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Matrix Formulation of the Force Method for a Structure Curved in Space

Expression matricielle de la méthode des forces appliquée à une structure tri-dimensionnelle courbe

Matrizielle Formulierung der Kräftemethode für ein räumlich gekrümmtes Tragwerk

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Introduction

The matrix formulation presented herein is designed for highspeed analysis, with a digital computer, of space structures. It is tremendously advantageous when a large number of load combinations must be considered. The method is applicable to various types of structures, but the present paper deals particularly with a single branch curved in space and to a lesser extent with systems of such interconnected branches.

The branches considered in this paper consist of a single flexural structure curved in any arbitrary fashion in space. This structure may be of variable cross section and may be subjected to loads in any direction in space. The effect of externally applied moments about any axis can also be included.

Besides moments and forces along the branch, the procedure yields fixed-end moments and stiffness coefficients of individual branches for use in the analysis, by a displacement method, of a structural system conisting of interconnected branches. The analysis of such systems has been illustrated in previous papers [1, 2].

Results for the interconnected system of branches can be obtained by superimposing, on results for single branches, corrections due to rotations and displacements at the connections joining the branches. In the previous papers [1, 2], however, the analysis of individual branches and the calculation of stiffness coefficients was performed by a numerical procedure [3, 4] that is not particularly suitable for computer operations.

Moment distribution methods have also been used for multibranch networks [2, 4, 5, 6].

Sign Convention

Fig. 1 shows a structure continuous between two supports. It may be of variable cross section and have any shape in space, and it may be subjected to any system of applied loading. One end is designated a front face and the other end a back face. Positive moments M about each of three orthogonal directions, and positive forces V along each of the same directions are shown at the front face and at the back face. Use the right-hand-screw rule for sense of moments.

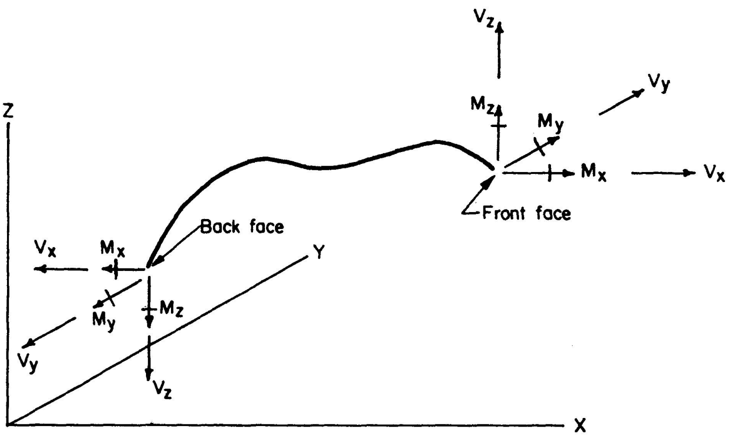

Fig. 1.

A right-hand system of axes is used in this paper. In a previous method of analysis [3, 4] a left-hand system was used, although this was later changed by Baron [7] who put that method of analysis in matrix form.

In an analysis the structure is subdivided into a number of relatively short segments with front and back faces. The back face of a segment is that first reached in traversing from the back face of the structure towards its front face. For all segments, moments and forces are positive on a front face when they are in the direction of the axes adopted. Moments and forces are positive on a back face when they are opposite to the direction of the axes.

The identical sign convention is used for rotations and displacements at the ends of the structure and for angle changes in each segment of the structure. Thus, positive rotations or angle changes correspond to the sense of positive moments shown in Fig. 1, and positive displacements correspond to the direction of positive forces in the same figure.

Segment Flexibilities

Before proceeding with the analytical method, a matrix formulation of the flexibilities of a differential segment, *ds*, is obtained. Such a segment of a space structure is shown in Fig. 2. Axes 1, 2, and 3 are principal (and orthogonal) axes of the cross section of the segment. Section $a-a$ is taken looking forward along *ds*. Axis Y' is perpendicular to axis 1 and parallel to the XY-plane. Axis X' is perpendicular to axis 1 and to axis Y'. The angle ψ is measured clockwise from axis X'.

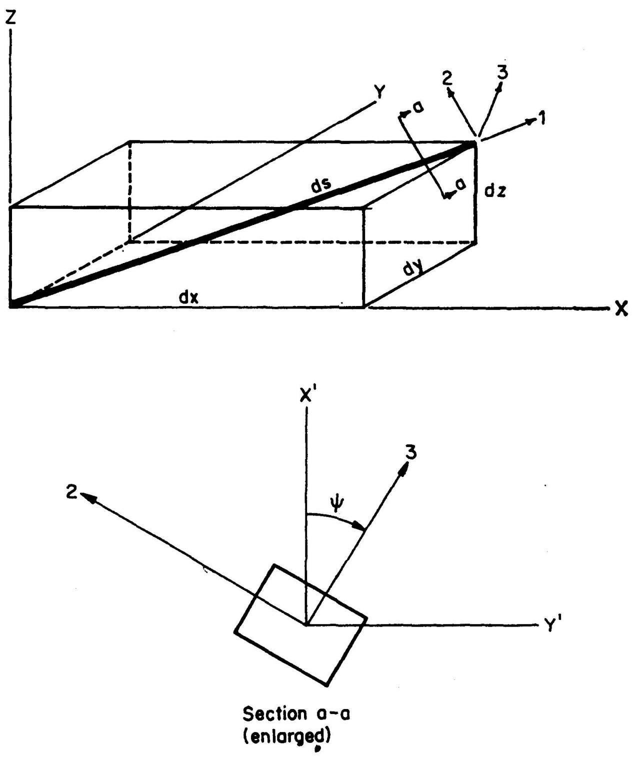

Fig. 2.

Let unit vectors along axes 1, 2, and 3 be q_1, q_2, and q_3 respectively. These unit vectors can be referred to the general XYZ-coordinate system of the structure through use of the transformation matrix

$$[t] = \begin{bmatrix} q_{1x} & q_{2x} & q_{3x} \\ q_{1y} & q_{2y} & q_{3y} \\ q_{1z} & q_{2z} & q_{3z} \end{bmatrix} \tag{1}$$

the elements of which are the x, y, and z components of the three unit vectors. Using Fig. 2 to substitute expressions for these components, Eq. (1) becomes

$$[t] = \begin{bmatrix} \dfrac{dx}{ds} & \dfrac{-dy}{\sqrt{(dx)^2+(dy)^2}} & \dfrac{-(dz)(dx)}{ds\sqrt{(dx)^2+(dy)^2}} \\ \dfrac{dy}{ds} & \dfrac{dx}{\sqrt{(dx)^2+(dy)^2}} & \dfrac{-(dy)(dz)}{ds\sqrt{(dx)^2+(dy)^2}} \\ \dfrac{dz}{ds} & 0 & \dfrac{\sqrt{(dx)^2+(dy)^2}}{ds} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}. \tag{2}$$

If $dx = dy = 0$ for any segment of the structure, Eq. (2) is not determinate. For such a case no angle ψ is defined. Instead, defining an angle α measured clockwise when looking forward along the segment (counterclockwise when looking backward, as in Fig. 3),

$$[t] = \begin{bmatrix} 0 & -\sin\alpha & -\cos\alpha \\ 0 & \cos\alpha & -\sin\alpha \\ 1 & 0 & 0 \end{bmatrix}. \tag{3}$$

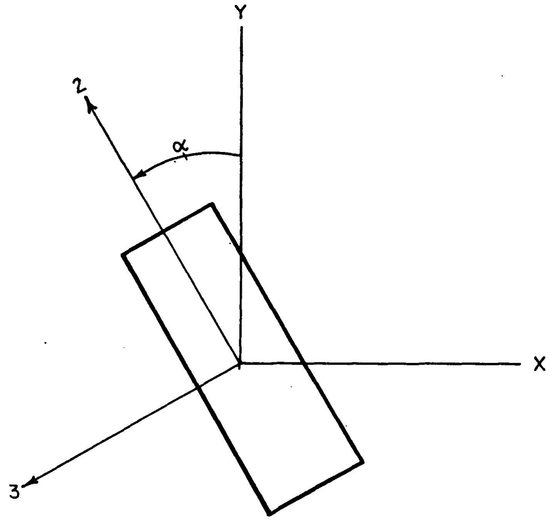

Fig. 3.

Since axes 1, 2, and 3 in Fig. 2 are principal axes, the segment flexibility matrix with respect to those axes is

$$[f]_{1,2,3} = \begin{bmatrix} \dfrac{ds}{GJ} & 0 & 0 \\ 0 & \dfrac{ds}{EI_2} & 0 \\ 0 & 0 & \dfrac{ds}{EI_3} \end{bmatrix}. \tag{4}$$

For numerical calculations, ds is replaced by finite segments, Δs, in all of the preceding formulations.

Transformation of the matrix of Eq. (4) to the general coordinate system is made as follows:

$$[f]_{x,y,z} = [t]\,[f]_{1,2,3}\,[t]^T, \tag{5}$$

in which $[t]^T$ is the transpose of $[t]$. The transformed matrix will then be in a form

$$[f]_{x,y,z} = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}, \tag{6}$$

in which, for example, f_{xx} represents, for a segment, the angle change about an x-axis fue to unit moment applied about the x-axis, and f_{yz} represents the angle change about the y-axis due to unit moment about the z-axis. Because of reciprocal relations, $[f]_{x,y,z}$ is a symmetric matrix.

The use of Eq. (5) for obtaining segment flexibilities has advantages over previous expressions [3, 4] when a digital computer is used. Those expressions will, however, give identical results because the difference in direction of measurement of ψ compensates for the left-hand coordinate system previously used.

Matrix Formulation of the Analysis

The column matrix of moments can be obtained as

$$\{M\} = \{M_0\} + [u]\{X\} \tag{7}$$

or

$$\{M\} = [b]\{Q\} + [u]\{X\}, \tag{8}$$

in which, as further defined subsequently, $\{M_0\}$ or its equivalent $[b]\{Q\}$ represents the moments consistent with a statically possible assumed solution, and $[u]\{X\}$ represents the corrections necessary to restore continuity. As indicated later, once $\{X\}$ is known, Eq. (8) can also be used to obtain forces $\{V\}$.

Let

k = number of segments into which the structure is divided,
p = total number of load points,
n = degree of statical indeterminateness (six for the fixed end space structure).

Then

$\{M\}$ = matrix partitioned into $k \times 1$ submatrices of final moments about x, y, and z axes,
$\{M_0\}$ = matrix partitioned into $k \times 1$ submatrices of assumed statically possible moments about x, y, and z axes,
$[u]$ = matrix partitioned into $k \times n$ submatrices of moments due to unit moment or force applied successively at the chosen redundants,
$\{X\}$ = $n \times 1$ matrix of redundant moments and forces,
$[b]$ = matrix partitioned into $k \times p$ submatrices of moments in the assumed statically determinate structure due to unit load applied successively in the x, y, and z directions at one load point at a time,
$\{Q\}$ = matrix of loads partitioned into $p \times 1$ submatrices of 3×1 order (one load in each of the three orthogonal directions).

By using p columns in $[b]$ and p rows in $\{Q\}$, a perfectly general solution results for loading applied in any possible combination.

In Eq. (8) the matrices $[b]$ and $[u]$ can be straightforwardly compiled as discussed in the next section of the paper. The values of X are obtained from the simultaneous equations that express the requirements of geometry. In matrix notation,

$$\{X\} = -[F]^{-1}\{\Delta\}, \tag{9}$$

in which $[F]^{-1}$ represents the inverse of the $n \times n$ flexibility matrix of the structure, and $\{\Delta\}$ is the $n \times 1$ column vector of the errors in geometry resulting from the assumed solution. The flexibility matrix can be obtained from

$$[F] = [u]^T [f] [u], \tag{10}$$

in which $[u]^T$ is the transpose of $[u]$, and $[f]$ is a $k \times k$ quasi-diagonal matrix of the flexibilities $[f]_{x,y,z}$ of the individual segments of the structure.

The errors in geometry are obtained as

$$\{\Delta\} = [u]^T [f] \{M_0\} = [u]^T [f] [b] \{Q\} \tag{11}$$

and if we introduce

$$[C] = [u]^T [f] [b] \tag{12}$$

the expression becomes

$$\{\Delta\} = [C]\{Q\}. \tag{13}$$

From Eqs. (9) and (13),

$$\{X\} = -[F]^{-1}[C]\{Q\} \tag{14}$$

and Eq. (8) can now be written as follows:

$$\{M\} = [b]\{Q\} - [u][F]^{-1}[C]\{Q\}. \tag{15}$$

Finally, introducing

$$[B] = [b] - [u][F]^{-1}[C] \tag{16}$$

we obtain

$$\{M\} = [B]\{Q\}. \tag{17}$$

With $\{X\}$ known, values of the forces $\{V\}$ acting on each segment can be obtained from Eq. (8) by using forces due to unit loads for the elements of $[b]$ and by using forces due to unit forces for the elements of $[u]$. Similarly, the non-redundant moments and forces at the end of the structure can be obtained from Eq. (8) by using the appropriate values of $[b]$ and $[u]$ for the end. See Eq. (23).

Note that $[B]$ of Eq. (16) is a matrix of influence values in the actual structure. Thus $[b]$ is a matrix of influence values for the assumed solution, and the matrix product $[u][F]^{-1}[C]$ represents the corrections due to statical indeterminateness. If the overall form and dimensions of the structure are set, only the matrix $[f]$ is changed in any successive analyses that may be required to arrive at an acceptable design. The matrices $[b]$ and $[u]$ are generated only once.

Matrices for the Analysis

The solution requires the compilation of three matrices. These are $[u]$, $[b]$, and $[f]$.

The $[u]$ matrix for a fixed-end space structure consists of $k \times 6$ submatrices or cells, with one row of cells for each segment, as follows:

$$[u] = \left[\begin{array}{c:c:c:c:c:c}
1 & 0 & 0 & 0 & z & -y \\
0 & 1 & 0 & -z & 0 & x \\
0 & 0 & 1 & y & -x & 0 \\
\hdashline
1 & 0 & 0 & 0 & z & -y \\
0 & 1 & 0 & -z & 0 & x \\
0 & 0 & 1 & y & -x & 0 \\
\hdashline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hdashline
1 & 0 & 0 & 0 & z & -y \\
0 & 1 & 0 & -z & 0 & x \\
0 & 0 & 1 & y & -x & 0
\end{array}\right], \tag{18}$$

with x, y, z measured from the redundant end. The first row of each submatrix represents moments about the x-axis of the segment, the second row represents moments about the y-axis, and the third row represents moments about the z-axis. All moments in the first column of submatrices are due to a unit moment applied about the x-axis at the redundant end, and the moments in the other five columns are due, respectively, to unit moment about the y-axis, unit moment about the z-axis, unit force along the x-axis, unit force along the y-axis, and unit force along the z-axis.

The $[f]$ matrix is a $k \times k$ quasi-diagonal matrix with each submatrix along the leading diagonal consisting of the matrix of Eq. (6) for the corresponding segment.

The matrix $[b]$ consists of $k \times p$ submatrices whose elements are moments in the assumed statically determinate structure due to unit load applied successively at all load points. Each submatrix consists of

$$\begin{bmatrix} 0 & b_{xy} & b_{xz} \\ b_{yx} & 0 & b_{yz} \\ b_{zx} & b_{zy} & 0 \end{bmatrix}, \tag{19}$$

in which, for example, b_{xy} represents the moment about the x-axis of a segment due to unit load applied along the positive y-axis of another segment. If, in the analysis, the structure is assumed cantilevered from one end, then for unit load applied in the positive x, y, and z directions at a segment j, the moments at a segment i are

$$\begin{bmatrix} b_{xy} \\ b_{xz} \\ b_{yx} \\ b_{yz} \\ b_{zx} \\ b_{zy} \end{bmatrix} = \begin{bmatrix} z_i - z_j \\ y_j - y_i \\ z_j - z_i \\ x_i - x_j \\ y_i - y_j \\ x_j - x_i \end{bmatrix}. \tag{20}$$

Note that b_{xy} is not equal to b_{yx}, b_{zx} is not equal to b_{xz}, and b_{yz} is not equal to b_{zy}.

Stiffness Coefficients and Fixed-end Moments and Forces for Analysis of Interconnected Space Structures

The stiffness coefficients for a branch are the moments and forces induced by successively applying unit rotation and unit displacement with respect to each axis at each end, in each case preventing all other possible rotations and displacements at both ends. This, in the general case of a branch AB fixed at both ends, results in a 12th order symmetrical matrix which can be partitioned into four 6th order submatrices as follows [1]:

$$[S^{AB}] = \left[\begin{array}{c|c} S_{AA} & S_{AB} \\ \hline S_{BA} & S_{BB} \end{array}\right], \tag{21}$$

in which $[S_{AA}]$ consists of the stiffness coefficients at end A due to unit displacements and rotations at end A, and $[S_{AB}]$ consists of the stiffness coefficients at end A due to unit displacements and rotations at end B. The submatrices $[S_{BB}]$ and $[S_{BA}]$ consist of the corresponding stiffness coefficients at B.

Submatrix $[S_{AA}]$ is the inverse of the flexibility matrix obtained by applying Eq. (10) to end A. Submatrix $[S_{BB}]$ is obtained by the same equation for end B. Because matrix $[S^{AB}]$ of Eq. (21) is symmetrical, submatrice $[S_{AB}]$ is the transpose of submatrice $[S_{BA}]$. Thus it is necessary to determine only one of these. For example, with submatrix $[S_{AA}]$ known,

$$[S_{BA}] = [u_B][S_{AA}], \tag{22}$$

in which $[u_B]$ is the 6×6 matrix of moments and forces at end B due to unit moments and forces applied at end A.

After the stiffness matrix of each branch is determined, the stiffness matrix of the entire system of interconnected branches can be compliled by addition of values for the individual branches meeting at each joint.

With the previously determined column vector $\{X_A\}$ of fixed-end moments and fixed-end forces at A, the three fixed-end moments M_B and the three fixed-end forces V_B at B can be obtained by modifying Eq. (8) as follows:

$$\begin{Bmatrix} M_B \\ V_B \end{Bmatrix} = [b_B]\{Q\} + [u_B]\{X_A\}, \tag{23}$$

in which $[b_B]$ is partitioned into $6 \times p$ submatrices of moments and forces at end B in the assumed statically determinate structure due to unit load applied successively at all load points.

Unbalanced moments and forces at each joint of an interconnected system are obtained by combining the fixed-end values of all branches framing into the joint. Analysis of the system by use of the displacement method can then be made [1, 2].

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Summary

A matrix formulation of the force method is presented for the analysis of a single-branch structure curved in space in any arbitrary manner and subjected to any possible static loading condition. The method, when programmed for a digital computer, is particularly advantageous when a large number of loading conditions must be considered.

The procedure is also used to determine the stiffness coefficients and the fixed-end moments and forces of single branches. This information makes possible the analysis, by the displacement method, of an interconnected system of branches curved in space.

Résumé

La méthode des forces est présentée sous forme matricielle en vue de calculer une structure à une barre ayant une courbure quelconque dans l'espace et soumise à des cas de charges statiques quelconques. Programmée pour être exploitée sur un calculateur numéral, cette méthode acquiert un intérêt tout

particulier lorsqu'il y a lieu de prendre en considération un grand nombre d'états de charges différents.

On l'applique également pour déterminer les coefficients de rigidité ainsi que les moments d'encastrement et les efforts dans les barres. Ces résultats obtenus, il est possible en appliquant la méthode des déplacements, de calculer des systèmes constitués par plusieurs barres solidaires courbes dans l'espace.

Zusammenfassung

Die Kräftemethode wird in matrizieller Form dargestellt für die Untersuchung eines durch einen einzigen räumlich gekrümmten Stab gebildeten Tragwerkes, das durch beliebige Lasten beansprucht wird. Wenn diese Methode für einen digitalen Rechner programmiert wird, erweist sie sich als besonders vorteilhaft, falls eine große Anzahl von Belastungsfällen betrachtet werden müssen.

Das Verfahren wird ebenfalls benutzt für die Bestimmung des Steifigkeitskoeffizienten sowie der Einspannmomente und der Kräfte in Einzelstäben. Mit diesen Werten kann die Untersuchung, unter Verwendung der Deformationsmethode, auch für ein aus verschiedenen räumlich gekrümmten Stäben zusammengesetztes System durchgeführt werden.