Paths of Rapid Growth of Entire Functions

by WILFRED KAPLAN, Ann Arbor, Mich. (USA)

In 1957 A. HUBER published a paper in which he deduced the following theorem ([1], p. 52):

Theorem. *Let* $f(z)$ *be an entire function, not a polynomial. Let* $\lambda > 0$. *Then there exists a locally rectifiable path* C_λ *tending to infinity, such that*

$$\int_{C_\lambda} | f(z) |^{-\lambda} | dz | < \infty . \tag{1}$$

HUBER's proof depends on a deep study of subharmonic functions and is quite involved. Because of the simplicity of the result, I have been seeking a simple proof. This I have succeeded in obtaining only for special values of λ: $\lambda \geqq 1$ or $\lambda = 1 - (1/n)$, $n = 2, 3, \ldots$ In this note I present the proof for these values of λ.

As remarked by HUBER, there is no difficulty if $f(z)$ has only a finite number of zeros, so that $f(z) = P(z) \exp [g(z)]$, where P is a polynomial and g is entire. The function

$$w = \Phi(z) = \int_0^z e^{-\lambda g(z)} dz$$

is then entire and without critical points. If the inverse function $\Phi^{-1}(w)$ has no singular points, then it is also entire, so that $\Phi(z)$ has form $az + b$ and $f(z)$ is a polynomial; hence $\Phi^{-1}(w)$ must have singularities. In particular there must be a functional element of $\Phi^{-1}(w)$ which can be continued from $w = 0$ along a finite segment ending at a singularity w_0. The segment is mapped by $\Phi^{-1}(w)$ on a path C_λ in the z-plane, on which $z \to \infty$ as $w \to w_0$. Then

$$| w_0 | = \int_{C_\lambda} \left| \frac{dw}{dz} \right| | dz | = \int_{C_\lambda} | e^{g(z)} |^{-\lambda} | dz | .$$

Thus C_λ is the desired path if $P(z) \equiv 1$; by removing a finite portion of C_λ, one can ensure that $| P(z) | \geqq 1$ on the remaining portion C'_λ, so that C'_λ is the desired path.

Now let us suppose that f has infinitely many zeros and let λ have form $1 - (1/n)$, $n = 2, 3, \ldots$ We can then assume without loss of generality that $f(z)$ is expressible as $z^{2n} G(z)$, where $G(z)$ is entire and $G(0) \neq 0$. For moving a zero of f from z_1 to the origin, or from the origin to z_1, is equivalent to multiplying f by $z/(z - z_1)$, or by $(z - z_1)/z$, a factor which approaches 1 as z approaches infinity and which has therefore no effect on the integral in (1). We select k such that $f(k) \neq 0$ and introduce

$$w = \Phi(z) = \int_k^z [f(z)]^{-\lambda} dz = \int_k^z z^{2-2n}[G(z)]^{\frac{1}{n}-1} dz \ . \tag{2}$$

This equation defines $\Phi(z)$ as a multiple-valued function of z. However, we remark that one branch (in fact, every branch) has a pole of order $2n - 3$ at $z = 0$. The inverse function $\Phi^{-1}(w)$ can be considered as the solution of the differential equation

$$\frac{dz}{dw} = z^{2n-2}[G(z)]^{1-\frac{1}{n}} \ , \tag{3}$$

such that $z = k$ when $w = 0$. We consider the solution along rays arg $w =$ const., starting with a given analytic branch at $w = 0$. By the theory of differential equations, the solution continues to exist as long as the value of z remains within the domain of analyticity of the right-hand member of (3). Trouble can arise as $w \to w_0$ ($w_0 \neq \infty$) only if, as $w \to w_0$, z approaches a zero of G or z approaches infinity. If $z \to z_0$, $G(z_0) = 0$, then z_0 must be a zero of first order of G, for by (2) at a multiple zero $w \to \infty$ as $z \to z_0$. Near a first order zero we obtain series expansions

$$w - w_0 = (z - z_0)^{1/n}[b_0 + b_1(z - z_0) + \ldots] \, , \; b_0 \neq 0 \, ,$$
$$z - z_0 = b_0^{-n}(w - w_0)^n + \ldots \, ;$$

that is, $\Phi^{-1}(w)$ is a single-valued analytic function in a neighborhood of w_0. [An illustration is provided by $z = \sin w$ as a solution of the differential equation $dz/dw = (1 - z^2)^{\frac{1}{2}}$].

Therefore continuation of $\Phi^{-1}(w)$ can be interrupted at a finite value w_0 only if, as $w \to w_0$, $z \to \infty$. If indefinite continuation were possible along all rays, then $\Phi^{-1}(w)$ would be an entire function of w. But we know that one branch of $\Phi^{-1}(w)$ approaches 0 as $w \to \infty$, because of the pole of $\Phi(z)$ at $z = 0$. Therefore $\Phi^{-1}(w) \to 0$ as $w \to \infty$. Accordingly, $\Phi^{-1}(w) \equiv 0$, and there is a contradiction. Hence continuation must be interrupted at at least one value w_0, and we obtain the path C_λ as in the first part of the proof.

For $\lambda \geqq 1$ we consider two cases: λ rational, equal to m/n; λ irrational. In the rational case the proof for the case $\lambda = 1 - (1/n)$ can be repeated with the simplification that, at each zero of $G(z)$, $w \to \infty$ as $z \to z_0$.

If λ is irrational, we do not need to normalize f at $z = 0$. The previous argument can be repeated with slight modification; the differential equation (3) is replaced by the equation $dz/dw = [f(z)]^\lambda$ and a solution $z(w)$ can be continued along a ray arg $w =$ const. unless z approaches the boundary of the domain of analyticity of $[f(z)]^\lambda$, a RIEMANN surface over the z-plane. Since $|f(z)|^\lambda$ has the same value on all sheets of this surface, we conclude that continuation can be interrupted for finite w_0 only if, as $w \to w_0$, z approaches ∞ or a zero of f. But since $\lambda > 1$, $w \to \infty$ as z approaches a zero of f. Hence, if $\Phi^{-1}(w)$ has no singularity at which $z \to \infty$, then $\Phi^{-1}(w)$ is single-valued, an entire function $\psi(w)$, and

$$\frac{dz}{dw} = \frac{d\psi}{dw} = [f(z)]^\lambda = [f(\psi(w))]^\lambda = [g(w)]^\lambda,$$

where $g(w)$ is entire. Therefore $[g(w)]^\lambda$ is also entire. This is possible with λ irrational only if $g(w)$ has no zeros—hence only if $f(z)$ has at most one zero. Again we have a contradiction. Therefore HUBER's theorem is proved for $\lambda \geqq 1$ and for $\lambda = 1 - (1/n)$ $(n = 2, 3, \ldots)$.

Remark 1. The theorem can be strengthened for functions having no zeros. For then $\log f(z)$ can be defined as an entire function; if $\log f(z)$ is not a polynomial, there exists a path C_λ on which

$$\int_{C_\lambda} |\log f(z)|^{-1} |dz| < \infty .$$

Remark 2. In his paper ([1], p. 52) HUBER raises the question: Suppose $f(z)$ is entire and that there exists $\lambda > 0$ such that

$$\int_1^\infty |f(re^{i\theta})|^{-\lambda} dr = \infty$$

for all θ, $0 \leqq \theta < 2\pi$; does this imply that $f(z)$ is a polynomial? In other words, in the preceding theorem, can C_λ be chosen to be a ray?

This question we answer negatively as follows. A theorem of KELDYS and MERGELYAN ([2], p. 37) implies that, if $g(z)$ is continuous on a closed set E and analytic on the interior of E, then for each $\epsilon > 0$ there exists an entire function $f(z)$ such that $|f(z) - g(z)| < \epsilon$ on E, provided the complement E' of E is locally connected at infinity. In particular, E can be chosen to be the closure of a domain bounded by a simple path γ which approaches infinity in both directions. On such a set E we can easily construct $g(z)$, not identically constant, such that $|g(z)| < \frac{1}{2}$ on E (for example, $g(z)$ can be obtained with the aid of conformal mapping from the function $\frac{1}{4}e^z$ in the left half-plane). Let $g(z_0) = a$, $g(z_1) = b \neq a$. We choose $\epsilon = |b - a|/2$ and $f(z)$ entire, so that $|f - g| < \epsilon$ on E. Then f is not identically constant and $|f| < 1$ on E. Since f is bounded on such a set, f cannot be a polynomial. By proper choice of γ, we can force every ray $C_\theta : \theta = \text{const.}$ to meet E in a set of infinite length; for example, γ can be formed of two spirals which approach each other as $|z| \to \infty$, and E' as the set between the spirals. Then

$$\int_{C_\theta} |f(z)|^{-\lambda} |dz| \geqq \int_{C_\theta \cap E} |dz| = \infty .$$

For such a function $f(z)$ it is clear that the path C_λ of HUBER's theorem must either lie between the spirals (that is, in E') or be asymptotic to E' in the sense that the length of the part of C_λ outside of E' must be finite; hence effectively there is only one path.

Remark 3. Although the paths $\arg z =$ const. are not generally allowable as a choice of C_λ, it appears reasonable that the paths $\arg w =$ const. can serve. For on such a path, not passing through a zero of f, $|f(z)|$ grows steadily in one direction. I conjecture that, for each $f(z)$, a path $\arg f(z) = c$ can serve as C_λ for almost all values of c. For a similar reason, it appears probable that the paths $\mathrm{Re}[f(z)] = c$, $\mathrm{Im}[f(z)] = c$ can also serve as C_λ for almost all c.

REFERENCES

[1] A. HUBER, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv., 32 (1957), 13–72.

[2] S. N. MERGELYAN, *Uniform approximations to functions of a complex variable* (in Russian), Uspehi Mat. Nauk (N.S.) 7, No. 2 (48) (1952), 31–122. Amer. Math. Soc. Transl. No. 101, Providence, 1954.

(Received April 3, 1959)