KOEBE Arcs and FATOU Points of Normal Functions.

Autor(en): **Bagemihl, F. / Seidel, W.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **36 (1961-1962)**

PDF erstellt am: **25.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-515613

Koebe Arcs and Fatou Points of Normal Functions

by F. Bagemihl and W. Seidel[1]), Detroit, Michigan (USA)

Let C be the unit circle and D be the open unit disk in the complex z-plane, and C_w, D_w be the corresponding entities in the complex w-plane. The closure of a point set S will be denoted by $\overline{S}$, and the Lebesgue measure of a measurable set E by $m(E)$.

We begin by setting down some definitions.

Definition 1. Let A be an open arc of C, possibly C itself. A Koebe *sequence of arcs (relative to A)* is a sequence of Jordan arcs $\{J_n\}$ in D such that (a) for some sequence $\{\varepsilon_n\}$ satisfying the conditions $0 < \varepsilon_n < 1$ $(n = 1, 2, 3, \ldots)$ and $\varepsilon_n \to 0$ as $n \to \infty$, J_n lies in the ε_n-neighborhood of A $(n = 1, 2, 3, \ldots)$, and (b) every open sector Δ of D subtending an arc of C that lies strictly interior to A has the property that, for all values of n except at most a finite number, the arc J_n contains at least one Jordan subarc lying wholly in Δ except for its two end points which lie on distinct sides of Δ.

The terminology in Definition 1 is suggested by the appearance of such arcs in Koebe's lemma [2, p. 19].

Definition 2. *A strong* Koebe *sequence of arcs* is a Koebe sequence of arcs $\{J_n\}$ with the property that, to every $\zeta \in C$, there corresponds a rectilinear segment extending from ζ to a point of D, which is intersected by infinitely many of the arcs J_n $(n = 1, 2, 3, \ldots)$.

It is easily verified that a strong Koebe sequence of arcs is a Koebe sequence of arcs relative either to C itself or to C minus a single point of C.

Definition 3. If $f(z)$ is a meromorphic function in D and c is a constant, finite or ∞, we say that $f(z) \to c$ *along a* Koebe *sequence of arcs* $\{J_n\}$, provided that, for some sequence of positive numbers $\{\eta_n\}$, where $\eta_n \to 0$ as $n \to \infty$, we have, for every $z \in J_n (n = 1, 2, 3, \ldots)$, $|f(z) - c| < \eta_n$ or $|f(z)| > 1/\eta_n$, according as c is finite or infinite.

Definition 4. If $f(z)$ is a meromorphic function in D, we say that $f(z)$ *is bounded by M on a* Koebe *sequence of arcs* $\{J_n\}$, provided that there

---

[1]) W. Seidel's contribution to this paper was supported by National Science Foundation Grant NSF G-9663 held at the University of Notre Dame, Indiana, U.S.A.

exists a finite positive constant M such that $|f(z)| < M$ for every $z \in J_n$ $(n = 1, 2, 3, \ldots)$.

Definition 5. Let $z' = S(z)$ denote an arbitrary one-to-one conformal mapping of D onto itself. A function $f(z)$, meromorphic in D, is said to be *normal* in D [5, p. 53], if the family of functions $\{f(S(z))\}$ is normal in D in the sense of MONTEL, where convergence is defined in terms of the spherical metric.

Definition 6. A FATOU *point* of a meromorphic function in D is a point $\zeta \in C$ such that, for some complex number c (possibly ∞), as $z \to \zeta$ in any STOLZ angle at ζ, $f(z) \to c$; c is then called a FATOU *value* of $f(z)$.

We show first (Theorem 1) that a normal meromorphic function that tends to a constant along a KOEBE sequence of arcs is identically constant. This generalizes a result due to GROSS [4, pp. 35–36] as well as a result due to the present authors [1, Corollary 1, p. 266]. Next we prove (Theorem 2) that a normal holomorphic function that is bounded on a strong KOEBE sequence of arcs must be a bounded function. This generalizes [1, Corollary 2, p. 266]. (The two results in [1] alluded to involve "boundary paths" instead of KOEBE sequences of arcs.)

Theorem 3 asserts that if the set of FATOU points of a normal holomorphic function in D is of measure zero on an arc of C, then that arc contains an everywhere dense set of FATOU points of the function at each of which the corresponding FATOU value is ∞. This generalizes [1, Theorem 5, p. 267]. It follows immediately that the set of FATOU points of a normal holomorphic function in D is everywhere dense on C, which sharpens [1, Theorem 4, p. 267]. This result is to be contrasted with one given in [5, p. 58], according to which there exist normal *meromorphic* functions in D possessing no FATOU points. (*Cf.* also [1, Remark 4, p. 267].) Theorem 4 shows that a normal holomorphic function in D can have its set of FATOU points of arbitrarily small positive measure without having ∞ as a FATOU value. This leads us to pose the following problem, which we have not solved.

Problem. Let $f(z)$ be a normal holomorphic function in D. Suppose that an arc A of C exists such that the measure of the set of FATOU points of $f(z)$ on every subarc of A is less than the length of that subarc. Does A contain a FATOU point of $f(z)$ at which the corresponding FATOU value is ∞?

We proceed now to the proofs of our theorems.

Theorem 1. *Let $f(z)$ be a normal meromorphic function in D. If $f(z) \to c$ along a* KOEBE *sequence of arcs $\{J_n\}$, then $f(z) \equiv c$.*

Proof. We may assume that $c = 0$, for otherwise we can replace the normal meromorphic function $f(z)$ by the normal meromorphic function $f(z) - c$ if c is finite, or $1/f(z)$ if $c = \infty$.

Let the given sequence $\{J_n\}$ be a KOEBE sequence relative to the arc A (see Definition 1), and consider an arc $B = \{z : |z| = 1,\ q_1 < \operatorname{arc} z < q_2\}$ strictly interior to A. Denote by Δ the open sector of D with vertex at the origin and vertex angle β, subtending the arc B. The sides of Δ will be called s_1, s_2, where these segments terminate in e^{iq_1}, e^{iq_2}, respectively. In view of (b) in Definition 1, there is no loss of generality in asserting now that for every n the arc J_n contains a JORDAN subarc Γ_n lying wholly in Δ except for its endpoints $P_n^{(1)}$, $P_n^{(2)}$ which lie on s_1, s_2, respectively. It is obvious that $\{\Gamma_n\}$ is a KOEBE sequence of arcs relative to B.

Set

$$r_n = \min_{z \in \Gamma_n} |z|, \quad R_n = \max_{z \in \Gamma_n} |z| \qquad (n = 1, 2, 3, \ldots).$$

It follows from (a) in Definition 1 that

$$\lim_{n \to \infty} r_n = \lim_{n \to \infty} R_n = 1. \tag{1}$$

For $n = 1, 2, 3, \ldots$, we now define a JORDAN curve K_n. Let the circle $|z| = R_n$ intersect s_1 and s_2 in the respective points $Q_n^{(1)}$, $Q_n^{(2)}$, and denote the radial segments $P_n^{(1)} Q_n^{(1)}$, $P_n^{(2)} Q_n^{(2)}$ by $t_n^{(1)}$, $t_n^{(2)}$, respectively (these segments may reduce to single points). Then, if B_n is the open arc of the circle $|z| = R_n$ which lies in Δ and B_n^* is the complementary arc, we put

$$K_n = t_n^{(1)} \cup B_n^* \cup t_n^{(2)} \cup \Gamma_n .$$

The interior of K_n will be called Ω_n, and we set $G_n = \{z : |z| < R_n\}$.

CARLEMAN's Extension Principle for harmonic measure implies [7, p. 70] that

$$\omega(0, t_n^{(1)} \cup \Gamma_n \cup t_n^{(2)}, \Omega_n) \geqq \omega(0, B_n, G_n) = \frac{\beta}{2\pi}.$$

We have [7, p. 26]

$$\omega(0, t_n^{(1)} \cup \Gamma_n \cup t_n^{(2)}, \Omega_n) = \omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) + \omega(0, \Gamma_n, \Omega_n).$$

An inequality due to OSTROWSKI [3, p. 42] shows that

$$\omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) \leqq \frac{4}{\pi} \operatorname{arc\,sin} \frac{2\sqrt{\dfrac{R_n - r_n}{2} \cdot \dfrac{R_n + r_n}{2}}}{\dfrac{R_n - r_n}{2} + \dfrac{R_n + r_n}{2}} = \frac{4}{\pi} \operatorname{arc\,sin} \frac{\sqrt{R_n^2 - r_n^2}}{R_n},$$

and (1) implies that $\lim_{n\to\infty} \omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) = 0$. Hence

$$\liminf_{n\to\infty} \omega(0, \Gamma_n, \Omega_n) \geqq \frac{\beta}{2\pi}.$$

Consequently, if D_w is mapped conformally onto Ω_n by means of the function $z = \psi_n(w)$, where $\psi_n(0) = 0$ and the point $w = e^{i\alpha_1}$ corresponds to the point $z = P_n^{(1)}$, then each arc Γ_n, for n sufficiently large, is the image of an arc of C_w of length at least $\beta/2$ with its end point of smaller argument at $e^{i\alpha_1}$.

If we set

$$g_n(w) = f(\psi_n(w)) \qquad (n = 1, 2, 3, \ldots), \tag{2}$$

then [5, p. 57] $g_n(w)$ is a normal meromorphic function in D_w. Since $f(z)$ is normal in D, there exists [5, p. 56] a finite positive constant γ such that for every $z \in D$,

$$\frac{|f'(z)|}{1 + |f(z)|^2}(1 - |z|^2) \leqq \gamma. \tag{3}$$

Now from (2) we obtain

$$\frac{|g'_n(w)|}{1 + |g_n(w)|^2}(1 - |w|^2) = \frac{|f'(\psi_n(w))| \cdot |\psi'_n(w)|}{1 + |f(\psi_n(w))|^2}(1 - |w|^2). \tag{4}$$

According to [9, p. 133], if $D_1(z)$ denotes the radius of univalence at the point $z = \psi_n(w)$ of the region Ω_n, we have

$$(1 - |w|^2)\,|\psi'_n(w)| \leqq 4D_1(z), \tag{5}$$

and since Ω_n lies in D,

$$D_1(z) \leqq 1 - |z| \leqq 1 - |z|^2. \tag{6}$$

Combining (3) to (6), we find that

$$\frac{|g'_n(w)|}{1 + |g_n(w)|^2}(1 - |w|^2) \leqq \frac{4|f'(z)|}{1 + |f(z)|^2} \leqq 4\gamma. \tag{7}$$

Let S denote the subarc of C_w whose end point of smaller argument is $e^{i\alpha_1}$ and whose length is $\beta/2$. The hypothesis that $f(z) \to 0$ along the KOEBE sequence $\{J_n\}$ implies that $\lim_{n\to\infty} g_n(w) = 0$ uniformly on S. This together with (7) shows, in view of [5, p. 64], that the sequence $\{g_n(w)\}$ tends uniformly to zero on every compact subset of D_w.

We shall now show that $f(z) \equiv 0$. Suppose that, on the contrary, $f(z_0) \neq 0$ for some $z_0 \in D$. By (a) in Definition 1, $z_0 \in \Omega_n$ for all sufficiently large values of n. Let $w = \varphi_n(z)$ be the inverse of the function $z = \psi_n(w)$. Then, according to (2),

$$g_n(\varphi_n(z_0)) = f(z_0)$$

for all sufficiently large values of n. Since $\{g_n(w)\}$ tends uniformly to zero on every compact subset of D_w, but $f(z_0) \neq 0$, we must have $\lim_{n \to \infty} |\varphi_n(z_0)| = 1$. But this is impossible; for if we fix ϱ so that $|z_0| < \varrho < 1$, then SCHWARZ's lemma yields

$$|\varphi_n(z_0)| \leqq \frac{|z_0|}{\varrho} < 1$$

for all sufficiently large values of n. Our supposition has thus led to a contradiction, and the theorem is proved.

Theorem 2. *Let $f(z)$ be a normal holomorphic function in D. If $f(z)$ is bounded by M on a strong* KOEBE *sequence of arcs $\{J_n\}$, then $f(z)$ is bounded by M throughout D.*

Proof. If $f(z)$ is bounded in D, then Definition 2 implies that none of its radial limits, except perhaps one, is greater than M in modulus, and the representation of $f(z)$ by its POISSON integral shows immediately that $|f(z)| < M$ throughout D.

We shall now suppose that $f(z)$ is unbounded in D, and show that this leads to a contradiction of the hypothesis that $\{J_n\}$ is a strong KOEBE sequence. The set of all points $z \in D$ at which $|f(z)| > M + 1$ is open and not empty; let R_1 be some component of this set. At all boundary points of R_1 that lie in D, we have $|f(z)| = M + 1$, and the maximum principle implies that R_1 cannot lie wholly in some disk $|z| < \varrho < 1$. Hence, the boundary of R_1 contains at least one point of C. The region R_1 cannot have more than one accessible boundary point on C, for if it had two such points ζ_1 and ζ_2, they could be connected by a JORDAN arc Γ lying, except for its end points ζ_1 and ζ_2, in R_1, and Γ would decompose D into two subregions. But R_1, and hence Γ, meets none of the arcs J_n $(n = 1, 2, 3, \ldots)$, and therefore infinitely many of these arcs would have to lie in one of the two subregions of D, contradicting the remark following Definition 2 and (b) in Definition 1.

We now map D_w conformally onto the universal covering surface R_1^* of R_1 by means of the single-valued function $z = \varphi(w)$, and set

$$g(w) = f(\varphi(w))$$

in D_w. We have $|\varphi(w)| < 1$ in D_w. The FATOU values of $\varphi(w)$ are of modulus 1 on at most a subset of measure zero of C_w; this follows from the RIESZ uniqueness theorem [7, p. 209] and the fact that R_1 has at most one accessible boundary point on C. Since R_1^* is unbranched over R_1, almost all the FATOU values of $\varphi(w)$ are points in D that lie on the boundary of R_1. Hence, $g(w)$ possesses limits of modulus $M + 1$ along almost all radii of C_w. It follows that $f(z)$ is unbounded in R_1, because otherwise we should have $|g(w)| < M + 1$ throughout D_w, contradicting the definition of R_1.

The set of all points $z \in R_1$, at which $|f(z)| > M + 2$ is open and not empty; let R_2 be some component of this set. Then $R_2 \subset R_1$, and if we apply to R_2 the foregoing argument for R_1, we arrive at the conclusion that $f(z)$ is unbounded in R_2. Proceeding in this manner, we obtain a sequence of nested regions

$$R_1 \supset R_2 \supset R_3 \supset \dots$$

such that, for $n = 1, 2, 3, \dots$,

$$|f(z)| > M + n \qquad (z \in R_n). \tag{8}$$

Now take

$$z_1 \in R_1,\ z_2 \in R_2 - \{z_1\},\ z_3 \in R_3 - \{z_1, z_2\}, \dots, z_n \in R_n - \{z_1, z_2, \dots, z_{n-1}\}, \dots,$$

and join z_1 to z_2 by means of a JORDAN arc K_1 lying in R_1, join z_2 to z_3 by means of a JORDAN arc K_2 lying in R_2 and having no point except z_2 in common with $K_1, \dots,$ join z_n to z_{n+1} by means of a JORDAN arc K_n lying in R_n and having no point except z_n in common with $K_1 \cup K_2 \cup \dots \cup K_{n-1}, \dots$. We thus obtain a path

$$P = \bigcup_{n=1}^{\infty} K_n$$

in D. Its initial point is z_1, and its "end" lies on C because, due to (8) and the fact that $K_n \subset R_n$ $(n = 1, 2, 3, \dots)$,

$$\lim_{n \to \infty} \min_{z \in K_n} |f(z)| = \infty,$$

and $f(z)$, by hypothesis, is holomorphic in D. The path P then is a "boundary path" in D along which $f(z) \to \infty$. According to [1, Corollary 1, p. 266], the end of P is a single point $\zeta \in C$. Since $f(z)$ is normal in D, ζ is a FATOU point of $f(z)$ with ∞ as the corresponding FATOU value [5, p. 53]. But, in view of Definition 2, this contradicts the hypothesis that $\{J_n\}$ is a strong KOEBE sequence, because $f(z)$ is bounded on $\{J_n\}$; and the theorem is proved.

Theorem 3. *Let $f(z)$ be a normal holomorphic function in D and A be an open subarc of C. If the set of* FATOU *points of $f(z)$ on A is of measure zero, then A contains a* FATOU *point of $f(z)$ at which the corresponding* FATOU *value is ∞.*

Proof. Take a point $\zeta \in A$. The function $f(z)$ cannot be bounded in any neighborhood of ζ, because otherwise, by a simple extension of FATOU's theorem, the set of FATOU points of $f(z)$ on A would be of positive measure, contrary to hypothesis. Hence, there exists a number $\delta > 0$ such that the region $H = D \frown \{z : |z - \zeta| < \delta\}$ satisfies the conditions that $\overline{H} \frown C \subset A$ and $f(z)$ is unbounded in H. Consequently there exists a sequence of points $\{z_n\}$ in D such that $z_n \to \zeta$ and $M_n = |f(z_n)| \to \infty$ as $n \to \infty$, where $1 < M_1 < M_2 < \cdots < M_n < \cdots$. For $n = 1, 2, 3, \ldots$, let V_n be the open set of all points of D at which $|f(z)| > M_n - 1$, and denote by R_n that component of V_n which contains the point z_n. Evidently $|f(z)| = M_n - 1$ *at all boundary points of R_n that lie in D.* The maximum principle implies that $\overline{R}_n \frown C$ is not empty. As $n \to \infty$, the diameter of R_n tends to zero. For if $r_n = \min_{z \in \overline{R}_n} |z|$, the hypothesis that $f(z)$ is holomorphic in D implies that $\lim_{n \to \infty} r_n = 1$, so that if the diameter of R_n did not tend to zero as $n \to \infty$, one could obtain a KOEBE sequence of arcs along which $f(z) \to \infty$, which is impossible in view of Theorem 1. Thus there exists a natural number N such that $R_N \subset H$, and we set $G_1 = R_N$.

We shall show that $f(z)$ is unbounded in G_1. Let G_1^* be the smallest simply connected region containing G_1, and $z = \varphi(w)$ be a function that maps D_w conformally onto G_1^*. The set $B^* = \overline{G_1^*} \frown C$ is not empty; we denote by B_1^* the set of all points of B^* that are accessible from the region G_1^*. According to FATOU's theorem, $\varphi(w)$ has a radial limit at almost all points of C_w; we put

$$\varphi^*(e^{i\mu}) = \lim_{r \to 1} \varphi(re^{i\mu})$$

for every μ for which the limit exists. The set

$$E_1 = \{e^{i\mu} : |\varphi^*(e^{i\mu})| = 1\}$$

is a BOREL set, and is therefore measurable, and we have

$$B_1^* = \{\varphi^*(e^{i\mu}) : e^{i\mu} \in E_1\}.$$

Consider the function

$$g(w) = f(\varphi(w))$$

in D_w. We are going to show that $g(w)$ is unbounded in D_w. Assume that $g(w)$ is bounded in D_w. We have either $m(E_1) > 0$ or $m(E_1) = 0$.

Suppose first that $m(E_1) > 0$. Let E_0 be the BOREL subset of positive measure of E_1 at each point of which $g(w)$ possesses a radial limit, and B_0^* be the image of E_0 under the mapping $z = \varphi(w)$. An application of an extension of LÖWNER's theorem [10, p. 322] shows that B_0^* is a measurable subset of B_1^* with $m(B_0^*) > 0$. Let $\zeta_0 \in B_0^*$. Then there is a path in G_1^* terminating in ζ_0, and this path is the image, under the mapping $z = \varphi(w)$, of a path in D_w that terminates in a point $e^{i\mu_0} \in E_0$. Now $\varphi^*(e^{i\mu_0}) = \zeta_0$, and $g(w)$ has a radial limit at the point $e^{i\mu_0}$; therefore $f(z)$ tends to a limit along a path in G_1^* terminating in ζ_0. By hypothesis, $f(z)$ is normal in D, and consequently [5, p. 53] ζ_0 is a FATOU point of $f(z)$. Since ζ_0 was an arbitrary point of B_0^*, and $m(B_0^*) > 0$, we have arrived at a contradiction of the hypothesis that the set of FATOU points of $f(z)$ on A is of measure zero.

Suppose next that $m(E_1) = 0$. Since every boundary point of G_1^* is a boundary point of G_1, the italicized remark in the first paragraph of the proof implies that the FATOU values of $g(w)$ are equal to $M_N - 1$ in modulus almost everywhere on C_w. The representation of $g(w)$ by its POISSON integral shows that $|g(w)| \leqq M_N - 1$ throughout D_w, which implies that $|f(z)| \leqq M_N - 1 = L$ throughout $G_1 = R_N$, contrary to the definition of R_N.

Thus $g(w)$ is unbounded in D_w, which implies that $f(z)$ is unbounded in G_1^* and hence in G_1. It follows that the open set of all points of G_1 at which $|f(z)| > L + 1$ is not empty, and letting G_2 denote a component of this set, we conclude as above that $f(z)$ is unbounded in G_2. Continuing in this manner, we obtain a sequence of nested subregions $G_1 \supset G_2 \supset G_3 \supset \ldots$ of H, and now an argument employed in the proof of Theorem 2 enables us to infer the existence of a FATOU point of $f(z)$ on A at which the corresponding FATOU value is ∞, thus completing the proof of the theorem.

Corollary 1. *The set of* FATOU *points of a normal holomorphic function in* D *is everywhere dense on* C.

Theorem 4. *Given* $\varepsilon > 0$, *there exists a normal holomorphic function* $f(z)$ *in* D *whose set of* FATOU *points is of measure less than* ε *but for which* ∞ *is not a radial limit.*

Proof. Consider first the function $\varphi(w) = g(w) + h(w)$ in D_w, where $g(w)$ is the elliptic modular function, holomorphic and normal in D_w, whose set of FATOU points E is enumerable and whose FATOU values are $0, 1, \infty$, and $h(w)$ is bounded and holomorphic in D_w and possesses a radial limit at every point of $C_w - E$ but no radial limit at any point of E [6, Theorem 6,

p. 14]. Now $\varphi(w)$ is holomorphic and normal in D_w [5, p. 53]; its set E_0 of FATOU points is enumerable, and ∞ is its only FATOU value.

Choose a positive number δ so small that, if $\varrho = \cos\frac{\delta}{2}$, then

$$\frac{L}{\left|\log\frac{\delta}{\varrho}\right| + 1} < \varepsilon, \tag{9}$$

where L is a certain positive absolute constant to be specified later. Let P be a perfect nowhere dense set on C_w that contains no point of E_0 and for which $m(P) > 2\pi - \delta$, and set $H = C_w - P$. Denote by R the simply connected subregion of D_w whose boundary consists of the points of P and the open chords of C_w that subtend the components of the open set H. The boundary of R is evidently a rectifiable JORDAN curve of length less than 2π. Since each component of H is of length less than δ, the region R contains the disk $|w| < \varrho$. Let the function $w = \lambda(z)$ map D conformally onto R so that $\lambda(0) = 0$, and let S be the set of all points on C that correspond under this mapping to points on the chords of C_w subtending components of H. Since the sum of the lengths of these chords is less than δ, we have, by a theorem of LAVRENTIEV [8, p. 125],

$$m(S) < \frac{L}{\left|\log\frac{\delta}{\varrho}\right| + 1}. \tag{10}$$

Now consider the function $f(z) = \varphi(\lambda(z))$ in D. It is holomorphic and normal in D [5, p. 57], does not have ∞ as a FATOU value, and its set of FATOU points is S. According to (9) and (10), $m(S) < \varepsilon$, and this completes the proof of the theorem.

REFERENCES

[1] F. BAGEMIHL and W. SEIDEL, *Behavior of meromorphic functions on boundary paths, with applications to normal functions.* Arch. Math. 11 (1960), 263–269.

[2] L. BIEBERBACH, *Lehrbuch der Funktionentheorie.* Bd. II, 2. Aufl., Leipzig und Berlin, 1931.

[3] C. GATTEGNO et A. OSTROWSKI, *Représentation conforme à la frontière; domaines généraux.* Paris, 1949.

[4] W. GROSS, *Über die Singularitäten analytischer Funktionen.* Monatsh. Math. Phys. 29 (1918), 3–47.

[5] O. LEHTO and K. I. VIRTANEN, *Boundary behaviour and normal meromorphic functions.* Acta Math. 97 (1957), 47–65.

[6] A. J. LOHWATER and G. PIRANIAN, *The boundary behavior of functions analytic in a disk.* Ann. Acad. Sci. Fenn. A I 239 (1957), 1–17.

[7] R. NEVANLINNA, *Eindeutige analytische Funktionen.* 2. Aufl., Berlin, Göttingen und Heidelberg, 1953.

[8] I. I. PRIWALOW, *Randeigenschaften analytischer Funktionen.* Berlin, 1956.
[9] W. SEIDEL and J. L. WALSH, *On the derivatives of functions analytic in the unit circle and their radii of univalence and of p-valence.* Trans. Amer. Math. Soc. 52 (1942), 128–216.
[10] M. TSUJI, *Potential theory in modern function theory.* Tokyo, 1959.

(Received October 10, 1960)