Essentials for discrete crack analysis

L'essentiel en ce qui concerne le calcul des fissures distinctes

Das Wichtigste über die Berechnung mit diskreten Rissen

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SUMMARY

Reinforced concrete structures sometimes display a failure behaviour which is dominated by one or a few discrete sharp cracks. An analysis can only predict such failure types if discrete cracks are allowed to develop in any direction, and to cross any element. It is argued that equilibrium models in the finite element method meet the requirements for an accurate simulation of this cracking problem.

RÉSUMÉ

Le comportement à la rupture de constructions en béton armé est souvent déterminé par quelques fissures distinctes. Seul une analyse permettant le développement de fissures dans toute direction, à travers tout élément, est capable de prévoir une telle rupture. Il est démontré que la méthode des éléments finis à condition d'équilibre, correspond aux exigences d'une simulation précise de la fissuration.

ZUSAMMENFASSUNG

Das Verhalten von Stahlbetonkonstruktionen wird manchmal von einem oder mehreren diskreten Rissen bestimmt. Um dieses Verhalten gut vorhersagen zu können, verwendet man am besten finite Elemente, die in verschiedenen Richtungen reissen können. Es wird gesagt, dass Gleichgewichtsmodelle die Anforderungen an die genaue Risssimulation erfüllen können.

1. SCOPE

Structural analysis of reinforced concrete is done, roughly speaking, in two distinct ways. One uses either an approach in which cracks are smeared out, or one uses single sharp interelement cracks. The first approach, using an average stiffness of cracked elements, seems to apply well in all cases where one can observe cracks in concrete which tend to be diffuse and spread over a large zone. The latter approach, handling interelement sharp cracks, is promising for the analysis of structures in which one or a few discrete cracks dominate the behaviour. Both approaches have their advantages and draw-backs.
The smeared out approach can give widely different results for crack propagation depending on the choice of the finite element mesh, and is, therefore, unobjective. Bažant [1] has proposed a propagation criterion to overcome this serious problem. Using the same concept as in fracture mechanics of sharp cracks and accounting for the effect of bond-slip across the crack, he refines the modelling in terms of element-wide blunt smeared crack band.
The paper presented here enters into the subject of an analysis with discrete cracks and aims for an enhancement of the single sharp interelement crack approach. Untill now, one splits each node in two when the crack advances, requiring node renumbering and changes in topological connectivity of the mesh with the necessary recalculations of the structural stiffness matrix. Moreover it can be necessary to vary the direction of the interface between two finite elements and move the location of the mode into which the crack is about to advance. The big advantage of this approach above the smeared out concept is the fact that actual crack spacings and widths can be predicted. The present method maintains these advantages and avoids the forementioned difficulties. The element mesh needs not be changed any more and single cracks are allowed to propagate in each possible direction regardless the chosen mesh. In this way the approach becomes suitable for the analysis of complete concrete structures. We will discuss the requirements to meet this goal.

2. PHENOMENA TO BE MODELLED

An adequate model to analyze a reinforced structure accounting for single cracks at arbitrary spacings, which is capable to predict correct crack widths and crack spacings, must meet the following requirements:

a. Constitutive laws for plain concrete must be available, including a crushing criterion and a cracking criterion. A similar requirement holds for the reinforcement bar.
b. A proper model has to be chosen for the description of the force transfer in a crack. This model must allow for shear stresses and normal stresses in the crack layer due to aggregate interlock. A relation between these stresses and the corresponding relative displacements of the crack faces (shift and dilatancy) must be available.
c. A suitable model has to be chosen for the bond-slip zone in between the reinforcement bar and the surrounding concrete. This model must provide a relation between the shear stress and normal (radial) stress on the one hand and the corresponding displacement components (parallel and radial) on the other hand.
d. The complex local state of stress at an intersection of a crack and a reinforcement bar, with typical high gradients for stresses, must be approached as close as possible. The reinforcement bar acts as a flexible connection between the two crack faces.

In order to make good predictions of the crack widths, the real bond spring behaviour near the crack must be modelled.
So the abrupt change of sign of bond shear stresses at the crack must be accounted for in this respect. A similar want applies for the dowel action of the reinforcement bar due to a parallel shift of the crack faces relative to each other. However this feature seams to be of much less influence on the behaviour of the structure than the bond-slip phenomenon. Dowel action will develop only noticeably after the structure has already started to fail.

e. The correct failure mechanism must be predicted. From experimental evidence, for instance [2], we know that the ultimate load capacity of a structure strongly depends on the crack pattern. This applies especially for failure modes due to combinations of bending moment and shear force. Failure is induced then by one or a few dominating cracks and displays a brittle character. An inaccurate destination of the crack position may influence the ultimate load seriously. The analysis model must also apply, of course, for ductile models of failure, as is the case in pure bending for low percentages of main reinforcement.

3. FINITE ELEMENT REALIZATION

The authors feel that the requirements listed in section 2 are best met if two decisions are made concurrently:

a. The finite element formulation must allow for single cracks at discrete arbitrary spacings. The specifications of the program must be so conceived, that no crack direction is enforced by the element mesh or by the way the program was written. A crack can come into being at any position in the mesh and can from there propagate in any direction through any element.
b. A proper description of the stress state is of much importance. Therefore, out of all known finite element formulations, an equilibrium model, which starts from assumed stress fields, is the most promising one [3]. The authors particularly prefer a modified version of the hybrid model [4] using (what has been called) *natural boundary displacements* [5]. This finite element model is used in combination with the *initial strain* concept to process nonlinear effects.

We will discuss these two decisions in more detail. The explanation will be restricted to states of plane stress.

3.1. Equilibrium model for the uncracked state

Let us start with the uncracked state of reinforced concrete. We basically adopt an approach of apart two-dimensional elements of plain concrete and apart one-dimensional elements of reinforcement. Therefore, concrete elements never contain reinforcement steel. The rebar elements run in between concrete elements. The rebar element is attached to the concrete element with the aid of a bond linkage element as described in section 2c. In physical reality full interelement equilibrium of surface tractions exists. It also holds that the integral of bond stresses along a rebar element yields the stress resultants in the cross-section of a rebar element. To model this properly, one needs a finite element formulation which basically is an equilibrium model. The wide-spread stiffness models concentrate on compatibility rather than on equilibrium, and are therefore less suited with regard to our requirements. The equilibrium model, used by the authors, has been derived from the hybrid model proposed by Pian [4].

This model starts from the following assumptions:

- stresses σ_{ij} are in equilibrium with the given load q_i all over the volume V_e of an element.
- it is not enforced a priori that the strains ε_{ij} are compatible with the displacement field u_i over the volume V_e of an element.
- on the interelement boundaries A_e nor continuity of tractions $(p_i^- = p_i^+)$ is enforced a priori, nor compatibility of displacements $(u_i^- = u_i^+)$.
- on the external boundary it is adopted that displacements u_i take the prescribed value u_i^o at the part A_u of the boundary. However, it is not enforced that the element tractions p_i equalize the given load p_i^o at the part A_p of the boundary.

All these appointments result in one variational condition to be fulfilled for all M elements:

$$\sum_{e=1}^{M} \left[\iiint_{V_e} \left\{ \varepsilon_{ij} - \frac{1}{2} (u_{i,j} + u_{j,i}) \right\} \delta\sigma_{ij} \, dV + \iint_{A_e} (u_i^- - u_i^+) \, \delta p_i \, dA - \iint_{A_e} (p_i^- + p_i^+) \, \delta u_i \, dA \right] + \iint_{A_p} (-p_i + p_i^o) \, \delta u_i = 0 \tag{1}$$

Application of the divergence theorem and re-ordering of terms yields

$$\sum_{e=1}^{M} \left\{ \iiint_{V_e} \varepsilon_{ij} \, \delta\sigma_{ij} \, dV - \oint\!\!\oint_{A_e} p_i \, \delta u_i \, dA \right\} + \iint_{A_p} p_i^o \, \delta u_i \, dA = 0 \tag{2}$$

In this expression the surface integral must be taken over the full boundary A_e of each element, also for the elements which join the external boundary A_p. We now introduce the constitutive law, using the flexibility relation for strains and stresses:

$$\varepsilon_{ij} = F_{ijkl} \, \sigma_{ij} \tag{3}$$

This enables us to make the following interpretation of the variational condition (2). When introducing the functional F:

$$F = \sum_{e=1}^{M} \left\{ - \iiint_{V_e} \frac{1}{2} F_{ijkl} \, \sigma_{ij} \sigma_{kl} \, dV + \oint\!\!\oint_{A_e} p_i \, u_i \, dA \right\} - \iint_{A_p} p_i^o \, u_i \, dA \tag{4}$$

the condition can be restated:

$$\delta F(\sigma_{ij}, u_i) = 0 \tag{5}$$

The functional F must be stationary in respect of variations of the element stresses σ_{ij} and the element boundary displacements u_i. This facilitates to choose a stress distribution over the whole element volume and to restrict ourselves to a distribution for the displacements which is defined along the element edges only.

Normally, this variational principle is used as follows. One choses nodes at the element corners and these nodes are common for all elements which join together at that position. In case of plane stress, one defines two degrees of freedoms for the displacements in each node and has to interpolate the displacements along the element boundaries. This means that the displacement fields for the several elements are interconnected. On the contrary, stress interpolations are full indepently chosen within each element. Execution of the variational process yields a set of equations similar to the standard stiffness method. This type of finite element method, referred to as *hybrid* method, appears to the user as a displacement method type of analysis. Fig. 1 shows the degrees of freedom in such an analysis (left part of figure).

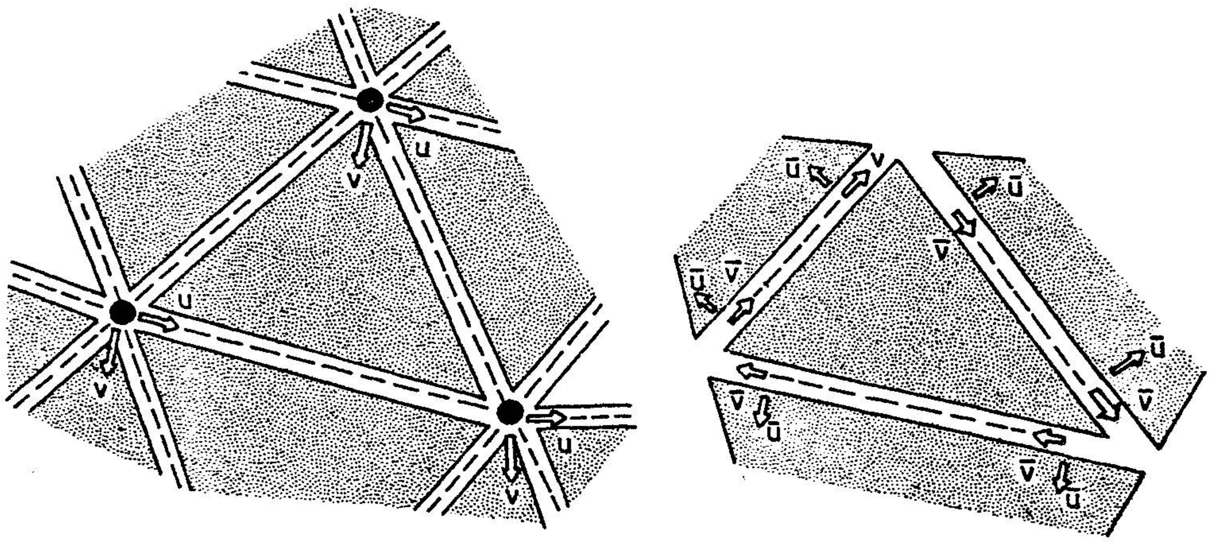

Fig. 1 Standard type of hybrid elements (left) and modified model with natural boundary displacements (right).

It will be clear that the hybrid method does not fully meet the requirements for a good crack analysis, stated in section 2. We still lack the wanted interelement stress continuity. The stress parameters can be eliminated per element without any assurance that stresses (tractions) will take the same values σ^-_{ij} and σ^+_{ij} at different sides of the interelement boundary.

This assurance does be got, if one so modifies the hybrid method, that degrees of freedom are not common to alle elements joining at a corner, but only common for two elements which share one single straight edge.
If one, additionally, uses the same distribution for the tractions (stresses) in both elements and above that also for the displacements along this edge, interelement stress continuity will be achieved automatically. Said in another way, one then gets a more natural stress transfer across element boundaries, and for that reason the name *natural boundary displacements* was introduced [5].Mathematically, the natural degrees of freedom are Lagrangian multipliers in a variational scheme. The method has been applied for the discrete crack analysis, reported in [6] and [7]. In this analysis a triangle is chosen and linear stress interpolations are applied, in connection with linearly varying natural boundary displacements. Fig. 1 shows in the right hand part a picture of the degrees of freedom which are needed in that case. It will be clear that special care must be taken for the programming of such a type of analysis, but using a frontal solution scheme, one will not have too big problems. The rebar element to be used in this method is of the special type shown in Fig. 2. The interaction between, concrete elements, and rebar elements is performed by natural boundary displacements, and the force transfer from one rebar element to another rebar element by standard degrees of freedom.

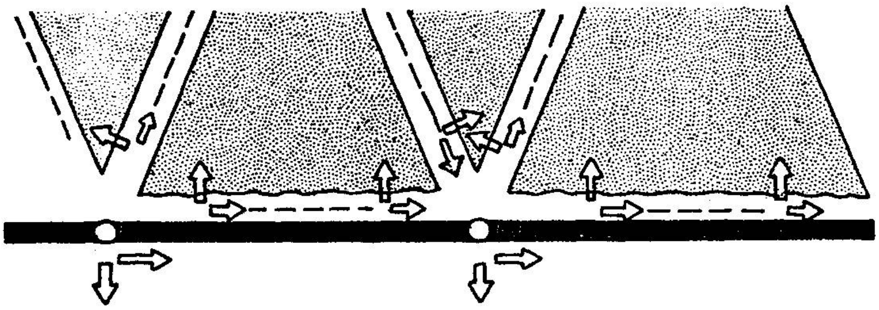

Fig. 2 *Special type of reinforcement bar element in the equilibrium model with natural boundary displacements*

3.2. Equilibrium model in the cracked state

We now consider the cracked state of a reinforced concrete structure. For the purpose of this paper we assume that just one crack occurs per element, but in general more cracks may cross an element.
Let us first observe a cracked concrete element that is not linked to a rebar element. Everyting is available in the formulation for elements with natural boundary displacements to handle this crack. A shear stress and normal stress in the crack can develop and full continuity of these stresses has been preserved between the faces of the two element parts after cracking.

Fig. 3 shows crack displacements u_i, u_j and v which have been chosen in [6]. These quantities, defining the crack opening and the crack shift, are processed as initial strains, as is done for all other nonlinear effects.
It is highly questionable if this approach would be possible in the standard stiffness method based on an assumption for the displacement field. Quite surely one needs additional degrees of freedom in that case.

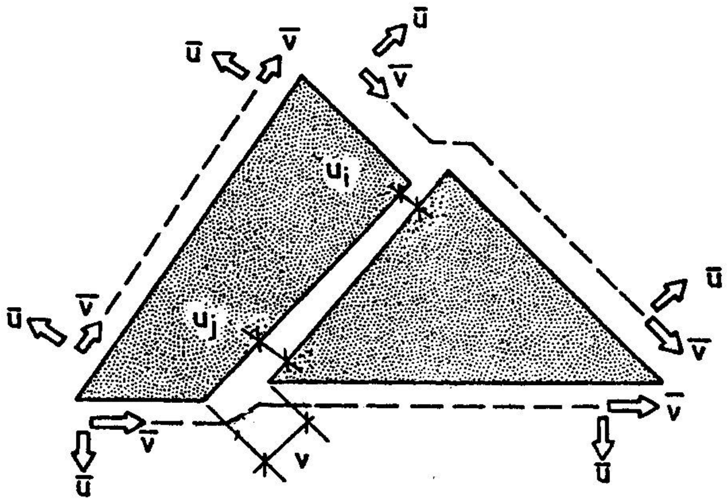

Fig. 3 *Crack displacements u_i, u_j and v are handled as initial strains. No additional degrees of freedom are needed.*

Now we move to observe a cracked concrete element that has been linked to a rebar element. In this case an intersection of the crack and the rebar element can occur, and abrupt gradients for the bond stresses and dowel stresses will come into being. Especially the dowel action of the reinforcement bar asks for additional features in the model. This is achieved by assuming an additional stress field in the cracked concrete element. This extra field must fullfil the continuity of stresses in the crack, but must additionally allow for the rapidly changing distribution of tractions between the element edge and the rebar. It has been shown in [6] that this can be achieved with the triangular elements for concrete described above. Now extra degrees of freedom are needed in the natural boundary displacements. The displacements discontinuities Δu° and Δv° in the natural displacements on the boundary have been introduced for this purpose, see Fig. 4.

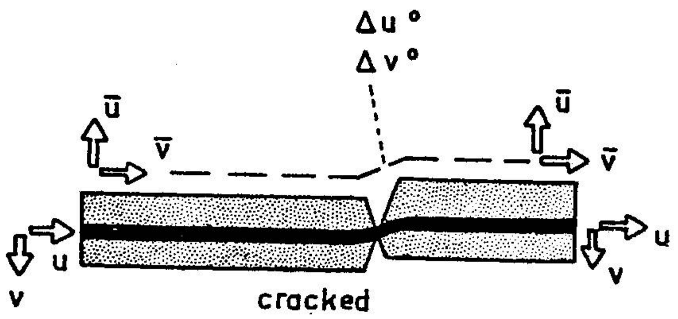

Fig. 4 *To account for dowel action, one needs additional degrees of freedom Δu° and Δv° and extra discontinuous stress fields*

The crack displacements u_i, u_j and v are treated as initial strains in the analysis and the discontinuities Δu^o and Δv^o in the natural boundary displacements determine an additional set of equations which has to be solved simultaneously with the global set of degrees of freedom which already exists in the uncracked state. This global set remains unchanged, no re-analysis of the stiffness matrix being needed. If dowel action is disregarded, the extra set of equations will not be needed.

We see that it is easy to handle internal element displacement discontinities. Obviously it is far more easy than in a formulation based on a displacement field interpolation over the whole element volume.

4. EVALUATION

In this final section some reflections of more comtemplative nature are presented. One may ask after the convergence of the process and put the question if the algorithm does yield correct internal crack displacements and the correct failure mechanism.

4.1. The advantage of an equilibrium model

To judge the chance for a correct crack displacement prediction, we may use an energy examination. When a crack crosses a structure, the two separated parts may move relative to each other. A displacement mechanism may occur in which the parts displace in a rigid body mode. Then, the energy dissipation in the crack, must be equal to the work performed by the external loading. Only when the stresses in the crack are in equilibrium with the external loading, we do get from this equality the correct internal crack displacement associated with a rigid body mode displacement of the structural parts. In the hybrid element formulation the stresses over an arbitrary line are in general not in equilibrium with the external loading, so we may not expect to get the correct internal crack displacements. And even when, after cracking, a rigid body mode is possible, we are not sure to find this mode in our calculation. In the element formulation with natural boundary displacements the stresses on the element boundaries are in equilibrium with the external loading. This applies however also for the stresses along any arbitrary line. So with this model we can expect to find the right crack displacement. And, what is even more important, we can expect to find the right failure mechanism.
Calculations performed with the MICRO-model have shown that with this model the different kinds of failure as bending failure, shear failure or combinations of these two can be analysed with great accuracy. This applies for ductile behaviour as well as brittle behaviour.
Fig. 5 shows a typical result for a brittle failure in shear.

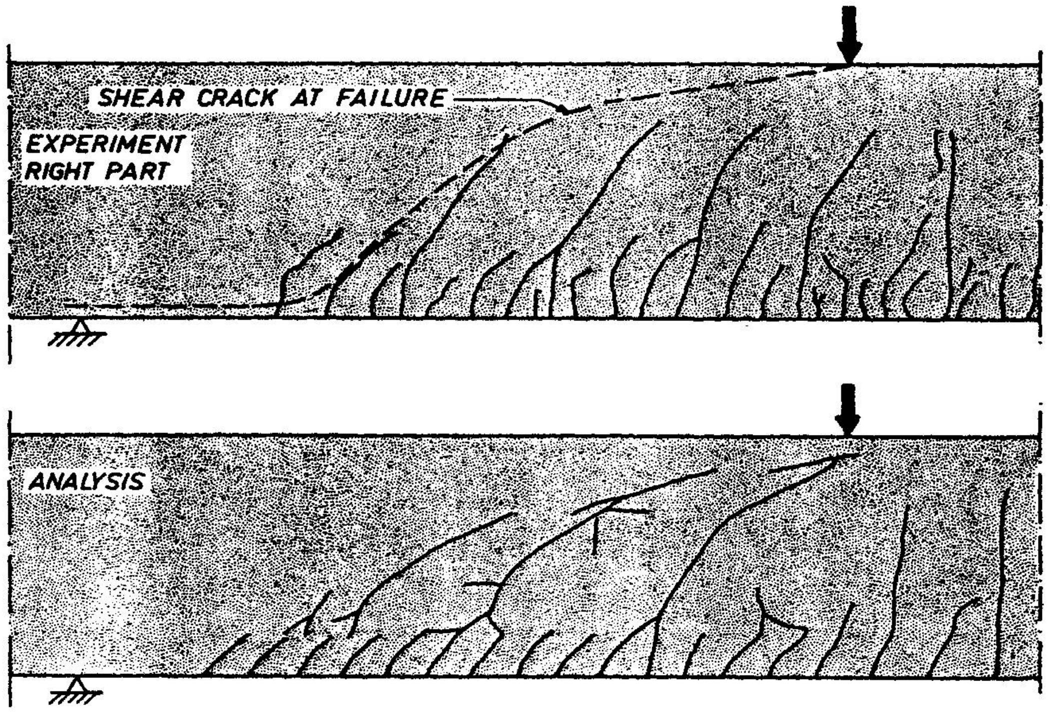

Fig. 5. A typical result for an analysis in which a few single sharp cracks dominate the brittle shear failure

4.2. Parellel stiffness versus serial stiffness

We want to make another comment on the use of an equilibrium model on the basis of an assumed stress field. Most research-workers apply the stiffness method on the basis of an assumed field of displacements and use the concept of smeared-out cracks with a scheme for numerical integration. This means, mostly, that they can make use of standard finite element programs for the analysis of cracked reinforced structures. It proves that such models are well suited for the prediction of ductile bending failure, but they yield bad predictions for brittle shear failure. The authors feel that this is due to the limited possibilities of the smeared-out concept with numerical integration to model a failure mechanism in which only a few cracks will be formed. Because of the assumed displacement field, and therefore an assumed strain field of limited freedom, the effect of the individual Gaussian points in one element is of a parallel nature and is not serial. Only an element, in which all integration points have been cracked, has the possibility of stressless deformation, but even when all integration points of the element are cracked, while the crack direction is not exactly the same in these points, the element has still a residual stiffness. This results in an over-estimation of the failure load and a much larger crack zone at failure than in reality. In a model with an assumed stress field such problems do not arise. The strain field is now free to develop and even can reach an infinite value at a crack. Because of the use of the constitutive law in a flexibility form, the stiffnesses are chained serially.

Then it is possible to develop a single sharp crack and a more correct failure mechanism is avaited for.

4.3. Convergence and crack propagation

Cracks tend to come in being abruptly, but in the MICRO/1 program crack displacements only can grow gradually. This has been reached by introducing a fictitious visco-plastic model for the behaviour of the crack layer. The convergence of this model does not only depend on the fictitious time integration procedure but also on the validity of the assumption that the unbalanced stresses in the crack will decline when the crack displacements are altered. From a physical point of view this seems obvious, but this will certainly not be the case for all finite element models. The best guarantee to get convergence is an accurate description of the internal crack displacements, as is the case in the MICRO/1 program.

When increasing the load on the structure, cracks may whether or not propagate. In fact one needs a proper criterion to decide on this matter. In each loading step one must correctly predict whether a new crack will come into being, or an existing crack will propagate, or maybe both at the same time. No doubt the real stress state around a crack tip shows a singularity with high stress peaks. One can choose for a suited energy criterion as is done for crack propagation in metals. Bažant has proposed such an approach in the framework of smeared out cracks. Until now the authors used a much simpler criterion. The stress field at the tip of a crack is not adapted. Instead, to account for the stress singularity, the allowable tensile stress is lowered down 20 percent in uncracked elements which are positioned at a crack tip. It has been found in this way that a good balance exists between the propagation of existing cracks and the origin of new ones.

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