On fibre spaces in which the fibre is contractible.

Autor(en): **Spanier, E.H. / Whitehead, J.H.C.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **29 (1955)**

PDF erstellt am: **24.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-23274

On fibre spaces in which the fibre is contractible

E. H. Spanier[1]) and J. H. C. Whitehead

Dedicated to H. Hopf on his 60th birthday

1. Let $f : X \to Y$ be a map of a space X in a space Y and let $A = f^{-1} y_0$, for some point $y_0 \in Y$. Let A be either a locally finite CW-complex, as defined in [16], or a compactum which is an ANR. Then so are $A \times A$ and[2])

$$A \vee A = (A \times a_0) \cup (a_0 \times A) \subset A \times A \qquad (a_0 \in A) .$$

We assume that the covering homotopy theorem is valid with respect to f for any map $g : A \times A \to X$ and any homotopy of $f \circ g$. This will be so if f determines a fibering of X with a local product representation ([13], § 11.7, and [7], § 5). Subject to the latter condition we describe f as a *bundle mapping.* In § 3 below we prove :

Theorem (1.1). *If A is contractible in X it is an H-space*[3]).

The method of proof is suggested by the following observation and an (unpublished) construction, due to M. G. Barratt, for defining the "generalized Whitehead product", $[\alpha, \beta] \in \pi_{m+n-1}(X, A)$, of given elements $\alpha \in \pi_m(X, A)$, $\beta \in \pi_n(A)$. Let f be a fibre mapping. Since $f_* \beta = 0$ it follows that $f_* [\alpha, \beta] = [f_* \alpha, 0] = 0$, where each f_* denotes the appropriate homomorphism induced by f. Therefore $[\alpha, \beta] = 0$. We have $\partial [\alpha, \beta] = \pm [\partial\alpha, \beta]$, where

$$\partial : \pi_{q+1}(X, A) \to \pi_q(A)$$

is the boundary homomorphism and $[\beta', \beta] \in \pi_{m+n-2}(A)$ is the ordinary Whitehead product of $\beta' \in \pi_{m-1}(A)$ and β. Hence it follows that

---

[1]) This note arose out of consultations during the tenure of a John Simon Guggenheim Memorial Fellowship by Spanier.

[2]) The fact that, if A is an ANR compactum, so is $A \vee A$, follows from Theorem 1 in [15].

[3]) i. e. there is a map $h : A \times A \to A$ such that $h(a, a_1) = h(a_1, a) = a$ for some a_1 and every $a \in A$.

$[\beta', \beta] = 0$ if $i_* \beta' = 0$, where $i_* : \pi_{m-1}(A) \to \pi_{m-1}(X)$ is the injection. In particular $\pi_1(A)$ is Abelian if X is simply connected (cf. [14], p. 289).

Before considering the consequences of (1.1), in its full generality, we draw a corollary from the preceding observation. Let X be a finite dimensional, locally compact, separable metric space, which is an AR (absolute retract). Let $f : X \to Y$ be a bundle mapping with a connected fibre A. Since X is an AR it follows from an argument on p. 467 of [12] that A is acyclic and from the above observation that $\pi_1(A)$ is Abelian. Therefore $\pi_1(A) \approx H_1'(A) = 0$, where $H_n'(A)$ is the n^{th} integral, singular homology group of A. Therefore $\pi_n(A) \approx H_n'(A) = 0$ for every $n \geqslant 1$. It follows from the local product representation that A is a neighbourhood retract of X and hence an ANR. Since X is locally compact so, obviously, is A and since $\dim A \leqslant \dim X < \infty$ it follows that A may be imbedded as a closed[4]) sub-set in some Euclidean space, E, of which it is a neighbourhood retract. Since A is connected and $\pi_n(A) = 0$ for every $n \geqslant 1$ it follows that A is a retract of E, and hence an AR. The map f is obviously open and it follows without difficulty that Y is a C_σ-space, as defined in § 11.3 of [13]. In particular Y is covered by a countable set of open sub-sets, $U_1, U_2, \ldots$ such that $\overline{U}_i$, the closure of U_i, lies in a coordinate neighbourhood V_i (i. e. a neighbourhood such that $f^{-1}V_i$ is represented as $A \times V_i$). Let $Y_n = \overline{U}_1 \cup \cdots \cup \overline{U}_n$ $(n \geqslant 1)$ and assume that there is a map $g_n : Y_n \to X$ such that $f g_n y = y$ for every $y \in Y_n$. It follows from the local product representation that this is so if $n = 1$. Let $T_{n+1} = \overline{U}_{n+1} \cap Y_n$. Then T_{n+1} is a closed sub-set of $\overline{U}_{n+1}$ and the latter is a separable metric space, since it is homeomorphic to a sub-set of X. Since A is an AR it follows that every map $T_{n+1} \to A$ has an extension $\overline{U}_{n+1} \to A$. Therefore it follows from the local product representation that g_n has an extension $g_{n+1} : Y_{n+1} \to X$ such that $f g_{n+1} y = y$ $(y \in Y_{n+1})$. Hence it follows by induction on n that f has a right inverse, $g : Y \to X$, by means of which Y may be imbedded in X in such a way that f becomes a retraction. Therefore Y is an AR and it follows from § 11.6 in [13] that X, as a bundle over Y, is equivalent to the product $A \times Y$. That is to say there is a homeomorphism $h : A \times Y \to X$, onto X, such that $h(A \times y) = f^{-1} y$ for every $y \in Y$. Thus we have :

---

[4]) By the addition of a single point, c, we can imbed A in a compactum, C, such that $\dim C \leqslant \dim A + 1$. We imbed C in a p-sphere, S^p, for some large value of p, and $E = S^p - c$.

Theorem (1.2). *Any fibre bundle with a connected fibre, which, as a space, is a finite dimensional, locally compact, separable metric AR, is equivalent to a product bundle.*

It follows from the arguments in [13] that, if G is a topological transformation group of A and if X is a bundle with G as its group, then (1.2) is valid if equivalence is interpreted as equivalence with respect to G.

We now turn to the consequences of (1.1). Let $f : X \to Y$ be a bundle mapping of a compactum X. Since f is an open mapping onto Y it follows that Y, A are compacta $(A = f^{-1} y_0)$. Let A be connected, contractible in X and an ANR. Since A is a neighbourhood retract of X it will certainly be an ANR if X is an ANR. Let $\dim X < \infty$. Then $\dim A$, $\dim Y \leqslant \dim X < \infty$ in consequence of the local product representation. Let $H^*(P, G)$, $H^n(P, G)$ denote the (discrete) Čech cohomology ring and the n^{th} Čech cohomology group of a given compactum P, with coefficients in a given ring G. We assume that Y is locally and globally pathwise connected and that $\pi_1(Y)$, operating on $H^*(A, G)$ as in [9], operates simply for every G. Since A is connected, and hence pathwise connected, this will certainly be the case if X is simply connected. For then $\pi_1(Y) = 1$. Let I_0, R and S^n denote respectively the ring of integers, the ring of rational numbers and an n-sphere. We write $H^*(P, I_0) = H^*(P)$, $H^i(P, I_0) = H^i(P)$. In § 3 we prove:

Theorem (1.3). *Let* $H^*(X) \approx H^*(S^n)$ $(n \geqslant 1)$. *Then:*

a) *either A is an*[5] *AR or $H^*(A, R) \approx H^*(S^q, R)$, for some odd value of q,*

b) *if A is homeomorphic to a topological product, $A_1 \times A_2$, then one of A_1, A_2 is an AR.*

In consequence of the second alternative in (1.3a) we have the exact sequence of Gysin ([3], [9], Ch. III)

$$\cdots \xrightarrow{f^*} H^{j-1}(X, R) \to H^{j-q-1}(Y, R) \xrightarrow{\theta} H^j(Y, R) \xrightarrow{f^*} \cdots ,$$

in which $\theta v = v \smile \Omega$ for some $\Omega \in H^{q+1}(Y, R)$. Since $\dim Y < \infty$ there is a $k \geqslant 0$ such that $\Omega^k \neq 0$, $\Omega^{k+1} = 0$, where $\Omega^0 = 1 \in H^0(Y, R)$, $\Omega^r = \Omega \smile \cdots \smile \Omega \in H^{r(q+1)}(Y, R)$. It may be verified that $k > 0$, since A is contractible in X, that $n = k(q+1) + q$ and that

$$\left.\begin{array}{lll} H^i(Y, R) \approx R & \textit{for} & i = 0, q+1, \ldots, k(q+1) \\ H^i(Y, R) = 0 & \textit{for all other values of } i. & \end{array}\right\} \qquad (1.4)$$

[5]) This will be the case, for example, if $X = Y$ and $f = 1$.

Let $S^n \to Y$ be a bundle mapping with a connected fibre F. Then F is an ANR, which is contractible in S^n except in the trivial case $F = S^n$. Therefore we have:

Corollary (1.5). *If F is homeomorphic to $A_1 \times A_2$, then one of A_1, A_2 is an AR.*

The results (1.2), (1.5) above extend the two theorems concerning the fibering of Euclidean spaces and spheres by tori which are proved in [4]. Also (1.5) extends a theorem due to A. Borel ([1], [2], p. 165). It will be seen that our (1.3) is an easy corollary of (1.1) together with this theorem of Borel's.

2. Let P, Q be topological spaces which are either locally finite CW-complexes or ANR compacta. Then so are $P \times Q$ and

$$P \vee Q = (P \times q_0) \cup (p_0 \times Q) \subset P \times Q \ ,$$

where p_0, q_0 are points in P, Q, which are 0-cells if P, Q are CW-complexes. Let P, Q be imbedded in $P \vee Q$ so that $p = (p, q_0)$, $q = (p_0, q)$ for each point $p \in P$ and each $q \in Q$. Let

$$P \overset{u}{\to} A \overset{v}{\leftarrow} Q$$

be given maps such that $up_0 = vq_0 = a_0$, say, and u is homotopic in X (and hence homotopic rel. p_0) to the constant map $P \to a_0$. Let

$$g_0 : P \times Q \to X$$

be defined by $g_0(p, q) = vq$. Then $g_0 P = a_0$ and there is a homotopy $u_t : P \to X$, rel. p_0, such that $u_0 \ p = g_0 \ p = a_0$, $u_1 \ p = up$ $(p \in P)$. This can be extended, first to a homotopy $u'_t : P \vee Q \to X$ such that $u'_t q = vq$ if $q \in Q$, and then ([16], p. 228, [15]) to a homotopy $g_t : P \times Q \to X$. Then $g_1 \ p = up$, $g_1 \ q = vq$. Let

$$h : (P \times Q, P \vee Q) \to (X, A)$$

be the map determined by g_1. We describe h as *inessential* if, and only if, it is related by a homotopy of the form $(P \times Q, P \vee Q) \to (X, A)$ to a map with values in A. We describe v as *inessential* if, and only if, it is homotopic, and hence homotopic rel. q_0, to the constant map $Q \to a_0$.

Lemma (2.1). *If v is inessential, so is h.*

Let $v_t : Q \to A$, rel. q_0, be a homotopy such that $v_0 = v$, $v_1 Q = a_0$. Let $g' : P \times Q \times I \to X$ be defined by

$$
\begin{aligned}
g'(p,q,t) &= g_{1-3t}(p,q) && \text{if} \quad 0 \leqslant t \leqslant 1/3 \\
&= v_{3t-1}(q) && \text{if} \quad 1/3 \leqslant t \leqslant 2/3 \\
&= u_{3t-2}(p) && \text{if} \quad 2/3 \leqslant t \leqslant 1 \ .
\end{aligned}
$$

Then $g'(p,q,0) = g_1(p,q) = h(p,q)$, $g'(Q\times I)\subset A$ and $g'(P\times Q\times 1)\subset A$. Also $g'(p_0\times q_0\times I) = a_0$ and since $g_s(p,q_0) = u_s(p)$ it follows that $g' \mid (P \vee Q)\times I$ is homotopic, rel. $(P\times 0) \cup (Q\times I) \cup (P\times 1)$, to a map in which $(p,t) \to up$. Therefore (2.1) follows from the homotopy extension theorem, applied to the pair $(P\times Q\times I, K)$, where

$$K = (P\times Q\times 0) \cup (P \vee Q)\times I \cup (P\times Q\times 1) \ .$$

Notice that we have used the form of the homotopy extension theorem in which the argument spaces are of a special sort and the image space is arbitrary. The definition of h and the proof of (2.1) apply unchanged if X is an ANR, of the sort appropriate to some general category of spaces to which $P, Q, P\times Q$ etc. belong (cf. [6]).

3. *Proof of* (1.1). Let $f' : (X, A) \to (Y, y_0)$ be the map determined by f. Then

$$f' \circ h : (P\times Q, P \vee Q) \to (Y, y_0)$$

is defined in the same way as h, in § 2, with g_t, v replaced by $f\circ g_t$ and the constant map $Q \to y_0$. Therefore it follows from (2.1) that $f'\circ h$ is homotopic, rel. $P\vee Q$, to the constant map c, where $c(P\times Q) = y_0$. Assuming that a homotopy $f'\circ h \simeq c$ can be lifted it follows that h is inessential. Therefore $h \mid P\vee Q$ has an extension $P\times Q \to A$ and (1.1) follows on taking $P = Q = A$ and $u = v =$ the identical map.

Let f be a bundle mapping and let X be a locally compact, separable metric space. Then X and likewise A and $A\times A$ are obviously C_σ-spaces. Therefore we have, in consequence of the concluding remarks in § 2 and § 11.3 in [13]:

Theorem (3.1). *If $f : X \to Y$ is a bundle mapping, if X is a locally compact, separable metric ANR and if a fibre, A, is contractible in X, then A is an H-space.*

4. *Proof of* (1.3). Let $g : E \to B$ be a fibre mapping, with fibre F, where E, B, and hence also F, are compacta. Let $H^i(P) = 0$ for $P = B, E, F$ and all sufficiently large values of i. This will be the case, for example, if $\dim P < \infty$. Also let $H^i(P)$ be finitely generated for all values of i. It follows from the theory of the spectral sequence associated with the mapping g that this will be the case if any two of $H^i(B)$, $H^i(E)$,

$H^i(F)$ are finitely generated for every[6] i. Therefore it will be the case if $H^*(E) \approx H^*(S^n)$ and F is an ANR. We quote the universal coefficient theorem[7])

$$H^r(Q, G) \approx H^r(Q) \otimes G + H^{r+1}(Q) * G \ , \tag{3.1}$$

for the (discrete) Čech cohomology groups of a compactum Q, with coefficients in G. It follows from (3.1) that, if

$$H^m(Q) \approx I_0 \ , \quad H^i(Q) = 0 \quad \text{for} \quad i > m \ , \tag{3.2}$$

then

$$H^m(Q, G) \approx G \ , \quad H^i(Q, G) = 0 \quad \text{for} \quad i > m \ . \tag{3.3}$$

Let $H^j(Q)$ be finitely generated for every $j \geqslant m$. Then (3.2) is true if (3.3) holds for every field, G, as group of coefficients.

Let E satisfy (3.2) for some $m \geqslant 0$ and let K be a given field. Then it follows from Theorem (9.1) on p. 189 of [9] that there are integers $r = r_K$, $s = s_K$ such that $r + s = m$ and

$$\left.\begin{array}{llll} H^r(B, K) \approx K \ , & H^i(B, K) = 0 & \text{if} & i > r \ , \\ H^s(F, K) \approx K \ , & H^j(F, K) = 0 & \text{if} & j > s \ , \end{array}\right\} \tag{3.4}$$

in which $\approx$ indicates isomorphism between vector spaces over K. Let $k = r_R$, $l = s_R$. Since $H^i(B)$, $H^i(F)$ are finitely generated it follows from (3.1) and (3.4), with $K = R$, that

$$H^k(B) \approx I_0 + T \ , \qquad H^l(F) \approx I_0 + T' \ ,$$

where T, T' are finite groups. Hence, and from (3.1), it follows that $H^k(B, K)$ and $H^l(F, K)$ each contains a summand which is isomorphic to K. Therefore $k \leqslant r$, $l \leqslant s$ and since $k + l = m = r + s$ we have $k = r$, $l = s$. Thus r, s are independent of the choice of K. Therefore B, F satisfy (3.2) with m replaced by r or s according as $Q = B$ or F.

Let $f : X \to Y$ and A be as in (1.3). Then $H^*(X) = H^*(S^n)$ and it follows from the preceding paragraph that

$$\left.\begin{array}{llll} \text{a)}\ H^p(Y) \approx I_0 \ , & H^i(Y) = 0 & \text{if} & i > p \\ \text{b)}\ H^q(A) \approx I_0 \ , & H^j(A) = 0 & \text{if} & j > q \ , \end{array}\right\} \tag{3.5}$$

---

[6]) The argument is essentially the same as the one on p. 465 of [12]. See also § 9 of [9]

[7]) See Theorem 44.2 on p. 823 of [5], in which the term $H^{r+1}(Q) * G$ is expressed differently. The only property of this "product" which we need is that $H * G = 0$ if either $H = 0$ or if G has no (non-zero) element of finte order. We use $+$ to indicate direct summation.

for some pair of integers p, q such that $p+q=n$. Moreover A, being contractible in X, is an H-space, according to (1.1).

First assume that $q=0$. Then it follows from (5.1) on p. 346 of [10] that $H_i(A)=0$ for every $i>0$, where $H_i(A)$ is the i^{th} discrete, integral Čech homology group of A. Since A is an ANR it follows that $H'_i(A)=0$ if $i>0$ where $H'_i(A)$ is the i^{th} singular homology group of A ([11], p. 107). Hence it follows, as in the proof of (1.2), that A is an AR. If A is homeomorphic to $A_1 \times A_2$, then A_1 is homeomorphic to a retract of A. Therefore it follows that A_1 is an AR. This proves (1.3) if $q=0$ and we proceed on the assumption that $q>0$.

Since A is an H-space and $H^j(A)=0$ if $j>q$ we have ([8], No. 24)

$$H^*(A, R) \approx H^*(S^{i_1} \times \cdots \times S^{i_p}, R)$$

for certain odd values of $i_1, \ldots, i_p$. Since $H^*(X) \approx H^*(S^n)$ it follows from [1] and [2], p. 165, that $p=1$. Thus

$$H^*(A, R) \approx H^*(S^q, R) , \tag{3.6}$$

where q is odd. This proves (1.3a).

Let A be homeomorphic to $A_1 \times A_2$. On taking $g: E \to B$ to be the projection $A_1 \times A_2 \to A_2$, with F homeomorphic to A_1, it follows from (3.5b) and (3.4) that

$$H^{q_j}(A_j) \approx I_0 , \qquad H^i(A_j)=0 \quad \textit{if} \quad i>q_j ,$$

for some pair of integers q_1, q_2 such that $q_1+q_2=q$. The group $H^{q_j}(A_1 \times A_2, R)$ contains a summand which is isomorphic to $H^{q_j}(A_j, R)$. Hence it follows from (3.6) that $q_j=0$ or q. Since $q_1+q_2=q$ it follows that either $q_1=0$ or $q_2=0$, say $q_1=0$. Since A_1 is homeomorphic to a retract of A it is an ANR. Since $\pi_1(A) \approx \pi_1(A_1) \times \pi_1(A_2)$ and $\pi_1(A)$ is Abelian, because A is contractible in X, it follows that $\pi_1(A_1)$ is Abelian. Therefore it follows from an argument similar to the one used above to dispose of the case $q=0$ that A_1 is an AR. This completes the proof.

REFERENCES

[1] *Armand Borel,* Impossibilité de fibrer une sphère par un produit de sphères, C. R. Acad. Sci. Paris 231 (1950) 943–45.

[2] *Armand Borel,* Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. Math. 57 (1953) 115–207.

[3] *Shiing-Shen Chern and E. Spanier,* The homology structure of sphere bundles, Proc. Nat. Acad. Sci. 36 (1950) 248–55.

[4] *B. Eckmann, H. Samuelson and G. W. Whitehead,* On fibering spheres by toruses, Bull. Amer. Math. Soc. 55 (1949) 433–38.

[5] *Samuel Eilenberg and Saunders MacLane,* Group extensions and homology, Ann. Math. 43 (1942) 757–831.

[6] *Olof Hanner,* Retraction and extension of mappings of metric and non-metric spaces, Ark. Mat. 2 (1952) 315–60.

[7] *I. M. James and J. H. C. Whitehead,* Note on fibre spaces, Proc. London Math. Soc. (3), 4 (1954), 129–137.

[8] *Jean Leray,* Sur la forme des espaces topologiques et sur les points fixes des représentations, J. Math. Pures Appl. 24 (1945) 95–167.

[9] *Jean Leray,* L'homologie d'un espace fibré dont la fibre est connexe, Ibid 29 (1950) 169–213.

[10] *S. Lefschetz,* Algebraic Topology, New York 1942.

[11] *S. Lefschetz,* Topics in Topology, Ann. Math. Studies No. 10, Princeton 1942.

[12] *Jean-Pierre Serre,* Homologie singulière des espaces fibrés, Ann. Math. 54 (1951) 425–505.

[13] *N. E. Steenrod,* The topology of fibre bundles, Princeton 1951.

[14] *J. H. C. Whitehead,* On the groups $\pi_r(V_{n,m})$ and sphere bundles, Proc. London Math. Soc. (2) 48 (1944) 243–91.

[15] *J. H. C. Whitehead,* Note on a theorem due to Borsuk, Bull. Amer. Math. Soc. 54 (1948) 1125–32.

[16] *J. H. C. Whitehead,* Combinatorial homotopy (I), Ibid. 55 (1949) 213–45.

(Received October 8, 1953.)