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## A partition formula connected with Abelian groups

By P. Hall, Cambridge (England)

Let $p$ be a given prime. The object of this note is to prove the following rather curious result.

The sum of the reciprocals of the orders of all the Abelian groups of order a power of $p$ is equal to the sum of the reciprocals of the orders of their groups of automorphisms.

It is well known that the Abelian groups of order $p^{n}$ stand in (1-1) correspondence with the $\omega(n)$ unrestricted partitions of $n$, the partition corresponding to a given Abelian group being called its type.

Thus the sum of the reciprocals of the orders of all the Abelian groups of order a power of $p$ is equal to

Writing

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{\omega(n)}{p^{n}} \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
f_{n}(x) \equiv(1-x)\left(1-x^{2}\right)\left(1-x^{3}\right) \ldots\left(1-x^{n}\right), \quad f_{0}(x)=1 \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\varrho=\frac{1}{p} \tag{3}
\end{equation*}
$$

the value of the sum (1) is easily seen to be

$$
\begin{equation*}
\frac{1}{f_{\infty}(\varrho)} \tag{4}
\end{equation*}
$$

But, by an identity due to Euler, this is the same as

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{\varrho^{n}}{f_{n}(\varrho)} \tag{5}
\end{equation*}
$$

And the theorem mentioned above will accordingly follow once we have shown that the sum of the reciprocals of the orders of the groups of automorphisms of the $\omega(n)$ Abelian groups of order $p^{n}$ is equal to $\varrho^{n} \mid f_{n}(\varrho)$.

For partitions we use the notation of Macmahon. Thus, the Abelian group $G$ of order $p^{n}$ and type

$$
\left(\begin{array}{llll}
1^{\lambda_{1}} & 2^{\lambda_{2}} & 3^{\lambda_{3}} & \cdots \tag{6}
\end{array}\right)
$$

is the direct product of cyclic groups, $\lambda_{1}$ of which are of order $p, \lambda_{2}$ of order $p^{2}, \lambda_{3}$ of order $p^{3}$, and so on. Clearly,

$$
\begin{equation*}
n=\lambda_{1}+2 \lambda_{2}+3 \lambda_{3}+\cdots \tag{7}
\end{equation*}
$$

The partition of $n$ which is associated with the partition (6) has the parts $\mu_{1}, \mu_{2}, \ldots$ given by

$$
\begin{equation*}
\mu_{i}=\lambda_{i}+\lambda_{i+1}+\lambda_{i+2}+\cdots \quad(i=1,2, \ldots) \tag{8}
\end{equation*}
$$

Thus, since $\lambda_{i} \geqslant 0$ for each $i$, we have

$$
\begin{equation*}
\mu_{1} \geqslant \mu_{2} \geqslant \cdots \quad \geqslant 0 \tag{9}
\end{equation*}
$$

And plainly, from (7) and (8),

$$
\begin{equation*}
\mu_{1}+\mu_{2}+\cdots \quad=n \tag{10}
\end{equation*}
$$

Conversely, given any partition of $n$ in the form (9), (10), the associated partition (6) is obtained at once by the rule that

$$
\begin{equation*}
\lambda_{i}=\mu_{i}-\mu_{i+1} \quad(i=1,2, \ldots) \tag{11}
\end{equation*}
$$

The associated partition has a simple meaning for the group $G$. Let $G_{k}$ denote the characteristic subgroup of $G$ which consists of all elements of $G$ of order $p^{k}$ or less. Then

$$
1=G_{0}<G_{1}<G_{2}<\cdots<G_{m}=G
$$

where $m$ is the largest of the type invariants ${ }^{1}$ ) of $G$, and the order of $G_{k} \mid G_{k-1}$ is precisely $p^{\mu_{k}}$.

Now the order of the group of automorphisms of $G$ can be expressed very simply in terms of the "associated invariants" $\mu_{k}$. It is ${ }^{2}$ )

$$
\begin{equation*}
\frac{f_{\mu_{1}-\mu_{2}}(\varrho) f_{\mu_{2}-\mu_{3}}(\varrho) \cdots}{\varrho^{\mu_{1}^{2}+\mu_{2}^{2}+\cdots}} \tag{12}
\end{equation*}
$$

And the result we require to prove is the case $x=\varrho$ of the identity

$$
\begin{equation*}
\frac{x^{n}}{f_{n}(x)}=\sum_{(\mu)} \frac{x^{\mu_{1}^{2}+\mu_{2}^{2}+\cdots}}{f_{\mu_{1}-\mu_{2}}(x) f_{\mu_{2}-\mu_{3}}(x) \ldots} \tag{13}
\end{equation*}
$$

the sum being taken over all $\omega(n)$ partitions (9), (10) of the number $n$.
The various terms of (13) may be regarded as the generating functions of partitions or compositions of certain definite kinds. For example, the coefficient of $x^{N}$ on the left of (13) is equal to the number of partitions of $N$ for which the greatest part is exactly $n$. As a first step in the proof of the identity, we shall connect every such partition of $N$ with a particular

[^0]one of the $\omega(n)$ partitions (9), (10) of $n$, and thereby with a particular one of the $\omega(n)$ summands on the right of (13).

This may be done most conveniently by means of the graph $^{3}$ ) of the partition of $N$ in question. Let the parts of this partition, arranged in descending order of magnitude be $N_{1}, N_{2}, \ldots$, so that we have

$$
\begin{gather*}
n=N_{1} \geqslant N_{2} \geqslant \cdots, \\
N_{1}+N_{2}+\cdots=N . \tag{14}
\end{gather*}
$$

Then its graph may be defined to consist of a set of $N$ coplanar latticepoints, viz. all those points whose Cartesian coordinates $(x, y)$ are positive integers satisfying

$$
\begin{equation*}
x \leqslant N_{y} \tag{15}
\end{equation*}
$$

(When $y$ exceeds the number of parts of (14), we take $N_{y}=0$.)
We are now able to define, successively, the numbers $\mu_{1}, \mu_{2}, \ldots$, which correspond to the partition (14).

We take $\mu_{1}$ to be the greatest integer such that the point $\left(\mu_{1}, \mu_{1}\right)$ belongs to the graph (15). Next, supposing that $\mu_{1}, \mu_{2}, \ldots, \mu_{i-1}$ have already been defined, and that their sum is less than $n$, we define $\mu_{i}$ to be the greatest integer such that $\left(\mu_{1}+\mu_{2}+\cdots+\mu_{i}, \mu_{i}\right)$ is a point of the graph.

It follows at once, from (14) and (15), that the numbers $\mu_{i}$ so defined satisfy (9) and (10). Plainly, also, the square of $\mu_{i}^{2}$ lattice-points having for opposite corners the points $\left(\mu_{1}+\cdots+\mu_{i-1}+1,1\right)$ and ( $\mu_{1}+\cdots+$ $\mu_{i}, \mu_{i}$ ) belongs entirely to the graph. Thus, if we write

$$
\begin{equation*}
M=N-\mu_{1}^{2}-\mu_{2}^{2}-\cdots, \tag{16}
\end{equation*}
$$

there remain, outside the squares just mentioned, precisely $M$ further points of the graph.

We divide these $M$ remaining points into sets, according to the values of their $x$-coordinates. Let the number of them which lie in the strip $0<x \leqslant \mu_{1}$ be $M_{1}$. And, for any $i>1$, let the number which lie in the strip $\mu_{i-1}<x \leqslant \mu_{i}$ be $M_{i}$. If the number of $\mu$ 's is $r$, we obtain in this way a definite composition ${ }^{4}$ ) of $M$,

$$
\begin{equation*}
M=M_{1}+M_{2}+\cdots+M_{r}, \tag{17}
\end{equation*}
$$

into $r$ non-negative integers, this composition, like the partition (9), (10), being uniquely determined by the original partition (14) of $N$.

[^1]As a final consequence of (14), (15), we remark that (for each $i=1,2, \ldots, r)$ the $M_{i}$ points of the $i$-th strip constitute, a translation apart, the graph of a certain partition $P_{i}$ of $M_{i}$, these partitions $P_{i}$ being, just as much as the numbers $M_{i}$ themselves, uniquely determined by (14). Further, for each $i$, the greatest part of $P_{i}$ is not greater than $\mu_{i}$. And, for each $i>1$, the number of parts of $P_{i}$ is not greater that $\mu_{i-1}-\mu_{i}$.

But, conversely, suppose that we choose any set of positive integers $\mu_{i}$ satisfying (9) and (10), the sum of whose squares does not exceed $N$, and define $M$ by (16); then choose any composition (17) of $M$, one part $M_{i}$ corresponding to each $\mu_{i}$, taking care that

$$
M_{i} \leqslant \mu_{i}\left(\mu_{i-1}-\mu_{i}\right) \quad(i>1) ;
$$

and finally, for each $M_{i}$, choose arbitrarily a partition $P_{i}$ having its greatest part not greater than $\mu_{i}$ and having (for $i>1$ ) not more than $\mu_{i-1}-\mu_{i}$ parts.

Then it is obvious that we can reverse our former construction at every step, and arrive at a definite partition (14) of $N$, which has $n$ as its greatest part, and for which the corresponding $\mu$ 's, $M$ 's and $P$ 's are precisely the ones we have chosen.

If, then, we denote by $\psi_{a, b}(x)$ the generating function for the partitions of $N$ into at most $a$ parts none of which exceed $b$, we have proved the identity

$$
\begin{equation*}
\frac{x^{n}}{f_{n}(x)}=\sum_{(\mu)} \psi_{\infty, \mu_{1}}(x) \psi_{\mu_{1}-\mu_{2}, \mu_{2}}(x) \psi_{\mu_{2}-\mu_{3}, \mu_{8}}(x) \ldots x^{\mu_{1}^{2}+\mu_{2}^{2}+\cdots} \tag{18}
\end{equation*}
$$

the sum being taken over all partitions (9), (10) of $n$. But it is known ${ }^{1}$ ) that, for finite $a$ and $b$,

$$
\psi_{a, b}(x)=\frac{f_{a+b}(x)}{f_{a}(x) f_{b}(x)}
$$

while

$$
\psi_{\infty, b}(x)=\frac{1}{f_{b}(x)}
$$

Substituting these values in (18), we obtain the required identity (13). This concludes the proof.
(Eingegangen den 17. September 1938.)

[^2]
[^0]:    ${ }^{1}$ ) I. e. $\lambda_{m}>0, \lambda_{m+1}=\lambda_{m+2}=\cdots=0$.
    ${ }^{2}$ ) Cf. A. Speiser, Theorie der Gruppen von endlicher Ordnung, 3er. Aufl., §43, Satz 114.

[^1]:    ${ }^{3}$ ) P.A. Macmahon, Combinatory Analysis, II, 3. Our graph reads upwards, not downwards as in Macmahon.
    ${ }^{4}$ ) A composition is a partition in which the order of the summands is important.

[^2]:    $\left.{ }^{5}\right)$ P. A. Macmahon, loc. cit., 5.

