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Groups of Transformations of Kaehler and almost Kaehler Manifolds

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In his Princeton lectures delivered in 1956, Lichnerowicz proved that the largest connected Lie group $C_0(E^{2n})$ of conformal transformations of a compact Einstein-Kaehler space $E^{2n}\ (n > 1)$ of positive Ricci curvature leaves the Kaehlerian structure invariant. This result is of particular interest in the study of homogeneous spaces, for, every compact homogeneous Kaehlerian space admits if the group of conformal transformations is semi-simple an invariant Einstein-Kaehler metric. In a subsequent paper, this result was extended to Kaehler manifolds[2]), in general, by employing an integral formula giving a characterization of an infinitesimal analytic transformation in terms of the Ricci curvature. Indeed, if $M^{2n}\ (n > 1)$ is a compact Kaehler manifold the largest connected Lie group $C_0(M^{2n})$ of conformal transformations coincides with the largest connected group $A_0(M^{2n})$ of automorphisms of the Kaehler structure. For $n = 1$ it coincides with the largest connected group of analytic homeomorphisms.

Now, if M is a compact almost Kaehlerian manifold, it is known that the largest connected group of affine transformations with respect to the Riemannian connection coincides with the largest connected group $A_0(M)$ of automorphisms of the almost Kaehlerian structure. It was recently shown that $C_0(M) = A_0(M)$ in the case where $\dim M = 4k$, so that for these dimensions a conformal map is an affine transformation and conversely [4]. It is the main purpose of this paper to extend this result so that it holds for all dimensions. We shall prove

Theorem 1. *The largest connected Lie group of conformal transformations of a compact almost Kaehler manifold $M^{2n}(n > 1)$ coincides with the largest connected group of automorphisms of the almost Kaehler structure, that is* $C_0(M^{2n}) = A_0(M^{2n})$.

If the almost complex structure is completely integrable and comes from a complex analytic structure we obtain as an immediate consequence the

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[2]) The manifolds, differential forms and tensorfields considered are assumed to be of class C^∞.

theorem of LICHNEROWICZ [6]. The proof given in [6] was recently extended to include the almost KAEHLER manifolds as well [9]. Indeed, an integral formula is established relating a certain tensorfield (whose vanishing, in the KAEHLERian case, gives a necessary and sufficient condition for an infinitesimal transformation to be analytic) with the RICCI curvature of the manifold. The derivation of this formula is rather complicated as it involves several lengthy non-trivial computations.

Our method of proof does not differ essentially from that given in a previous paper [4] for the special case previously mentioned, that is, it stems from the same general formula (cf. equation 4 below) upon which the results of this paper depend. For the sake of completeness and because of its intrinsic interest we shall reproduce this result thereby giving an alternate proof for the dimensions $4k$.

In § 8, Theorem 1 is generalized to include those compact orientable RIEMANNian manifolds carrying an harmonic form of a given degree whose coefficients satisfy a certain relation.

2. In the noncompact case it is shown that an infinitesimal analytic conformal map of a pseudo-KAEHLER manifold is a homothetic map, and hence also an affine transformation corresponding to the RIEMANNian connection (cf. Theorem 5). We shall also consider infinitesimal maps whose covariant forms are closed. Indeed, let X be a vector field on a pseudo-KAEHLERian manifold whose image by the almost complex structure operator J is an infinitesimal conformal map preserving the pseudo-KAEHLERian structure. The vector field JX is then "closed," that is its covariant form (by the duality defined by the metric) is closed. We shall show that, in general, a "closed conformal map" is a homothetic transformation. In fact, the following theorem is proved:

Theorem 2. *If M^{2n} $(n > 1)$ is a complete pseudo-KAEHLERian manifold which is not locally flat a closed infinitesimal conformal map is an automorphism of the pseudo-KAEHLERian structure.*

In the locally flat case we may prove

Theorem 3. *Let M^{2n} $(n > 1)$ be a complete locally flat pseudo-KAEHLERian manifold. Then, a closed infinitesimal conformal transformation of bounded length is an automorphism of the pseudo-KAEHLERian structure.*

Remark. The linear space A of all closed infinitesimal conformal maps of a compact KAEHLER manifold M^{2n} is an abelian subalgebra of the LIE algebra L of infinitesimal conformal transformations of M^{2n} $(n > 1)$. An alternate way of expressing this is to say that any two elements of A are in involution. If the first betti number vanishes $A = (0)$. To see this, we observe first that the image by J of an element X of A is isometric (cf. proofs of Theorems 2 and 3). Hence, $\delta C \xi = 0$, where ξ is the covariant form of X and C the complex structure operator applied to forms. By Theorems 2 and 3, $X \in A_0(M)$, that is $dC\xi$ vanishes. It follows that $C\xi$ is harmonic and therefore since M is a KAEHLER manifold ξ is also harmonic. The condition $\delta \xi = 0$ implies that X is an infinitesimal isometry. But a harmonic vector field which is at the same time a KILLING field must be a parallel field of vectors. For $X, Y \in A$, $X = \xi^i \frac{\partial}{\partial x^i}$, $Y = \eta^i \frac{\partial}{\partial x^i}$

$$\begin{aligned}[X, Y]^i &= \xi^k \frac{\partial \eta^i}{\partial x^k} - \eta^k \frac{\partial \xi^i}{\partial x^k} \\ &= \xi^k D_k \eta^i - \eta^k D_k \xi^i \\ &= 0\end{aligned}$$

where D_k denotes the covariant derivative with respect to the canonical connection.

(The summation convention is used throughout.)

If we consider the subalgebra of closed bounded conformal maps the compactness condition may be replaced by completeness.

3. A real manifold M^{2n} of even dimension $2n$ is said to be *almost complex* if there is a linear transformation J defined on the tangent space at each point which varies differentiably with respect to local coordinates and whose square is minus the identity, that is if there is a real tensorfield F^i_j on M^{2n} satisfying

$$F^i_j F^j_k = -\delta^i_k, \qquad (i, j, k = 1, \ldots, 2n).$$

In a coordinate neighborhood of an even dimensional real manifold with coordinates $x^1, \ldots, x^{2n}$ complex coordinates may be introduced by setting $z^j = x^j + i x^{j+n}$, $j = 1, \ldots, n$. The almost complex structure given by J is called *completely integrable* if the manifold can be made into a complex manifold with local coordinates $z^1, \ldots, z^n$, so that operating with J is equivalent to transforming dz^j and $d\bar{z}^j$ into $i dz^j$ and $-i d\bar{z}^j$. In this case multiplication by i in the tangent space has an invariant meaning.

Consider a manifold M^{2n} admitting a 2-form

$$\omega = F_{ij} dx^i \wedge dx^j$$

of rank $2n$ everywhere. If ω is closed the manifold is said to be *symplectic*. Let g be a metric on M^{2n} with the property

$$F_{ik} F_{jl} g^{kl} = g_{ij} \qquad (i, j, k, l = 1, \ldots, 2n).$$

Such a metric always exists. The operator J acting in the tangent space at each point

$$J : \xi^k \rightarrow (i(X)\,\omega)^k$$

(where $i(X)$ is the interior product by X operator) defines on M^{2n} an almost complex structure and together with g an *almost HERMITian* structure. If the manifold is symplectic with respect to the fundamental form ω the almost HERMITian structure is called *almost KAEHLERian*. In this case, M^{2n} is said to be an *almost KAEHLER manifold*. In such a manifold the fundamental 2-form ω is both closed and co-closed, that is harmonic. If J defines a completely integrable almost complex structure M^{2n} is said to be *pseudo-KAEHLERian*. If the almost complex structure comes from a complex structure M^{2n} is said to be a *KAEHLER manifold*. A KAEHLER manifold is thus an HERMITian manifold which is symplectic for the fundamental 2-form of the HERMITian structure.

4. A transformation f of a complex manifold is said to be *analytic* if it preserves the complex analytic structure. The corresponding almost complex structure J is therefore invariant by f. If two complex analytic structures induce the same almost complex structure they must coincide. Hence, in order that f be analytic it is necessary and sufficient that J be preserved. If the manifold is compact it is well known that the largest group of analytic transformations is a complex LIE group [2].

Let G denote a connected LIE group of analytic transformations of the complex manifold M. To each element A of the LIE algebra of G is associated the 1-parameter subgroup a_t of G generated by A. The corresponding 1-parameter group of transformations R_{a_t} on M $(R_{a_t} \cdot P = P \cdot a_t,\ P \in M)$ induces a (right invariant) vector field X on M. From the action on the almost complex structure J it follows that $\theta(X)J$ vanishes where $\theta(X)$ is the LIE derivative operator applied to J. An *infinitesimal analytic transformation* is an infinitesimal transformation defined by a vector field X satisfying $\theta(X)J = 0$. On the other hand, a vector field on M satisfying this equation generates a local 1-parameter group of local transformations of M. In order that a connected LIE group G of transformations be a group of analytic transformations it is necessary and sufficient that the vector fields on M induced by the 1-parameter subgroups of G define infinitesimal analytic transformations. If M is complete every infinitesimal analytic transformation generates a 1-parameter global group of analytic transformations of M.

Consider a RIEMANNian manifold M with metric g. By a *conformal transformation* of M is meant a differentiable homeomorphism f of M onto itself with the property that

$$f^* g = a^2 g$$

where f^* is the induced map in the bundle of frames over M and a is a real differentiable function on M. If a is a constant f is said to be *homothetic*. If the metric is preserved $(a = 1)$ f is an *isometry*. The group of all the isometries of M is a LIE group with respect to the natural topology. Let G denote a connected LIE group of conformal transformations of M and L its LIE algebra. To each element A of L is associated the 1-parameter subgroup a_t of G generated by A. The corresponding 1-parameter group of transformations of M induces a differentiable vector field X on M. From the action on the metric tensor g we conclude that

$$\theta(X)g = \lambda g$$

where λ is a real differentiable function depending on X. A vector field satisfying this equation is called an *infinitesimal conformal transformation*. If M is complete every solution of $\theta(X)g = \lambda g$ generates a 1-parameter global group of conformal transformations of M.

In terms of a system of local coordinates the (symmetric) tensor $\theta(X)g$ has the components

$$(\theta(X)g)_{ij} = D_j\xi_i + D_i\xi_j \, .$$

Hence, if $\dim M = m$ an infinitesimal conformal map is a solution of the equation

$$\theta(X)g + \frac{2}{m}\delta\xi\cdot g = 0 \, .$$

5. Let α and β be any two p-forms on a compact and orientable RIEMANNian manifold M^m. Then, for any vector field X on M^m it follows from STOKES' theorem and the identity

$$\theta(X) = di(X) + i(X)d$$

that

$$\int_{M^m} \theta(X)(\alpha \wedge *\beta) = \int_{M^m} di(X)(\alpha \wedge *\beta) = 0$$

where $\theta(X), i(X)$ and d denote the operations of LIE derivation, contraction (interior product) and exterior derivation, resp. and $*$ denotes the duality (star) operator of HODGE. We employ the notation $(\ ,\ )$ for the global scalar product

$$(\alpha, \beta) = \int_{M^m} \alpha \wedge *\beta \, .$$

Since $\theta(X)$ is a derivation

$$(\theta(X)\alpha, \beta) = -\int_{M^m} \alpha \wedge \theta(X)*\beta \, .$$

If, therefore, we put

$$*\bar{\theta}(X)\beta = -\,\theta(X)*\beta \, ,$$

that is

$$\overline{\theta}(X) = (-1)^{np+p+1} * \theta(X) * , \tag{1}$$

we have

$$(\theta(X)\alpha, \beta) = (\alpha, \overline{\theta}(X)\beta) . \tag{2}$$

It follows that the operator $\overline{\theta}(X)$ is the dual of $\theta(X)$. One therefore obtains

$$\overline{\theta}(X) = \delta\varepsilon(\xi) + \varepsilon(\xi)\delta \tag{3}$$

where $\xi = X_i dx^i$ is the covariant form for X and $\varepsilon(\xi)$ is the dual of $i(X)$:

$$i(X) = (-1)^{n(p-1)} * \varepsilon(\xi) * ,$$

that is

$$\varepsilon(\xi)\alpha = \xi \wedge \alpha$$

for any p-form α. The operators $\theta(X)$ and d commute and, clearly, so do their duals as one may also infer from (3). Moreover, if g denotes the metric tensor of M^m

$$(\theta(X) + \overline{\theta}(X))\alpha = \alpha \cdot \delta\xi + \sum_{r=1}^{p} g^{jk}(\theta(X)g)_{ki_r} \cdot \cdot \alpha_{i_1, \ldots i_{r-1} j i_{r+1} \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} . \tag{4}$$

Lemma 1. *The harmonic forms on a compact and orientable* Riemann*ian manifold M are invariant under the* Lie *algebra K of the largest connected group of isometries of M.*

The proof depends on the fact that $\theta(X) + \overline{\theta}(X), X \in K$ annihilates harmonic forms [3]. Since X is an infinitesimal motion, $\theta(X)g = 0$ from which it follows that $\delta\xi$ vanishes also. If α is a harmonic form $d\theta(X)\alpha = \theta(X)d\alpha = 0$ and $\delta\theta(X)\alpha = -\delta\overline{\theta}(X)\alpha = -\overline{\theta}(X)\delta\alpha = 0$. Hence $\theta(X)\alpha$ is a harmonic form. But $\theta(X)\alpha = di(X)\alpha$ from which by the Hodge decomposition of a form $\theta(X)\alpha$ vanishes.

Lemma 2. *The harmonic forms of degree $p = \frac{m}{2}$ of a compact, orientable* Riemann*ian manifold M of even dimension m are invariant under the* Lie *algebra L of the largest connected group of conformal transformations of M.*

Indeed, let X be an element of L. Then,

$$\theta(X)g = -\frac{2}{m}\delta\xi \cdot g \tag{5}$$

and, from (4)

$$\theta(X)\alpha + \overline{\theta}(X)\alpha = \left(1 - \frac{2p}{m}\right)\delta\xi \cdot \alpha .$$

Lemma 3. *If* $\dim M = 2$, *the inner product of a harmonic vector field and a vector field defining an infinitesimal conformal transformation is a constant on M.*

This is clearly the case if M is a RIEMANN surface.

6. In a compact almost KAEHLER manifold M the fundamental 2-form ω which is canonically defined by the almost HERMITian metric is closed and co-closed. The LAPLACE-BELTRAMI operator $\Delta = d\delta + \delta d$ and the operator $L : \alpha \to \alpha \wedge \omega$ (α: a p-form) do not commute in general. In fact,

$$L\Delta - \Delta L = dCdC + CdCd .$$

However, since $C\omega = \omega$, $\Delta L\omega = L\Delta\omega = 0$ from which we may conclude that ω^k is a harmonic $2k$-form. Hence, if $\dim M = 4k$ and X is an infinitesimal conformal map it follows from Lemma 2 and compactness that $\theta(X)\omega^k$ vanishes. Now, $\theta(X)\omega^k = k\theta(X)\omega \wedge \omega^{k-1}$, and so since the map L defines an isomorphism between the spaces $\Lambda^p(M)$ of p-forms over M and $\Lambda^{p+2}(M)$ for $p \leqq 2k - 2$ we conclude that $\theta(X)\omega$ vanishes, that is X defines an automorphism of the almost KAEHLER structure. That the automorphisms are isometries is seen as follows: Since ω is closed, $\theta(X)\omega = = di(X)\omega$ and hence $i(X)\omega$ is closed. Thus, by the HODGE-DE RHAM decomposition of a form $i(X)\omega = d\Phi + H[i(X)\omega]$ for some real C^∞ function Φ; the operator H is the harmonic projector. Since $i(X)\omega = C\xi$, $\xi = -Cd\Phi + CH[C\xi]$, from which we conclude that $\delta\xi$ vanishes.

Remark. In an almost KAEHLER manifold the operators C and H do not commute. Nevertheless, if h is a harmonic 1-form its image by C has zero divergence. However, if α is an effective closed p-form $(p \leqslant 1)$, $\delta C\alpha$ vanishes.

Proof of Theorem 1. This is an application of equation (6) applied to the fundamental 2-form ω:

$$\theta(X)\omega + \bar{\theta}(X)\omega = \left(1 - \frac{2}{n}\right)\delta\xi \cdot \omega .$$

Applying δ to both sides of this relation we derive since $\bar{\theta}(X)$ and δ commute and the fact that $\delta\omega$ vanishes

$$\begin{aligned} \delta\theta(X)\omega &= \left(1 - \frac{2}{n}\right)\delta(\delta\xi \cdot \omega) \\ &= -\left(1 - \frac{2}{n}\right) D_k(\delta\xi \cdot F^k_i)dx^i \\ &= -\left(1 - \frac{2}{n}\right) Cd\delta\xi . \end{aligned}$$

Taking the global scalar product with $C\xi$ we have since the manifold is compact

$$(\delta\theta(X)\omega, C\xi) = (\theta(X)\omega, dC\xi) = (\theta(X)\omega, \theta(X)\omega)$$

and

$$(Cd\delta\xi, C\xi) = (d\delta\xi, \xi) = (\delta\xi, \delta\xi) .$$

Hence,

$$\| \theta(X)\omega \|^2 = -\left(1 - \frac{2}{n}\right) \| \delta\xi \|^2$$

where we have employed the notation $\| \alpha \|^2 = \int_M \alpha \wedge * \alpha$. The l.h.s. being non-negative and the r.h.s. non-positive we conclude for $n > 1$ that $\theta(X)\omega$ vanishes.

For $n > 2$ it is immediate that $\delta\xi = 0$, that is X is an infinitesimal isometry. For $n = 2$ a previous argument gives the same result. Hence, $C_0(M^{2n}) = A_0(M^{2n})$.

7. Throughout this section we assume that M is a pseudo-Kaehlerian manifold. Let X be a vector field on M whose image by the almost complex structure operator J is closed. Then, X is an infinitesimal automorphism of the pseudo-Kaehler structure since the fundamental form ω of M is closed. Denote by $t(X)$ the tensorfield $\theta(X)J$ modulo $i(X)D\omega$ where D is the covariant differential operator. For pseudo-Kaehler manifolds $D\omega$ vanishes, that is $t(X)$ and $\theta(X)J$ coincide. If J is induced by a complex structure the vanishing of $t(X)$ characterizes the infinitesimal analytic maps. Let t be a covariant real tensor of order 2 and denote by J again the operator

$$J : t_{ij} \to t_{ir} F^r_j .$$

Since $F_{ik} F_{jl} g^{kl} = g_{ij}$, $J\omega = g$ where we denote by J once again the induced map on 2-forms. The tensorfield $t(X)$ associated with a given tangent vector field X is given by

$$(t(X))^i_j = F^i_k D_j \xi^k - F^k_j D_k \xi^i .$$

It is easily checked that $t(JX) = Jt(X)$. Therefore, if X is an analytic map so is JX and the dimension of the group of analytic homeomorphisms of any complex manifold is even.

Lemma 4. *For any vector field* X

$$t(X) = \theta(X)\omega + J\theta(X)g .$$

Indeed,

$$\begin{aligned} -(t(X))_{ij} &= F^k_j (D_k \xi_i - D_i \xi_k + D_i \xi_k) + F^k_i D_j \xi_k \\ &= F^k_j (\theta(X)g)_{ik} + D_j(C\xi)_i - D_i(C\xi)_j \\ &= -(J\theta(X)g)_{ij} - (\theta(X)\omega)_{ij} . \end{aligned}$$

Theorem 4. *A vector field X defines an infinitesimal analytic transformation of a KAEHLER manifold if and only if $J\,\theta(X)\,\omega = \theta(X)g$, that is when applied to the fundamental 2-form the operators $\theta(X)$ and J commute.*

Let $\tilde{t}(X)$ denote the 2-form corresponding to the skew-symmetric part of $t(X)$.

Lemma 5. *For any vector field X on a pseudo-KAEHLER manifold*

$$\overline{\theta}(X)\,\omega - \theta(X)\,\omega = \delta\xi\cdot\omega - 2\tilde{t}(X)\ .$$

This is a straightforward application of equation (4) and Lemma 4.

As an immediate application of equation (6) and lemma 5 we obtain

Theorem 5. *An infinitesimal analytic conformal map of a pseudo-KAEHLER manifold is necessarily homothetic.*

Lemma 6. *For any vector field X on a compact pseudo-KAEHLER manifold M*

$$\|\,\theta(X)\,\omega\,\|^2 = \|\,\delta\xi\,\|^2 + 2\,(\tilde{t}(X),\ \theta(X)\,\omega)\ .$$

The proof is based on that of Theorem 1.

Corollary. *An infinitesimal transformation of M satisfying $\widetilde{J\,\theta(X)g} = 0$ is an automorphism of the pseudo-KAEHLERian structure.*

Theorem 6. *Let X be an infinitesimal analytic transformation of a compact KAEHLER manifold. Then*

$$\|\,\theta(X)\,\omega\,\| = \|\,\delta\xi\,\|\ .$$

Hence, a divergence free analytic map is an infinitesimal automorphism of the KAEHLER structure.

This follows immediately from Lemma 6. An application of Lemma 4 together with Theorem 4 results in

Theorem 7. *Let M be a KAEHLER manifold. Then, in order that an infinitesimal analytic transformation be the image by J of an infinitesimal isometry it is necessary and sufficient that it be closed.*

This generalizes to the non-compact case a theorem of LICHNEROWICZ [7].

Proof of Theorem 2. Since ξ is closed,

$$\begin{aligned} -\,(t(X))_{ij} &= F^k_j(D_k\xi_i - D_i\xi_k + D_i\xi_k) + F^k_i D_j\xi_k \\ &= F^k_j D_i\xi_k + F^k_i D_j\xi_k \\ &= (\theta(C\xi)g)_{ij}\,, \end{aligned}$$

that is $t(X)$ is a symmetric tensorfield. Since $\theta(X)g = -\frac{1}{n}\delta\xi \cdot g$ it follows from Lemma 4 that

$$t(X) = \theta(X)\omega + \frac{1}{n}\,\delta\xi\cdot\omega\,. \tag{7}$$

Hence, $t(X)$ is also skew-symmetric and must therefore vanish. It follows that $d\delta\xi \wedge \omega = 0$ and for $n > 1$ we may conclude that $d\delta\xi$ vanishes, that is X defines a homothetic transformation. But a homothetic map of a complete connected RIEMANNian manifold which is not locally flat is isometric [5], hence volume preserving and therefore from equation (7) we conclude that the fundamental form is preserved.

Proof of Theorem 3. Every homothetic transformation of a RIEMANNian manifold is also an affine transformation corresponding to the RIEMANNian connection. Moreover, an infinitesimal affine transformation of a complete locally flat RIEMANNian manifold is an isometry if and only if its length is bounded.

Remarks. (a) Clearly, if the manifold is compact every vector field has bounded length.

(b) Every conformal map of a complete flat space is homothetic but this is not sufficient to insure that it is an automorphism of the pseudo-KAEHLERian structure.

(c) It is known that every affine transformation of a complete pseudo-KAEHLERian manifold whose RICCI curvature is non-degenerate is an automorphism of the pseudo-KAEHLERian structure.

(d) If X is a homothetic transformation of an almost KAEHLERian manifold M the 2-form $\theta(X)\omega$ is harmonic. Indeed, $\theta(X)\omega$ is closed. Moreover, from the proof of Theorem 1 it is also co-closed. If $\dim M = 4$ this is so for any infinitesimal conformal transformation.

(e) M. OBATA has communicated to us the following result (unpublished): "A closed infinitesimal conformal transformation of a (locally) reducible RIEMANNian manifold is homothetic." This means that only an (absolutely) irreducible RIEMANNian manifold can admit closed non-homothetic maps.

For non-compact KAEHLER manifolds an infinitesimal isometry X need not be analytic. Indeed, the condition $\theta(X)\omega = di(X)\omega$ where $\theta(X)\omega$ is a harmonic 2-form does not imply that $\theta(X)\omega$ vanishes. However, if the manifold is complete the proof of Theorem 2 shows that a closed infinitesimal conformal transformation X is analytic and, in this case X is isometric. Without the assumption of completeness we may conclude in any case that an infinitesimal isometry which is closed is analytic and, in fact by Lemma 4 preserves the fundamental form.

Let X and Y be infinitesimal automorphisms of an almost KAEHLER manifold M. Since $\theta([X, Y])\omega$ vanishes where ω is the fundamental 2-form of M, $[X, Y]$ is also an infinitesimal automorphism. Moreover,

$$\begin{aligned} i([X, Y])\omega &= \theta(X)i(Y)\omega - i(Y)\theta(X)\omega \\ &= i(X)\mathrm{d}i(Y)\omega + \mathrm{d}i(X)i(Y)\omega \\ &= \mathrm{d}i(\xi \wedge \eta)\omega \ . \end{aligned}$$

Put $Z=[X, Y]$. Then, $C\zeta = i(Z)\omega = \mathrm{d}i(\xi\wedge\eta)\omega$. Hence, if the LIE algebra of infinitesimal automorphisms is abelian

$$i(\xi \wedge \eta)\omega = \text{const.}$$

for any infinitesimal automorphisms X and Y.

Remark. We have assumed throughout that the manifolds under consideration are of class C^∞. It is known that on an almost complex manifold M^{2n} of class C^{2n+1} with completely integrable almost complex structure defined by a tensorfield of class C^{2n} it is possible to introduce complex analytic coordinates [8]. Under these conditions every pseudo-KAEHLERian manifold is differentiably homeomorphic with a KAEHLER manifold. Our results are valid if the C^∞ condition is replaced by only C^3. Then, only in the case $n = 1$ is it known that a pseudo-KAEHLERian manifold is KAEHLERian.

8. In this section we assume that M^m is a compact orientable RIEMANNian manifold on which there is defined an harmonic p-form

$$\alpha = \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

with the property

$$(P) : \alpha_i^{\,i_1 \dots i_{p-1}} \alpha_{j i_1 \dots i_{p-1}} = g_{ij} \ .$$

For $p = 2$, LICHNEROWICZ has shown that the metric of any RIEMANNian space with an almost complex structure can be modified so that (P) is satisfied. It is well-known that a compact semi-simple LIE group carries an harmonic 3-form whose coefficients satisfy the property (P). More generally, let $H(m, p)$ denote a compact orientable RIEMANNian manifold satisfying the property (P)[3]. Then, the proof of theorem 1 yields

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[3]) These spaces were defined and studied by R. S. CLARK in his paper "*On conformal equivalence of Riemannian manifolds which admit an exterior form,*" Proc. Kon. Ned. Akad. v. Wet. A, *19* (1956), 198–203.

Theorem 8. *An infinitesimal conformal map of an* $H(m, p)$, $m > 2p$ *is an infinitesimal isometry.*

Corollary. *An infinitesimal conformal map of an m-dimensional compact semi-simple* LIE *group* ($m \geqq 6$) *with the canonical (left-invariant) metric is an infinitesimal isometry.*

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REFERENCES

[1] S. BOCHNER, *Vector fields and* RICCI *curvature.* Bull. Amer. Math. Soc. *52* (1946), 776–797.

[2] S. BOCHNER and D. MONTGOMERY, *Groups of differentiable and real or complex analytic transformations.* Ann. of Math. *49* (1948), 379–390.

[3] S. BOCHNER and K. YANO, *Curvature and* BETTI *Numbers.* Ann. Math. Stud. *32*, p. 49.

[4] S. I. GOLDBERG, *Groups of automorphisms of* KAEHLER *Manifolds.* AFOSR technical note 59–727.

[5] S. ISHIHARA and M. OBATA, *Affine transformations in a Riemannian manifold.* Tôhoku Math. J. *7* (1955), 146–150.

[6] A. LICHNEROWICZ, *Sur les transformations analytiques des variétés kählériennes compactes.* C. R. Acad. Sc., Paris, *244* (1957), 3011–3013.

[7] A. LICHNEROWICZ, Col. de Géom. Diff. Globale, Brussels (1959), 11–26.

[8] A. NEWLANDER and L. NIRENBERG, *Complex analytic coordinates in almost complex manifolds.* Ann. of Math. *65* (1957), 391–404.

[9] S. TACHIBANA, *On almost-analytic vectors in almost-Kaehlerian manifolds.* Tôhoku Math. J. *11* (1959), 247–265.

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