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# The class number of an imaginary quadratic field

By L. Carlitz

**1. Introduction.** Let $h(D)$ denote the number of classes of properly primitive forms $ax^2 + 2bxy + cy^2$ with $b^2 - ac = -D$, $D > 0$. In his paper [3] Hurwitz expressed the residue (mod $p$) of $h(D)$ in terms of the coefficients of certain power series; here $p$ is an odd prime divisor of $D$ such that $D \not\equiv 0 \pmod{p^2}$. In some cases the residue was expressed explicitly in terms of Bernoulli numbers.

Let now $h(d)$ denote the class number of the imaginary quadratic field $R(\sqrt{d})$ of discriminant $d$. (We prefer to use the terminology of quadratic fields rather than quadratic forms in order to stress the analogy with certain other results on the class number of cyclic fields [1]). Let $p$ be an odd prime divisor of $d$ and $n \geq 0$; then we show that for $d < -4$.

$$h(d) \equiv -2c\left(\frac{q}{p}\right) \sum_{1 \leq s < q/2} \left(\frac{q_0}{s}\right) B_k\left(\frac{s}{q}\right) \pmod{p^{n+1}}, \tag{1.1}$$

where $d = (-1)^{(p-1)/2} p q_0$, $q = |q_0|$, $(q_0/s)$ is the Kronecker symbol, $B_k(x)$ is the Bernoulli polynomial of degree $k = \frac{1}{2}(p-1)p^n + 1$ and $c = 1 + \frac{1}{2}p^n$ for $n \geq 1$, while $c = 2$ for $n = 0$. A particularly simple special case of (1.1) is

$$h(-4p) \equiv \tfrac{1}{2} E_{k-1} \pmod{p^{n+1}}, \tag{1.2}$$

where $p \equiv 1 \pmod 4$ and $E_{k-1}$ is an Euler number.

**2. Kronecker's symbol.** We recall a few properties of Kronecker's symbol (see for example [4, p. 51]). If $d \equiv 0$ or $1 \pmod 4$ and is not a square and if $m > 0$, we define

$$\left(\frac{d}{1}\right) = 1, \quad \left(\frac{d}{p}\right) = 0 \quad (p \mid d),$$

$$\left(\frac{d}{2}\right) = \begin{cases} 1 \ (d \equiv 1 \pmod 8) \\ -1 \ (d \equiv 5 \pmod 8), \end{cases}$$

$$\left(\frac{d}{p}\right) = \text{the Legendre symbol when } p > 2, \ d \not\equiv 0 \pmod p,$$

$$\left(\frac{d}{p_1 \cdots p_r}\right) = \left(\frac{d}{p_1}\right) \cdots \left(\frac{d}{p_r}\right).$$

It follows from the definition that

$$\left(\frac{d}{m}\right) = 0 \quad \text{for} \quad (d, m) > 1\,; \quad \left(\frac{d}{m_1 m_2}\right) = \left(\frac{d}{m_1}\right)\left(\frac{d}{m_2}\right); \tag{2.1}$$

also for odd $d$ we have

$$\left(\frac{d}{m}\right) = \left(\frac{m}{|\,d\,|}\right), \tag{2.2}$$

where the quantity on the right is a Jacobi symbol. Another useful property is

$$\left(\frac{d}{m_1}\right) = \left(\frac{d}{m_2}\right) \quad (m_1 \equiv m_2 (\text{mod } d)). \tag{2.3}$$

We shall also require

$$\left(\frac{d}{|\,d\,| - m}\right) = \left(\frac{d}{m}\right) \text{sgn } d \quad (m < |\,d\,|). \tag{2.4}$$

It is sometimes convenient to define $\left(\frac{d}{0}\right) = 0$.

Thus it is clear that if $d$ is the discriminant of a quadratic field then $\left(\frac{d}{m}\right)$ is a character (mod $d$).

The letter $p$ will denote a positive odd prime. We define

$$p_0 = (-1)^{(p-1)/2} p, \tag{2.5}$$

so that $p_0 \equiv 1 \pmod 4$. If $p \mid d$ we put

$$d = p_0 q_0, \quad q = |\,q_0\,|. \tag{2.6}$$

It follows that the Kronecker symbols $(p_0/m)$ and $(q_0/m)$ are defined; moreover

$$\left(\frac{d}{m}\right) = \left(\frac{p_0}{m}\right)\left(\frac{q_0}{m}\right). \tag{2.7}$$

Hereafter $d$ will denote the discriminant of an imaginary quadratic field. Hence $d < 0$ and is not divisible by the square of any odd prime.

**3. Bernoulli polynomials.** We use the notation of Nörlund [5, Chapter 2] for the Bernoulli polynomials. The following formulas will be needed.

$$\sum_{a=0}^{m-1} (x + a)^{n-1} = \frac{B_n(x + m) - B_n(x)}{n}, \tag{3.1}$$

$$B_n(x + y) = \sum_{s=0}^{n} \binom{n}{s} x^{n-s} B_s(y)\,; \tag{3.2}$$

$$B_n(1 - x) = (-1)^n B_n(x). \tag{3.3}$$

In addition we recall a special case of Kummer's congruence [2, Theorem 5]

$$\frac{B_{n+t}(a)}{n+t} \equiv \frac{B_n(a)}{n} \pmod{p^e}, \tag{3.4}$$

where $p^{e-1}(p-1) \mid t$, $n \not\equiv 0 \pmod{p-1}$, $n > e$ and the rational number $a$ is integral (mod $p$). The following divisibility property will also be used.

$$B_m(a) \equiv 0 \pmod{p^r} \qquad (p^r \mid m,\ m \not\equiv 0 \pmod{p-1}) \tag{3.5}$$

where again $a$ is integral (mod $p$).

For some purposes it is convenient to define the Bernoulli function $\overline{B}_m(x)$:

$$\overline{B}_m(x) = \overline{B}_m(x) \qquad (0 \leq x < 1)$$
$$\overline{B}_m(x+1) = B_m(x).$$

Then $\overline{B}_m(x)$ satisfies (3.3) as well as the multiplication formula

$$\sum_{s=0}^{r-1} \overline{B}_m\left(x + \frac{s}{r}\right) = r^{1-m}\, \overline{B}_m(rx); \tag{3.6}$$

the polynomial $B_m(x)$ also satisfies (3.6).

**4. The main result.** Let $d < -3$. It is familiar that

$$h(d) = \frac{1}{d} \sum_m m\left(\frac{d}{m}\right), \tag{4.1}$$

where $m$ runs through a complete residue system (mod $d$). We assume $p \mid d$ and make use of (2.5), (2.6), (2.7). Let $q > 1$. Then (4.1) becomes

$$\begin{aligned} h(d) &= \sum_{r=0}^{p-1} \sum_{s=1}^{q-1} (rq+s)\left(\frac{p_0}{rq+s}\right)\left(\frac{q_0}{rq+s}\right) \\ &= \frac{1}{d} \sum_{s=1}^{q-1} \left(\frac{q_0}{s}\right) \sum_{r=0}^{p-1} (rq+s)\left(\frac{rq+s}{p}\right). \end{aligned} \tag{4.2}$$

Now it follows from

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$$

that

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)p^n/2} \pmod{p^{n+1}}. \tag{4.3}$$

Then using (4.3) we have

$$\sum_{r=0}^{p-1} (rq+s)\left(\frac{rq+s}{p}\right) \equiv \sum_{r=0}^{p-1} (rq+s)^k \pmod{p^{n+1}}, \tag{4.4}$$

where for brevity we put

$$k = \tfrac{1}{2}(p-1)\,p^n + 1\,. \tag{4.5}$$

Then using (3.1) we get

$$\sum_{r=0}^{p-1}(rq+s)^k = q^k \sum_{r=0}^{p-1}\left(r+\frac{s}{q}\right)^k = q^k \frac{B_{k+1}\left(p+\frac{s}{q}\right) - B_{k+1}\left(\frac{s}{q}\right)}{k+1}\,. \tag{4.6}$$

In the next place by (3.2)

$$B_{k+1}\left(p+\frac{s}{q}\right) - B_{k+1}\left(\frac{s}{q}\right) = (k+1)\,pB_k\left(\frac{s}{q}\right) + \binom{k+1}{2} p^2 B_{k-1}\left(\frac{s}{q}\right) + \sum_{r=3}^{k+1}\binom{k+1}{r} p^r B_{k-r+1}\left(\frac{s}{q}\right).$$

But it is easily verified that

$$\binom{k+1}{r} p^r B_{k-r+1}\left(\frac{s}{q}\right) \equiv 0 \pmod{p^{n+1}} \quad (r \geq 3)\,.$$

Thus (4.6) becomes

$$\sum_{r=0}^{p-1}(rq+s)^k \equiv q^k\left\{pB_k\left(\frac{s}{q}\right) + \tfrac{1}{2}kp^2 B_{k-1}\left(\frac{s}{q}\right)\right\} \equiv pq\left(\frac{q}{p}\right) B_k\left(\frac{s}{q}\right) \pmod{p^{n+1}}\,,$$

where we have used (4.3) and (3.5).

Substituting in (4.4) and (4.2) we therefore get

$$h(d) \equiv -\left(\frac{q}{p}\right)\sum_{s=1}^{q-1}\left(\frac{q_0}{s}\right) B_k\left(\frac{s}{q}\right) \pmod{p^n}\,.$$

Finally using (3.4) this becomes

$$h(d) \equiv -\frac{1}{1-\frac{1}{2}p^{n-1}}\left(\frac{q}{p}\right)\sum_{s=1}^{q-1}\left(\frac{q_0}{s}\right) B_l\left(\frac{s}{q}\right) \pmod{p^n}\,.$$

where $l = \frac{1}{2}(p-1)\,p^{n-1} + 1$. Replacing $n$ by $n+1$ we get

$$h(d) \equiv -c\left(\frac{q}{p}\right)\sum_{s=1}^{q-1}\left(\frac{q_0}{s}\right) B_k\left(\frac{s}{q}\right) \pmod{p^{n+1}}\,, \tag{4.7}$$

where $k$ is defined by (4.5) and $c = 1 + \frac{1}{2}p^n$ for $n > 0$, $c = 2$ for $n = 0$.

In the next place if $p \equiv 1 \pmod 4$, then $k$ is odd so that by (3.3)

$$B_k\left(\frac{q-s}{q}\right) = -B_k\left(\frac{s}{q}\right).$$

Also since $p_0 > 0, q_0 < 0$, so that by (2.4) $\left(\frac{q_0}{q-s}\right) = -\left(\frac{q_0}{s}\right)$.

It follows that

$$\left(\frac{q_0}{q-s}\right) B_k\left(\frac{q-s}{q}\right) = \left(\frac{q_0}{s}\right) B_k\left(\frac{s}{q}\right). \tag{4.8}$$

If $p \equiv -1 \pmod 4$, then $k$ is even and $q > 0$, so that

$$B_k\left(\frac{q-s}{q}\right) = B_k\left(\frac{s}{q}\right), \quad \left(\frac{q_0}{q-s}\right) = \left(\frac{q_0}{s}\right).$$

and again (4.8) follows. Since for $q_0$ even the value $s = q_0/2$ in (4.7) may be ignored we get

$$h(d) \equiv -2c\left(\frac{q}{p}\right) \sum_{1 \le s < q/2} \left(\frac{q_0}{s}\right) B_k\left(\frac{s}{q}\right) \pmod{p^{n+1}}, \tag{4.9}$$

where $k$ and $c$ are the same as in (4.7).

**5. The case $q = 1$.** While (4.9) does not hold for $q = 1$, it is easy to obtain a similar result in that case. We now have $d = -p$, where $p \equiv 3 \pmod 4$. Thus (4.1) becomes

$$h(-p) = -\frac{1}{p} \sum_m m\left(\frac{-p}{m}\right) = -\frac{1}{p} \sum_m m\left(\frac{m}{p}\right).$$

Now

$$\sum_{m=1}^{p-1} m\left(\frac{m}{p}\right) \equiv \sum_{m=1}^{p-1} m^k \equiv \frac{B_{k+1}(p) - B_{k+1}}{k+1} \pmod{p^{n+1}},$$

where $k$ is the same as in (4.5); note that $k$ is even. A little manipulation leads to

$$h(-p) \equiv -B_k \pmod{p^n}. \tag{5.1}$$

In particular for $n = 1$, (5.1) becomes

$$h(-p) \equiv -B_{(p-1)p/2+1} \pmod p,$$

which by (3.4) reduces to

$$h(-p) \equiv -2B_{(p+1)/2} \pmod p. \tag{5.2}$$

**6. Some special cases.** Returning to (4.9) we consider first the special case $d = -3p, p \equiv 1 \pmod 4$. Thus $q_0 = -3, q = 3$ and (4.9) reduces to

$$h(-3p) \equiv -2c\left(\frac{3}{p}\right) B_k\left(\frac{1}{3}\right) \pmod{p^{n+1}}. \tag{6.1}$$

Since $k$ is odd it does not seem possible to further simplify the right member of (6.1). For $n = 0$, (6.1) becomes

$$h(-3p) \equiv -4\left(\frac{3}{p}\right) B_{(p+1)/2}\left(\frac{1}{3}\right) \pmod p. \tag{6.1'}$$

Next for $d = -4p, p \equiv 1 \pmod 4$, $q_0 = -4, q = 4$, so that (4.9) becomes

$$h(-4p) \equiv -2cB_k(\tfrac{1}{4}) \pmod{p^{n+1}}. \tag{6.2}$$

Now we may use the formula [4, p. 29]

$$B_k\left(\frac{1}{4}\right) = -k\frac{E_{k-1}}{4^k} \quad (k \text{ odd}),$$

where $E_{k-1}$ is an Euler number. Since

$$4^k = 2^{2k} = 2^{(p-1)p^n+2} \equiv 4 \pmod{p^{n+1}},$$

it is easily verified that (6.2) gives

$$h(-4p) \equiv \tfrac{1}{2}E_{k-1} \pmod{p^{n+1}}; \tag{6.3}$$

in particular for $n = 0$, we get

$$h(-4p) \equiv \tfrac{1}{2}E_{(p-1)/2} \pmod p. \tag{6.3'}$$

For example for $p = 5, h(-20) = 2, E_2 = -1$.

For $d = -5p, p \equiv 3 \pmod 4$, $q = q_0 = 5$, we have

$$h(-5p) \equiv -2c\left(\frac{5}{p}\right)\left\{B_k\left(\frac{1}{5}\right) - B_k\left(\frac{2}{5}\right)\right\} \pmod{p^{n+1}}; \tag{6.4}$$

in particular

$$h(-5p) \equiv -4\left(\frac{5}{p}\right)\left\{B_{(p+1)/2}\left(\frac{1}{5}\right) - B_{(p+1)/2}\left(\frac{2}{5}\right)\right\} \pmod p. \tag{6.4'}$$

For $d = -8p$, we have either (i) $p \equiv 1 \pmod 4$, $q_0 = -8, q = 8$, or (ii) $p \equiv 3 \pmod 4$, $q_0 = q = 8$. The two possibilities may be combined in the single formula

$$h(-8p) \equiv -2c\left(\frac{2}{p}\right)\left\{B_k\left(\frac{1}{8}\right) + \left(\frac{-1}{p}\right)B_k\left(\frac{3}{8}\right)\right\} \pmod{p^{n+1}}, \tag{6.5}$$

which does not seem to reduce further. Using (3.3) we may however write

$$h(-8p) \equiv -2c\left(\frac{2}{p}\right)\left\{B_k\left(\frac{1}{8}\right) - B_k\left(\frac{5}{8}\right)\right\} \pmod{p^{n+1}}, \tag{6.6}$$

as is easily verified.

We may also mention the case $d = -12p$, where $p \equiv 3 \pmod 4$, $q = q_0 = 12$. Thus (4.9) becomes

$$h(-12p) \equiv -2c\left(\frac{3}{p}\right)\left\{B_k\left(\frac{1}{12}\right) - B_k\left(\frac{5}{12}\right)\right\} \pmod{p^{n+1}}. \tag{6.7}$$

**7. Some additional formulas.** Formula (4.7) becomes somewhat more symmetrical if we introduce the Bernoulli function $\bar{B}_k(x)$ defined in § 3. For we may now write

$$h(d) \equiv -c\left(\frac{q}{p}\right)\sum_{s}\left(\frac{q_0}{s}\right)\overline{B}_k\left(\frac{s}{q}\right) \pmod{p^{n+1}}, \tag{7.1}$$

where $s$ runs through a complete residue system (mod $q$). In the next place, using (3.6), we have

$$\sum_{s=0}^{q-1}\overline{B}_k\left(\frac{s}{q}\right) = q^{1-k}\,\overline{B}_k(0) = q^{1-k}B_k\,. \tag{7.2}$$

Also

$$q^{1-k} = q^{-(p-1)/2\,p^n} \equiv \left(\frac{q}{p}\right) \pmod{p^{n+1}},$$

so that (7.2) becomes

$$\sum_{s=1}^{q-1}\overline{B}_k\left(\frac{s}{q}\right) \equiv \left\{\left(\frac{q}{p}\right) - 1\right\} B_k\,.$$

Combining with (7.1) we get

$$h(d) \equiv -2c\left(\frac{q}{p}\right)\sum_{s}{}'\,\overline{B}_k\left(\frac{s}{q}\right) + c\left\{1 - \left(\frac{q}{p}\right)\right\} B_k \pmod{p^{n+1}}, \tag{7.3}$$

where the sum is now restricted to such $s$ that $(q_0/s) = 1$. If $p \equiv 1 \pmod 4$, so that $k$ is odd, (7.3) reduces to

$$h(d) \equiv -2c\left(\frac{q}{p}\right)\sum_{s}{}'\,\overline{B}_k\left(\frac{s}{q}\right) \pmod{p^{n+1}}; \tag{7.4}$$

if $(q/p) = 1$ then (7.4) holds for all $p$. If $q$ is a prime then (7.3) may also be written in the form

$$h(d) \equiv -c\left(\frac{q}{p}\right)\sum_{s=0}^{q-1}\overline{B}_k\left(\frac{s^2}{q}\right) + cB_k \pmod{p^{n+1}}. \tag{7.5}$$

The last formula suggests that it may be interesting to consider the sum

$$S_k(h, q) = \sum_{s=0}^{q-1}\overline{B}_k\left(\frac{s^2 h}{q}\right) - q^{1-k}B_k \tag{7.6}$$

for arbitrary positive $k$ and $q$. In particular if $q$ is an odd prime power then it follows from (7.2) and (7.6) that

$$S_k(h, q) = \sum_{s=1}^{q-1}\left(\frac{s}{q}\right)\overline{B}_k\left(\frac{sh}{q}\right) \quad ((h, q) = 1)$$

and therefore

$$S_k(h, q) = \left(\frac{h}{q}\right)S_k(1, q)\,. \tag{7.7}$$

In the next place if $q = p^r, r \geq 3$, then

$$S_k(1, p^r) = \sum_{b=0}^{p^{r-1}-1} \sum_{a=0}^{p-1} \overline{B}_k\left(\frac{b^2}{p^r} + \frac{2ab}{p}\right)$$

$$= p \sum_{b=0}^{p^{r-2}-1} \overline{B}_k\left(\frac{b^2}{p^{r-2}}\right) + \sum_{\substack{b=0 \\ p+b}}^{p^{r-1}-1} \sum_{a=0}^{p-1} \overline{B}_k\left(\frac{b^2}{p^r} + \frac{2ab}{p}\right)$$

$$= p\{S_k(1, p^{r-2}) + p^{(r-2)(1-k)} B_k\} + p^{1-k} \sum_{\substack{b=0 \\ p+b}}^{p^{r-1}-1} \overline{B}_k\left(\frac{b^2}{p^{r-1}}\right)$$

$$= p\{S_k(1, p^{r-2}) + p^{(r-2)(1-k)} B_k\} + p^{1-k} \sum_{b=0}^{p^{r-1}-1} \overline{B}_k\left(\frac{b^2}{p^{r-1}}\right)$$

$$- p^{1-k} \sum_{b=0}^{p^{r-2}-1} \overline{B}_k\left(\frac{b^2}{p^{r-3}}\right)$$

$$= p\{S_k(1, p^{r-2}) + p^{(r-2)(1-k)} B_k\}$$

$$+ p^{1-k}\{S_k(1, p^{r-1}) + p^{(r-1)(1-k)} B_k\}$$

$$- p^{2-k}\{S_k(1, p^{r-3}) + p^{(r-3)(1-k)} B_k\},$$

so that

$$S_k(1, p^r) = p^{1-k} S_k(1, p^{r-1}) + p S_k(1, p^{r-2}) - p^{2-k} S_k(1, p^{r-3}) + p^{r(1-k)} B_k. \tag{7.8}$$

For $r = 2$, we find that

$$S_k(1, p^2) = p^{1-k} S_k(1, p) + (p - p^{1-k} + p^{2(1-k)}) B_k. \tag{7.9}$$

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