathfrak{C}_h$ with any category $\mathfrak{C}$. Then one may define fibrations and cofibrations in $\mathfrak{C}$ and our results apply to any category $\mathfrak{C}$ in which maps may be expressed as compositions of homotopy equivalences and fibrations (cofibrations and homotopy equivalences). Thus we have deliberately stressed the simplicity, indeed triviality, of our arguments, in order to achieve clarification. The universality thereby achieved does in fact enable us to apply the results to abelian categories furnished with the notions of p-homotopy or i-homotopy; see [3]. It is worth remarking that the explicit constructions of generators of equalizer ideals, for example, are quite different in the case of the category $\mathfrak{T}_h$ and the homotopy category associated with an abelian category; but their existence admits a common, and elementary proof.

The desire to elucidate some of the algebraic structure of the category $\mathfrak{T}_h$ did not constitute the only motive for assembling the facts presented here. In his paper [2], BROWN pushes through a rather subtle 5-lemma argument (in the absence of group structure in all the terms) by means of a construction of a certain space W_f auxiliary to a map f. In reporting on BROWN's paper, one of the authors found that this construction, together with the crucial property enjoyed by the space W_f, rested on the facts that equalizer ideals were principal (in the category $\mathfrak{T}_h$ and in the category of sets) and that the «cohomology» functors BROWN considers preserve equalizers. Thus it seemed desirable to present the necessary category-theoretical background to this approach to BROWN's argument. We should further explain that, having BROWN's construction in mind, we were led to give preference in our exposition to cokernels, unions, and right equalizers over their duals.

The last two sections of the paper lead in a different direction. In section 4 we look at the relation between the very general homotopy notion of the previous three sections and the homotopy notion which arises from a homotopy system in the category. In section 5 we consider the circumstances under which a MAYER-VIETORIS sequence may be valid in a category with homotopy.

The notations and terminology of this paper are based on those of [5, 6], but the inverse product $A * B$ of two objects A, B is written $A \vee B$ in the categories $\mathfrak{T}$ and $\mathfrak{T}_h$. – A monomorphism $f: A \to B$ in $\mathfrak{C}$ is sometimes also called an «embedding». By convention, since the categories we consider are always pointed, functors will always be supposed to transform zero morphisms to zero morphisms.

2. Categories and associated homotopy categories

Let $\mathfrak{C}$ be an I-category; i.e., a category with inverse products. We call $\mathfrak{C}$ an h-category (or, more strictly, we furnish $\mathfrak{C}$ with an h-structure) if, for each $A, B \in \mathfrak{C}$, the set of maps $M_{\mathfrak{C}}(A, B)$ is furnished with an equivalence relation $\simeq$ such that

(i) if $g \simeq g' : A \to B$, $f: B \to C$, $h: D \to A$, then

$$fgh \simeq fg'h : D \to C; \tag{2.1}$$

and (ii) if $g_i \simeq g_i' : A_i \to B$, $i = 1, 2$, then

$$\langle g_1, g_2 \rangle \simeq \langle g_1', g_2' \rangle : A_1 * A_2 \to B. \tag{2.2}$$

Plainly if $\mathfrak{C}$ is an h-category we may form a new category $\mathfrak{C}_h$ whose objects are those of $\mathfrak{C}$ and whose maps are equivalence classes of maps in $\mathfrak{C}$; moreover $\mathfrak{C}_h$ is again an I-category and the evident functor $\mathfrak{C} \to \mathfrak{C}_h$, which we will also designate by H, is an I-functor. We will sometimes write $[g]$ for $H(g)$, where g is a map in $\mathfrak{C}$. We call $\mathfrak{C}_h$ *the homotopy category associated with the h-category* $\mathfrak{C}$. An I-functor T from the h-category $\mathfrak{C}$ to the h-category $\mathfrak{D}$ is called an *h-functor* if it transforms equivalent maps into equivalent maps. Evidently H is itself an

h-functor if we furnish $\mathfrak{C}_h$ with the trivial h-structure given by the equality relation. A map g in $\mathfrak{C}$ is called a *homotopy equivalence* if $[g]$ is an equivalence in $\mathfrak{C}_h$.

Let $g : A \to B$ be a map in the h-category $\mathfrak{C}$. We call g a *cofibration* if, for all $X \in \mathfrak{C}$ and all maps $f_0 : B \to X$, $h_1 : A \to X$ with $f_0 g \simeq h$, there exists $f_1 : B \to X$ with $f_0 \simeq f_1$ and $f_1 g = h_1$.

Proposition 2.3. *If A can be embedded in a contractible object*[1]*), then every cofibration $g : A \to B$ is a monomorphism.*

Proof: Let h_1 embed A in the contractible object X and let $f_0 = 0 : B \to X$. Then $f_0 g \simeq h_1$ since X is contractible. Since g is a cofibration, there exists $f_1 : B \to X$ with $f_1 g = h_1$. Since h_1 is a monomorphism so is g. –

Any map $g : A \to B$ has a *cokernel ideal*, coker g. This is the family of maps f, with domain B, such that $fg = 0$. We say that coker g is *principal* if there exists $c \in$ coker g such that every $f \in$ coker g is of the form $f'c$; we then call c a *generator* of coker g. Note that d is also a generator of coker g if and only if $d = uc$, $c = vd$ for some u, v. If g is a cofibration then coker g may also be called the *cofibre ideal* of g, and written cofib g.

Now let T be a covariant functor from $\mathfrak{C}$ to $\mathfrak{D}$ and let $g : A \to B$ in $\mathfrak{C}$. Then plainly T (coker g) $\subset$ coker Tg. We say that T *preserves cokernels* if

$$T\,(\operatorname{coker} g) = \operatorname{coker}\,(Tg) \tag{2.4}$$

for all g in $\mathfrak{C}$. If coker g is principal then (2.4) holds for g if and only if coker (Tg) is principal and T maps generators of coker g to generators of coker Tg; indeed T maps every generator of coker g to a generator of coker Tg if it so maps one generator.

Proposition 2.5. *Let $g : A \to B$ be a cofibration in the h-category $\mathfrak{C}$ and let $H : \mathfrak{C} \to \mathfrak{C}_h$ be the homotopy classification functor. Then*

$$H\,(\operatorname{cofib} g) = \operatorname{coker}\,(Hg).$$

Proof: Let $[f] \in$ coker (Hg). Thus $fg \simeq 0$. Since g is a cofibration, there exists $f' \simeq f$ with $f'g = 0$. Thus $f' \in$ cofib g and $[f] = [f'] \in H$ (cofib g).

Corollary 2.6. *If g is a cofibration and* cofib g *is principal, generated by c, then* coker (Hg) *is principal, generated by Hc.*

Proposition 2.7. *Let $g \simeq qu : A \to B$ where q is a homotopy equivalence with homotopy inverse j. Then*

$$(\operatorname{coker} Hu)Hj = \operatorname{coker} Hg.$$

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[1]) X is *contractible* if $1 \simeq 0 : X \to X$.

We omit the proof but draw the immediate conclusion:

Theorem 2.8. *All morphisms in $\mathfrak{T}_h$ have principal cokernels and kernels.*

Proof: The mapping cylinder functor provides a factorization $g = qu$ of every map g in $\mathfrak{T}$, where q is a homotopy equivalence and u is a cofibration. All cokernels in $\mathfrak{T}$ are principal. Thus by Corollary 2.6 coker Hu is principal and then by Proposition 2.7 coker Hg is principal.

The dual assertion about kernels in $\mathfrak{T}_h$ follows immediately from the observation that every g may be factorized in $\mathfrak{T}$ as $u'q'$, where q' is a homotopy equivalence and u' a fibration. –

We now consider unions in $\mathfrak{C}$ and $\mathfrak{C}_h$; we are content to discuss a pair of maps (g_1, g_2) in $\mathfrak{C}$, where $g_i : A \to B_i$, and leave generalizations to the reader. The *union ideal*, $UN(g_1, g_2)$, of (g_1, g_2) is the family of pairs (f_1, f_2) such that f_i has domain B_i and $f_1 g_1 = f_2 g_2$. We say that $UN(g_1, g_2)$ is *principal* if there exists a pair $(c_1, c_2) \in UN(g_1, g_2)$ such that every pair (f_1, f_2) in $UN(g_1, g_2)$ is of the form $(f' c_1, f' c_2)$; we then call (c_1, c_2) a *generator* of $UN(g_1, g_2)$. The concept of *union-preserving* functor is easily formulated and remarks analogous to those made for principal cokernels may be made for principal unions. We prove

Proposition 2.9. *Let $g_i : A \to B_i$, $i = 1, 2$, be maps in the h-category $\mathfrak{C}$ and let g_1 be a cofibration. Then*

$$H(UN(g_1, g_2)) = UN(Hg_1, Hg_2).$$

Proof: Of course, for any covariant functor $T : \mathfrak{C} \to \mathfrak{D}$,

$$T(UN(g_1, g_2)) \subset UN(Tg_1, Tg_2). \tag{2.10}$$

Thus it remains to establish the converse inclusion. Let $([f_1], [f_2]) \in UN([g_1], [g_2])$, so that $f_1 g_1 \simeq f_2 g_2$. Since g_1 is a cofibration, $f_1 \simeq f_1'$ with $f_1' g_1 = f_2 g_2$. Thus $(f_1', f_2) \in UN(g_1, g_2)$ and $([f_1], [f_2]) = ([f_1'], [f_2]) = H(f_1', f_2) \in H(UN(g_1, g_2))$.

Corollary 2.11. *If $UN(g_1, g_2)$ is principal generated by (c_1, c_2) and g_1 is a cofibration then $UN(Hg_1, Hh_2)$ is principal generated by (Hc_1, Hc_2).*

Proposition 2.12. *Let $g_i \simeq q_i u_i : A \to B_i$, $i = 1, 2$, where q_i is a homotopy equivalence with homotopy inverse j_i. Then*[2])

$$(UN(Hu_1, Hu_2))\,(Hj_1, Hj_2) = UN(Hg_1, Hg_2).$$

We omit the proof but draw the immediate conclusion:

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[2]) The composition on the left of the equality if that of maps in $\mathfrak{C}_h \times \mathfrak{C}_h$.

Theorem 2.13.

(i) *If* $g_i : A \to B_i$, $i = 1, 2$, *in* $\mathfrak{T}$ *then* $([g_1), [g_2])$ *has a principal union in* $\mathfrak{T}_h$.

(ii) *If* $g_i : A_i \to B$, $i = 1, 2$, *in* $\mathfrak{T}$ *then* $([g_1], [g_2])$ *has a principal intersection in* $\mathfrak{T}_h$.

The proof follows the lines of that of Theorem 2.8, using the fact that (provided their ranges and domains are appropriate) pairs of maps in $\mathfrak{T}$ have principal unions and also have principal intersections. In fact, of course, pairs of maps in $\mathfrak{T}$ have *unions*, that is, essentially unique canonical generators for their union ideals. In this case we may, given g_1, g_2, construct canonical generators for $UN(Hg_1, Hg_2)$ by expressing g_i as $q_i u_i$ through the mapping cylinder functor, taking the union (v_1, v_2) of (u_1, u_2) in $\mathfrak{T}$ and setting $c_i = v_i j_i$, $i = 1, 2$, where j_i is homotopy inverse to q_i. Then $UN(Hg_1, Hg_2)$ is generated by $([c_1], [c_2])$. Similarly a canonical procedure may be applied to obtain a generator of the intersection ideal, $IN(Hg_1, Hg_2)$, of the homotopy classes of maps $g_i : A_i \to B$.

Before closing this section we exemplify the main ideas by considering categories other than $\mathfrak{T}_h$ to which the results apply. Let $\mathfrak{A}$ be an abelian category and let us declare (see [3]) two maps $g, g' : A \to B$ to be homotopic if $g - g'$ may be factored through an injective object of $\mathfrak{A}$. Then $\mathfrak{A}$ is thereby furnished with an h-structure. Moreover every monomorphism is a cofibration and (Proposition 2.3) if $\mathfrak{A}$ has sufficient injectives every cofibration is a monomorphism. Also if $\mathfrak{A}$ has sufficient injectives every map $g : A \to B$ may be factored as $g = qu$ where q is a homotopy equivalence and u is a monomorphism. Thus we infer

Theorem 2.14. *Let* $\mathfrak{A}$ *be an abelian category with sufficient injectives and let* $\mathfrak{A}_h$ *be the associated injective-homotopy category. Then in* $\mathfrak{A}_h$ *cokernels and unions are principal.*

Note that the set of morphisms $M_{\mathfrak{A}_h}(A, B)$ has the natural abelian group structure of the quotient of $M_{\mathfrak{A}}(A, B)$ by the subgroup of nullhomotopic maps; and, of course, composition is distributive over addition.

3. Equalizers and a Lemma of E. H. Brown

We now consider (right) equalizers in the h-category $\mathfrak{C}$ and its associated homotopy category $\mathfrak{C}_h$. We are again content to discuss a pair of maps $g_i : A \to B$, $i = 1, 2$, in $\mathfrak{C}$. Then their *right equalizer ideal*, $RE(g_1, g_2)$, is the family of maps f, of domain B, such that $fg_1 = fg_2$. It is clear what is meant by the statement that $RE(g, g_2)$ is *principal, generated by* c, and what is meant by a right-equalizer-preserving functor. We prove

Lemma 3.1. *Let $\mathfrak{C}$ be an I-category in which union ideals are principal. Then right equalizer ideals in $\mathfrak{C}$ are principal.*

Proof: Given $g_i : A \to B$ in $\mathfrak{C}$, let (c_1, c_2) generate $UN(g_1, g_2)$ and let (c, s) generate $UN(\langle 1, 1\rangle, \langle c_1, c_2\rangle)$, where $c_i : B \to U, \langle 1, 1\rangle : B * B \to B$, $\langle c_1, c_2\rangle : B * B \to U$. We assert that $c : B \to C$ generates $RE(g_1, g_2)$. First $\langle c, c\rangle = c\ \langle 1, 1\rangle = s\langle c_1, c_2\rangle = \langle sc_1, sc_2\rangle$, so $c = sc_1 = sc_2$, and $cg_1 = = sc_1 g_1 = sc_2 g_2 = cg_2$. Second, let $fg_1 = fg_2$, Then $f = f'c_1 = f'c_2$ so that $f\langle 1, 1\rangle = \langle f, f\rangle = \langle f'c_1, f'c_2\rangle = f'\langle c_1, c_2\rangle$. Thus $f = f''c$ (and $f' = f''s$) and the lemma is proved.

We note that Lemma 3.1 may also be proved by showing that if (t, c) generates $UN(\langle 1, 1\rangle, \langle g_1, g_2\rangle)$, then c generates $RE(g_1, g_2)$.

Corollary 3.2. *Right and left equalizer ideals in $\mathfrak{T}_h$ are principal.*

Proof: Apply Theorem 2.13.

Remark 3.3. (i) Theorem 2.8 is, of course, a special case of Corollary 3.2. Note however that Theorem 2.8, restricted to cokernels, generalizes to any $\mathfrak{C}_h$ in which $\mathfrak{C}$ has principal *cokernels* and maps in $\mathfrak{C}$ factorize as qu where q is a homotopy equivalence and u a cofibration. On the other hand Corollary 3.2, restricted to right equalizers, generalizes to any $\mathfrak{C}_h$ in which $\mathfrak{C}$ has principal *unions* and maps in $\mathfrak{C}$ factorize as qu.

(ii) We may, of course, construct a generator of $RE(Hg_1, Hg_2)$ in $\mathfrak{T}_h$ quite explicitly. Namely, we consider the union $(A \times I) \vee B$, where I is the unit interval, and construct C by way of the identifications $g_1 a = (a, 0), g_2 a = (a, 1)$. Then $c : B \to C$ is induced by the embedding of B in $(A \times I) \vee B$.

Before proceeding to discuss Brown's lemma, we record a further consequence of Lemma 3.1, for abelian categories.

Corollary 3.4. *Let $\mathfrak{A}_h$ be as in Theorem 2.14. Then in $\mathfrak{A}_h$ right equalizers are principal.*

In [2] Brown considers a contravariant functor $H : \mathfrak{C} \to \mathfrak{S}$ where $\mathfrak{C}$ is a category of based spaces and $\mathfrak{S}$ is the category of sets[3]); and he imposes certain axioms, including two axioms designated by h) and e), on the functor H. Interpreting H as a covariant functor $T : \mathfrak{C} \to \mathfrak{S}^*$, where $\mathfrak{C}$ is structured as an h-category in the usual way and $\mathfrak{S}^*$ is the category dual to $\mathfrak{S}$ furnished with the trivial h-structure given by the equality relation, it is easy to see that axioms h) and e) together assert that T is an h-functor with the further property that

$$T(UN(u_1, u_2)) = UN(Tu_1, Tu_2) \tag{3.5}$$

if u_1, u_2 are cofibrations.

[3]) Here we prefer to regard $\mathfrak{S}$, as we may, as the category of *based* sets.

We wish to fit Lemma 2.6 of [2] into the category-theoretic framework set up in this paper. First we prove a general result whose applicability to the situation described above is evident.

Let $T_h : \mathfrak{C}_h \to \mathfrak{D}_h$ be the I-functor induced by an h-functor $T : \mathfrak{C} \to \mathfrak{D}$ having property (3.5). Assume (i) that every map g in $\mathfrak{C}$ may be factored as $g = qu$, where q is a homotopy equivalence and u is a cofibration, and (ii) that the classifying functor $H : \mathfrak{D} \to \mathfrak{D}_h$ preserves unions. We prove

Theorem 3.6. *Under these hypotheses T_h preserves unions and right equalizers.*

Proof: Observe first that since T is an h-functor it transforms homotopy equivalences into homotopy equivalences. Now let $g_i : A \to B_i$ in $\mathfrak{C}$, $i = 1, 2$; factorize g_i as $q_i u_i$ and let j_i be homotopy inverse to q_i. Then

$$\begin{aligned}
T_h(UN(Hg_1, Hg_2)) &= T_h((UN(Hu_1, Hu_2))(Hj_1, Hj_2)) && \text{(Prop. 2.12)}\\
&= T_h(H(UN(u_1, u_2)))(T_hHj_1, T_hHj_2) && \text{(Prop. 2.9)}\\
&= H(T(UN(u_1, u_2)))(HTj_1, HTj_2), && \text{since } T \text{ is an } h\text{-functor}\\
&= H(UN(Tu_1, Tu_2))(HTj_1, HTj_2), && \text{by (3.5)}\\
&= UN(HTu_1, HTu_2)(HTj_1, HTj_2), && \text{by assumption (ii)}\\
&= UN(HTg_1, HTg_2) && \text{(Prop. 2.12)}\\
&= UN(T_hHg_1, T_hHg_2),
\end{aligned}$$

and thus T_h preserves unions.

To show that T_h preserves right equalizers, it is convenient to remark that, in any category $\mathfrak{C}$, $RE(g_1, g_2)$ is characterized as follows: $f \in RE(g_1, g_2)$ if and only if there exist maps f_1, f_2, s such that $(f_1, f_2) \in UN(g_1, g_2)$ and $(f, s) \in UN(\langle 1, 1\rangle, \langle f_1, f_2\rangle)$. We omit the proof of this assertion[4]).

Of course $T_h(RE(\alpha_1, \alpha_2)) \subset RE(T_h\alpha_1, T_h\alpha_2), \alpha_1, \alpha_2 : A \to B$ in $\mathfrak{C}_h$. To prove the opposite inclusion, let $\beta \in RE(T_h\alpha_1, T_h\alpha_2)$. Then there exist β_1, β_2, σ with $(\beta_1, \beta_2) \in UN(T_h\alpha_1, T_h\alpha_2)$ and $(\beta, \sigma) \in UN(\langle 1, 1\rangle, \langle \beta_1, \beta_2\rangle)$. Since T_h preserves unions, $\beta_i = T_h\gamma_i$, where $(\gamma_1, \gamma_2) \in UN(\alpha_1, \alpha_2)$. Then since T_h is an I-functor, $\langle 1, 1\rangle = T_h\langle 1, 1\rangle, \langle \beta_1, \beta_2\rangle = T_h\langle \gamma_1, \gamma_2\rangle$ and

$$(\beta, \sigma) \in UN(T_h\langle 1, 1\rangle, T_h\langle \gamma_1, \gamma_2\rangle).$$

Again, it follows that $\beta = T_h\gamma, \sigma = T_h\tau$, where $(\gamma, \tau) \in UN(\langle 1, 1\rangle, \langle \gamma_1, \gamma_2\rangle)$. Thus $\gamma \in RE(\alpha_1, \alpha_2)$, $T_h\gamma = \beta$, and the theorem is proved.

Remark 3.7. The second part of the proof simply shows that if an I-functor preserves unions it preserves right equalizers; this was already known if maps of the category have unions in the precise sense.

[4]) Cf. the proof of Lemma 3.1.

We now recapitulate rapidly the background to Lemma 2.6 of [2]; Brown's category $\mathfrak{C}$ is essentially our category $\mathfrak{T}$, so we discuss a covariant functor $H:\mathfrak{T}\to\mathfrak{S}$.

Let $f:A\to B$ be a map in $\mathfrak{T}$ and let Q be obtained by attaching the cone on A to B by means of f. Let $q:B\to Q$ be the embedding. Then there is a cooperation[5]) $r:Q\to Q\vee\Sigma A$ (where Σ denotes suspension), obtained by pinching the cone CA half-way up, and, as observed by Brown, an induced exact sequence of sets

$$\ldots\to H(\Sigma B)\xrightarrow{H(\Sigma f)} H(\Sigma A)\xrightarrow{\partial} H(Q)\xrightarrow{H(q)} H(B)\xrightarrow{H(f)} H(A).$$

Moreover $H(\Sigma B), H(\Sigma A)$ are groups and $H(\Sigma f)$ is a homomorphism. Also $H(\Sigma A)$ operates on $H(Q)$ through $H(r):H(Q)\times H(\Sigma A)=H(Q\vee\Sigma A)\to H(Q)$; and two elements $u,u'\in H(Q)$ satisfy $H(q)u=H(q)u'$ if and only if $u'=u\alpha$ for some α in $H(\Sigma A)$.

Let $v\in H(B)$ and $H(f)v=0$ so that $v=H(q)u$ for some $u\in H(Q)$, and let H_u be the subgroup of $H(\Sigma A)$ consisting of those $\alpha\in H(\Sigma A)$ such that $u\alpha=u$. Plainly if $H(\Sigma A)$ is abelian (e.g. if A is itself a suspension) then H_u depends only on v; we may then write H_v for H_u. Brown proves the following lemma.

Lemma 3.8. *If $H(\Sigma A)$ is abelian, then there is a space W and maps $h:B\to W$, $k:\Sigma A\to W$, depending only on f, such that $H_v=H(k)H(h)^{-1}(v)$ for all $v\in\ker H(f)$.*

This lemma is an obvious consequence of

Proposition 3.9. *There is a space W and maps $j:Q\to W, k:\Sigma A\to W$, depending only on f, such that $H_u=H(k)H(j)^{-1}(u)$ for $u\in H(Q)$.*

For we take $h=jq$. Then $H(h)^{-1}v=\bigcup_{H(q)u=v} H(j)^{-1}u$, so

$$\begin{aligned} H(k)H(h)^{-1}v &= \bigcup_{H(q)u=v} H(k)H(j)^{-1}u \\ &= \bigcup_{H(q)u=v} H_u, && \text{if Prop. 3.9 is assumed,} \\ &= Hv && \text{if } H(\Sigma A) \text{ is abelian.}\end{aligned}$$

Proof of Proposition 3.9. Let $i:Q\to Q\vee\Sigma A$ be the inclusion. Let $[c]:Q\vee\Sigma A\to W$ generate $RE([r],[i])$ in $\mathfrak{T}_h$ (Corollary 3.2). We set $c=\langle j,k\rangle$ and must show that $u\alpha=u$ if and only if there is $\beta\in H(W)$ with $H(j)\beta=u, H(k)\beta=\alpha$.

[5]) See [4, 9].

Now $H(r)(u, \alpha) = u\alpha$, $H(i)(u, \alpha) = u$ and, by Theorem 3.6 (recall that H, as a functor from $\mathfrak{T}$ to $\mathfrak{S}$, is contravariant and homotopy-invariant and that $H: \mathfrak{S} \to \mathfrak{S}_h$ is the identity functor), $LE(H(r), H(i))$ is generated by $H(c)$. Thus $u\alpha = u$ if and only if there is $\beta \in H(W)$ with $H(c)\beta = (u, \alpha)$, and $H(c)\beta = (H(j)\beta, H(k)\beta)$.

Remark 3.10. Proposition 3.9 clearly has a dual in the same category $\mathfrak{T}$.

4. Homotopy systems, cofibrations, unions

We have worked, in the previous two sections, with a very general – and emasculated – homotopy concept. In this section we relate this concept to the richer notion of a homotopy system (see [6]).

We recall that a (left) homotopy system S on a category $\mathfrak{C}$ consists of a *cylinder* I-functor $Z: \mathfrak{C} \to \mathfrak{C}$ together with certain natural transformations

$$t, b: I \to Z,$$
$$p: Z \to I,$$
$$r: Z \to Z,$$

such that[6]) $pt = pb = 1, rt = b, rb = t$. We may then introduce the notion of strict homotopy in $\mathfrak{C}$ by declaring that $g \sim g': A \to B$ if there exists $F: ZA \to B$ such that $Fb_A = g, Ft_A = g'$. The relation $\sim$ is then reflexive and symmetric. Let $\simeq_s$ be the equivalence relation generated by $\sim$. Then (Proposition 6.2 of [6])

Proposition 4.1. *If S is a left homotopy system on $\mathfrak{C}$ then $\mathfrak{C}$ acquires an h-structure through the relation $\simeq_s$.*

We suppose henceforth that $\mathfrak{C}$ is furnished with a fixed homotopy system S. We then call the h-structure of Prop. 4.1 the *canonical* h-structure on $\mathfrak{C}$.

Let $g: A \to B$ be a map in $\mathfrak{C}$. We say that g has the *HE-property* if (b_B, Zg) generates $UN(g, b_A)$. We justify this terminology by observing that g *has the HE-property if and only if it has the homotopy extension property in the following sense:* for all X and all $F: ZA \to X, h: B \to X$, with $hg = Fb_A$, there exists $F': ZB \to X$ with $F'b_B = h, F'Zg = F$.

Theorem 4.2. *If g has the HE-property then g is a cofibration with respect to the canonical h-structure on $\mathfrak{C}$.*

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[6]) I is the identity functor. We did not explicitly require the *reversal* r in [6] but it is convenient to have it here.

Proof: Let $f_0 : B \to X, h_1 : A \to X$ be maps such that $f_0 g \simeq_s h_1$; we seek $f_1 : B \to X$ with $f_0 \simeq_s f_1$ and $f_1 g = h_1$. Since $\sim$ is reflexive and symmetric we may suppose there are maps $u^0, u^1, \ldots, u^n : A \to X$ with $f_0 g = u^0 \sim \sim u^1 \sim \ldots \sim u^n = h_1$. Thus there exists $F : ZA \to X$, with $f_0 g = F b_A$ and $u^1 = F t_A$. Since g has the HE-property there exists $F' : ZB \to X$ with $F' b_B = f_0$, $F' Zg = F$. Set $F' t_B = v^1$. Then $f_0 \sim v^1$ and $v^1 g = u^1$. We repeat the argument with v^1, u^2 replacing f_0, u_1 to obtain $v^1 \sim v^2$ and $v^2 g = u^2$. Thus we obtain a sequence of maps $v^1, v^2, \ldots, v^n$ with $f_0 \sim v^1 \sim v^2 \sim \ldots \sim v^n$ and $v^n g = u^n = h_1$. Set $f_1 = v^n$. –

Now let $g_i : A \to B_i, i = 1, 2,$ in $\mathfrak{C}$ and let (c_1, c_2) be *the union* of (g_1, g_2) in the strict sense of [6]: that is to say, (c_1, c_2) generates the $UN(g_1, g_2)$ *and* $\langle c_1, c_2 \rangle : B_1 * B_2 \to C$ *is an epimorphism.* We then call

$$\begin{tikzcd} & B_1 \arrow[dr, "c_1"] & \\ A \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & & C \\ & B_2 \arrow[ur, "c_2"'] & \end{tikzcd} \tag{4.3}$$

a *union-diagram,* and prove

Theorem 4.4. *Let the homotopy system S be faithful*[7]) *and let g_1 in the union-diagram* (4.3) *have the HE-property. So then does c_2.*

Proof. Consider, in addition to (4.3), the diagrams

$$\begin{tikzcd} & B_1 \arrow[dr, "b_{B_1}"] & \\ A \arrow[ur, "g_1"] \arrow[dr, "b_A"'] & & ZB_1 \\ & ZA \arrow[ur, "Zg_1"'] & \end{tikzcd}$$

(4.5)

$$\begin{tikzcd} & C \arrow[dr, "b_C"] & \\ B_2 \arrow[ur, "c_2"] \arrow[dr, "b_{B_2}"'] & & ZC \\ & ZB_2 \arrow[ur, "Zc_2"'] & \end{tikzcd}$$

(4.6)

$$\begin{tikzcd} & ZB_1 \arrow[dr, "Zc_1"] & \\ ZA \arrow[ur, "Zg_1"] \arrow[dr, "Zg_2"'] & & ZC \\ & ZB_2 \arrow[ur, "Zc_2"'] & \end{tikzcd}$$

(4.7)

Then (4.7) is a union-diagram since S is faithful; in (4.5) (b_{B_1}, Zg_1) generates $UN(g_1, b_A)$; and we wish to prove, in (4.6), that (b_C, Zc_2) generates

[7]) See [7]; Z is union-preserving, in the strict sense of [6].

$UN(c_2, b_{B_2})$. Thus let $k: C \to X$, $l: ZB_2 \to X$ be maps with $kc_2 = lb_{B_2}$. Consider the maps $kc_1: B_1 \to X$, $lZg_2: ZA \to X$. Then

$$kc_1g_1 = kc_2g_2 = lb_{B_2}g_2 = lZg_2b_A.$$

Thus (since g_1 has the HE-property) there exists $u: ZB_1 \to X$ with $ub_{B_1} = kc_1$, $uZg_1 = lZg_2$. This last equality yields, from the union-diagram (4.7) a map $v: ZC \to X$ with $vZc_1 = u$, $vZc_2 = l$.

It remains to show that $vb_C = k$. Since (4.3) is a union-diagram, it suffices to show that $vb_Cc_1 = kc_1$, $vb_Cc_2 = kc_2$. Now

$$vb_Cc_1 = vZc_1b_{B_1} = ub_{B_1} = kc_1,$$

and

$$vb_Cc_2 = vZc_2b_{B_2} = lb_{B_2} = kc_2,$$

so the theorem is proved.

We remark that we have only used here the fact that (Zc_1, Zc_2) generates $UN(Zg_1, Zg_2)$. It would thus appear to be asking too much that Z preserve strict unions. But any I-functor preserving unions (in the sense of this paper) certainly preserves right-equalizers (see Remark 3.7) and hence zero-cokernel maps; thus it is asking little more that Z preserve strict unions and S be faithful.

Of course the usual homotopy system (and its adjoint system) in $\mathfrak{T}$ is faithful. The *right* homotopy system in the category of c.s.s. groups is also faithful and the dual of Theorem 4.4 naturally holds for that category.

5. The Mayer-Vietoris Theorem

We revert to the union-diagram (4.3) and prove

Proposition 5.1. *The rule $v \to vc_1$ sets up a one-to-one correspondence between the ideals coker c_2 and coker g_1 under which epimorphisms are mapped to epimorphisms.*

Proof: Certainly if $v \in$ coker c_2 then $vc_1 \in$ coker g_1. Set $P(v) = vc_1$ so $P: \text{coker}\, c_2 \to \text{coker}\, g_1$. Now let $u \in \text{coker}\, g_1$. Then $ug_1 = 0 = 0g_2$ so there is a unique v such that $vc_1 = u$, $vc_2 = 0$. Set $Q(u) = v$, so $Q: \text{coker}\, g_1 \to \text{coker}\, c_2$. It is plain that $PQ = 1, QP = 1$.

Now suppose v an epimorphism and let $fvc_1 = f'vc_1$. But $fvc_2 = f'vc_2 (= 0)$, so that $fv = f'v$, and, v being an epimorphism, $f = f'$. This shows that vc_1 is an epimorphism when v is.

Corollary 5.2. *The map g_1 has a strict cokernel if and only if the map c_2 has a strict cokernel. Moreover, if v is the cokernel of c_2 then vc_1 is the cokernel of g_1.*

Now let $\mathfrak{C}$ be an h-category and let $T_n : \mathfrak{C} \to \mathfrak{A}$ $(-\infty < n < \infty)$ be a sequence of functors from $\mathfrak{C}$ to the abelian category $\mathfrak{A}$ with the property that if

$$A \xrightarrow{g} B \xrightarrow{q} Q$$

is a cofibre-sequence there is a natural transformation

$$\omega = \omega_n(g, q) : T_n(Q) \to T_{n-1}(A)$$

such that

$$\ldots \to T_n(A) \xrightarrow{T_n(g)} T_n(B) \xrightarrow{T_n(q)} T_n(Q) \xrightarrow{\omega} T_{n-1}(A) \to \ldots$$

is an exact sequence. Then we infer from Corollary 5.2

Theorem 5.3. (Mayer-Vietoris sequence) *Suppose in* (4.3) *that* g_1 *and* c_2 *are cofibrations with strict cokernels. Then there is an exact sequence*

$$\ldots \to T_n(A) \xrightarrow{\alpha} T_n(B_1) \oplus T_n(B_2) \xrightarrow{\beta} T_n(C) \xrightarrow{\gamma} T_{n-1}(A) \to \ldots \tag{5.4}$$

where $\alpha = \{T_n(g_1), T_n(g_2)\}$, $\beta = \langle T_n(c_1), -T_n(c_2)\rangle$, *and* $\gamma = \omega_n(g_1, kc_1)T_n(k)$, $k : C \to Q$ *being the cofibre of* c_2.

Proof: We have a commutative diagram

$$\begin{array}{ccccccccc}
\ldots \to & T_n(A) & \xrightarrow{T_n(g_1)} & T_n(B_1) & \xrightarrow{T_n(kc_1)} & T_n(Q) & \xrightarrow{\omega} & T_{n-1}(A) & \to \ldots \\
& \downarrow{\scriptstyle T_n(g_2)} & & \downarrow{\scriptstyle T_n(c_1)} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle T_{n-1}(g_2)} & \\
\ldots \to & T_n(B_2) & \xrightarrow[T_n(c_2)]{} & T_n(C) & \xrightarrow[T_n(k)]{} & T_n(Q) & \xrightarrow[\omega]{} & T_{n-1}(B_2) & \to \ldots
\end{array}$$

Since there is an exact functor from $\mathfrak{A}$ to the category of abelian groups we may suppose that the functors T_n themselves take values in the category of abelian groups and Theorem 5.3 is then a consequence of the theorem of Barratt-Whitehead (see [1]).

We may readily combine Theorems 4.4 and 5.3 to obtain

Theorem 5.5. *Let* $\mathfrak{C}$ *be a category furnished with a faithful left homotopy system. Then if* g_1 *has the HE-property and possesses a strict cokernel, the sequence* (5.4) *is exact.*

Remark 5.6. The topological context in which Theorem 5.3 (or Theorem 5.5) becomes applicable is that in which the functors T_n are actually defined on the *maps* of the category $\mathfrak{C}$ and pass to the objects of $\mathfrak{C}$ through an embedding functor $\mathfrak{C} \to \mathfrak{C}^2$. Thus the assumptions made on the system of functors T_n would arise from hypothesizing the usual exact sequence of a cofibration. Systems of functors T_n satisfying this latter exact sequence hypothesis are studied in detail in [10]. The additional excision hypothesis is also satisfied by any homology theory in $\mathfrak{T}$, ordinary or extraordinary, so that the Mayer-Vietoris sequence is valid in any homology theory. If we replace $\mathfrak{A}$ by the dual category we obtain a Mayer-Vietoris sequence for any cohomology theory in $\mathfrak{T}$. Also we may pass to the dual category of $\mathfrak{T}$ and obtain a Mayer-Vietoris sequence for any homotopy theory in $\mathfrak{T}$. In more detail, this would relate to an *intersection-diagram*,

$$\begin{tikzcd} & B_1 \arrow[dr, "g_1"] & \\ C \arrow[ur, "c_1"] \arrow[dr, "c_2"'] & & A \\ & B_2 \arrow[ur, "g_2"'] & \end{tikzcd} \tag{5.7}$$

in $\mathfrak{T}$, in which g_1 is a fibration; the functors T_n are required to convert a fibre-sequence

$$Q \xrightarrow{q} B \xrightarrow{g} A$$

into an exact sequence

$$\ldots \to T_n(Q) \to T_n(B) \to T_n(A) \to T_{n-1}(Q) \to \ldots$$

We would then infer from Theorem 5.5 – and (5.7) – the exact sequence

$$\ldots \to T_n(C) \to T_n(B_1) \oplus T_n(B_2) \to T_n(A) \to T_{n-1}(C) \to \ldots, \tag{5.8}$$

and the hypotheses would apply to the homotopy groups $T_n = \Pi_n(X, \quad)$.

Remark 5.9. On the other hand Theorem 5.3 also applies in purely algebraic contexts in which it appears less natural to regard the functors T_n as defined on maps of the category. Indeed we would then take $\mathfrak{C}$ to be itself an abelian category and formally replace the notion of cofibration by monomorphism[8]). Thus the T_n are to be a connected sequence of functors and we infer, just as for Theorem 5.5

Theorem 5.10. *Let $\mathfrak{C}$ be an abelian category and let* (4.3) *be a union-diagram in $\mathfrak{C}$ in which g_1 is a monomorphism. Then, if T_n is a connected sequence of functors from $\mathfrak{C}$ to the abelian category $\mathfrak{A}$, there is an exact* (Mayer-Vietoris) *sequence*

$$\ldots \to T_n(A) \to T_n(B_1) \oplus T_n(B_2) \to T_n(C) \to T_{n-1}(A) \to \ldots$$

[8]) Recall that these notions in fact coincide if $\mathfrak{C}$ has sufficient injectives.

For it is only necessary to observe that if g_1 is a monomorphism and (4.3) is a union-diagram in an abelian category $\mathfrak{C}$, then c_2 is also a monomorphism. As examples we may take $T_n = Ext^{-n}(K, \ )$, $n \leq 0$, $T_n = 0$, $n > 0$; or, if $\mathfrak{C}$ has sufficient injectives, $T_n = Ext^{-n}(K, \ )$, $n < 0$, $T_n = \bar{\pi}_n(K, \ )$, the n^{th} injective homotopy group functor, $n \geq 0$. There are also the obvious duals.

Eidgenössische Technische Hochschule, Zürich
Cornell University, Ithaca N. Y.

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