Out of Plane Bending of a Uniform Circular Ring

Flexion latérale des anneaux circulaires à section constante

Seitliche Biegung von Kreisringen mit konstantem Querschnitt

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Consider a thin curved beam (circular axis) whose cross section is any open contour. The beam is supported on a transverse elastic foundation which will produce transverse deflectional restraint proportional to the local deflection. There is also a torsional foundation (5) which will produce rotational restraint proportional to the local torsional rotation of the ring cross section.

Choose the principal x-axis in the plane of the circular z-axis and place the principal y-axis in the direction of transverse bending (Fig. 1). The z-axis is the axis of the center of gravity of the cross section and $0'\,0'$ is the axis of the shear center.

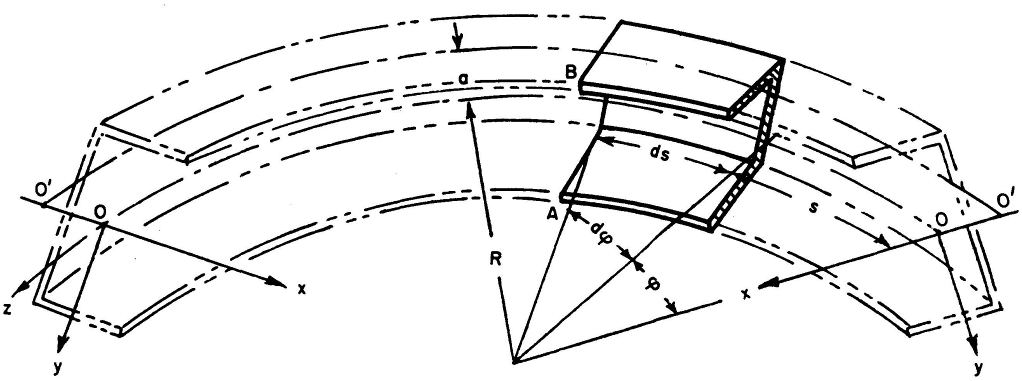

Fig. 1.

Any small element of the bent beam will be as shown in Fig. 2, where m_z is a distributed external torsional moment. Let k_y and k_z be the vertical and torsional constants of the elastic foundations. Projecting all the forces on to the three axes yields the three differential equations.

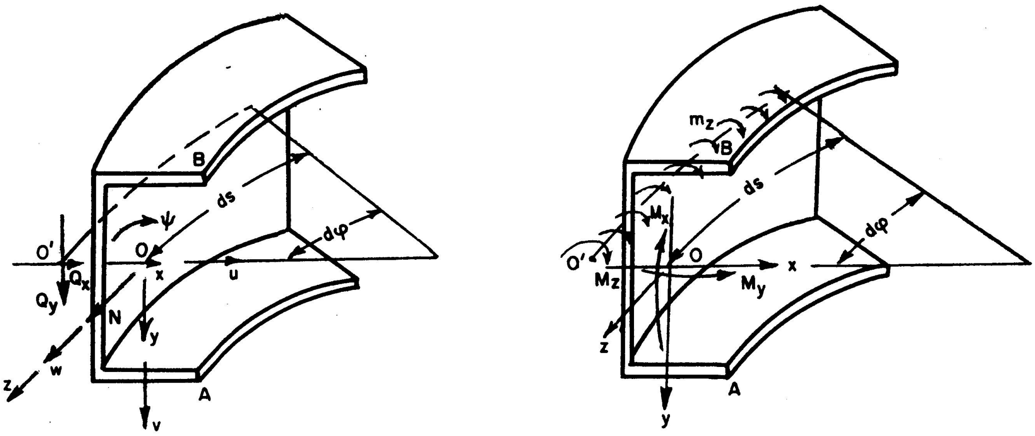

Fig. 2.

$$\frac{dQ_x}{ds}+\frac{N}{R}+q_{(x)}=0\,, \tag{1}$$

$$\frac{dQ_y}{ds}-k_y\,v+q_{(y)}=0\,, \tag{2}$$

and

$$\frac{dN}{ds}-\frac{Q_x}{R}+q_{(z)}=0\,, \tag{3}$$

where q_x, q_y, q_z are the components of the distributed external load in the three directions, and v is the displacement in the y direction. Summation of moments about the three axes produces

$$\frac{dM_x}{ds}-Q_y+\frac{M_z}{R}=0\,, \tag{4}$$

$$\frac{dM_y}{ds}+Q_x=0\,, \tag{5}$$

$$\frac{dM_z}{ds}=\frac{M_x}{R}-m_z+k_z\psi\,, \tag{6}$$

where ψ is the angle of rotation of any cross section. Eqs. (4)—(6) apply only to beams whose centers of gravity coincide with their shear center, and to beams with a large radius, because in those instances e/R is a small quantity which may be neglected.

Designate the displacement components in the x, y, z directions by u, v, w positive as shown in Fig. 2. The state of deformation of the beam may be described by, (a) the strain of the z axis, (b) changes of curvature of the z-axis in the x and y directions and, (c) the angle of twist of any cross section. The strain of the z axis is given by the formula

$$\epsilon_z=\frac{dw}{ds}-\frac{u}{R}. \tag{7}$$

The curvatures of the deformed beam k_x^0, k_y^0, k_z^0 relative to the undeformed axes are

$$k_x^0 = -\frac{d^2 v}{ds^2}, \qquad k_y^0 = \frac{1}{R} + \frac{d^2 u}{ds^2} + \frac{u}{R^2}, \qquad k_z^0 = \frac{d\psi}{ds}.$$

The curvatures of the deformed beam k_x^*, k_y^*, k_z^* relative to the deformed axes (Fig. 3) $0'' x'$ $0'' y'$ $0'' z'$ are

$$k_x^* = k_x^0 \cos(x x') + k_y^0 \cos(y x') + k_z^0 \cos(z x'),$$
$$k_y^* = k_x^0 \cos(x y') + k_y^0 \cos(y y') + k_z^0 \cos(z y'),$$
$$k_z^* = k_x^0 \cos(x z') + k_y^0 \cos(y z') + k_z^0 \cos(z z').$$

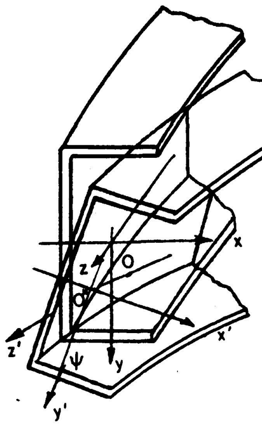

Fig. 3.

	X	Y	Z
X'	1	ψ	-u'
Y'	-ψ	1	-v'
Z'	u'	v'	1

Table 1

The direction cosines between the deformed and the undeformed axes may be obtained from Table 1, so that

$$k_x^* = -\frac{d^2 v}{ds^2} + \left(\frac{1}{R} + \frac{d^2 u}{ds^2} + \frac{u}{R^2}\right)\psi - \frac{d\psi}{ds}\frac{du}{ds},$$
$$k_y^* = \frac{d^2 v}{ds^2}\psi + \left(\frac{1}{R} + \frac{d^2 u}{ds^2} + \frac{u}{R^2}\right) - \frac{d\psi}{ds}\frac{dv}{ds},$$
$$k_z^* = -\frac{d^2 v}{ds^2}\frac{du}{ds} + \left(\frac{1}{R} + \frac{d^2 u}{ds^2} + \frac{u}{R^2}\right)\frac{dv}{ds} + \frac{d\psi}{ds}.$$

The curvature changes and the angle of twist may now be obtained by neglecting the small non-linear terms as:

$$\chi_x = -\frac{d^2 v}{ds^2} + \frac{\psi}{R}, \tag{8}$$

$$\chi_y = \frac{d^2 u}{ds^2} + \frac{u}{R^2}, \tag{9}$$

$$\tau = \frac{d\psi}{ds} + \frac{1}{R}\frac{dv}{ds}. \tag{10}$$

Since for a straight beam we had

$$N = E A \frac{dw}{ds}, \qquad M_x = -E I_x \frac{d^2 v}{ds^2}, \qquad M_y = E I_y \frac{d^2 u}{ds^2}$$

and

$$M_z = -D_1{}^{1)} \frac{d\psi^3}{ds^3} + G I_T \frac{d\psi}{ds}$$

1) See reference [6] for definition of D_1.

for a curved beam or a ring we must have

$$N = E A\left(\frac{dw}{ds} - \frac{u}{R}\right), \tag{11}$$

$$M_x = -E I_x\left(\frac{d^2 v}{ds^2} - \frac{\psi}{R}\right), \tag{12}$$

$$M_y = E I_y\left(\frac{d^2 u}{ds^2} + \frac{u}{R^2}\right), \tag{13}$$

$$M_z = -D_1\left(\frac{d^3\psi}{ds^3} + \frac{1}{R}\frac{d^3 v}{ds^3}\right) + G I_T\left(\frac{d\psi}{ds} + \frac{1}{R}\frac{dv}{ds}\right). \tag{14}$$

In-plane bending of rings may be found in reference [2].

For out of plane bending of uniform ring, with no foundation present, ψ may be eliminated from Eqs. (12) and (14) to give

$$(v''' + v')'' - \alpha(v''' + v') = \frac{R^2}{E I_x}[-\beta M_z + \alpha M_x' - M_x'''], \tag{15}$$

where $\alpha = \dfrac{G I_T}{R D}$, $\beta = \dfrac{E I_x}{R D}$, $D = \dfrac{D_1}{R^3}$

and all derivatives will now be with respect to the angular coordinate ϕ. The complementary solution of (15) is

$$v = A\cos\phi + B\sin\phi + C\cosh\sqrt{\alpha}\,\phi + E\sinh\sqrt{\alpha}\,\phi + F. \tag{16}$$

Use of Eq. (15) will be illustrated later.

Combining Eqs. (2), (4), (6), (12), and (14) gives

$$v^{\mathrm{VIII}} - (\alpha - 2)v^{\mathrm{VI}} + (\rho + \gamma + 1 + \beta\gamma - 2\alpha)v^{\mathrm{IV}} - \alpha(\rho + \gamma + 1)v'' + (1 + \gamma)\beta\rho v = -\frac{R^2}{D\beta}(m_z^{\mathrm{IV}} - R q_y^{\mathrm{IV}}) - \frac{\alpha R^3}{\beta D}q_y'' + (\alpha + \beta)\frac{R^2 m_z''}{D\beta} + (1 + \gamma)\frac{R^3 q_y}{D} \tag{17}$$

and

$$g\psi = -v^{\mathrm{VI}} + c_4 v^{\mathrm{IV}} + c_2 v'' + c_0 v - \frac{R^2}{D\beta}(m_z'' - R q_y'') - (\alpha + \gamma + 1)\frac{R^3 q_y}{D\beta} + (\alpha + \beta + \gamma + 1)\frac{R^2 m_z}{D\beta}, \tag{18}$$

where $g = R(1+\gamma)(\alpha+\beta+\gamma+1)$, $c_4 = (\alpha+\gamma-1)$, $c_2 = (2\alpha+\beta+\gamma+1-\rho)$,

$c_0 = \rho(\alpha+\gamma+1)$, $\gamma = \dfrac{R k_z}{D\beta}$, $\rho = \dfrac{R^3 k_y}{D\beta}$.

If the roots of the equation

$$m^8 - (\alpha - 2)m^6 + (\gamma + \rho + 1 + \beta\gamma - 2\alpha)m^4 - \alpha(\rho + \gamma + 1)m^2 + (1 + \gamma)\beta\rho = 0 \tag{A}$$

are of the form $\pm a \pm i b$ and $\pm p \pm i q$ [2]), then the complementary solution of (17) may be written

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[2]) This is true for some rings supported by cylinders where k_y and hence $\rho\beta$ in equation (A) is very large. Other type roots are also possible.

$$v = X S_\phi + Y T_\phi + Z K_\phi + W L_\phi + I U_\phi + J V_\phi + O M_\phi + G N_\phi, \tag{19}$$

where

$$\begin{aligned} S &= \cosh a\phi \cos b\phi, & T &= \sinh a\phi \sin b\phi, \\ K &= \cosh p\phi \cos q\phi, & L &= \sinh p\phi \sin q\phi, \\ U &= \cosh a\phi \sin b\phi, & V &= \sinh a\phi \cos b\phi, \\ M &= \cosh p\phi \sin q\phi, & N &= \sinh p\phi \cos q\phi, \end{aligned}$$

and $X, Y, \ldots 0, G$ are constants of integration. To have v and M_x symmetrical with the origin we must let I, J, 0, and G be zero. The v displacement and its derivatives may now be expressed in the form

$$\begin{aligned} (\text{even } n)\quad v^n &= X(a_n S - b_n T) + Y(a_n T + b_n S) + Z(p_n K - q_n L) \\ &\quad + W(p_n L - q_n K)\, n = 0 \quad \text{gives } v, \end{aligned}$$

$$\begin{aligned} (\text{odd } n)\quad v^n &= X(a_n V - b_n U) + Y(a_n U + b_n V) + Z(p_n N - q_n M) \\ &\quad + W(p_n M + q_n N). \end{aligned}$$

ψ and its derivatives are

$$\begin{aligned} (\text{even } n)\quad g\psi^n &= X(S e_n - T f_n) + Y(T e_n + S f_n) + Z(K r_n - L t_n) \\ &\quad + W(L r_n + K t_n)\, n = 0 \quad \text{gives } \psi, \end{aligned}$$

$$\begin{aligned} (\text{odd } n)\quad g\psi^n &= X(V e_n - U f_n) + Y(U e_n + V f_n) + Z(N r_n - M t_n) \\ &\quad + W(M r_n + N t_n), \end{aligned}$$

where

$$\begin{aligned} &a_0 = 1, \quad a_1 = a, \quad a_2 = (a^2 - b^2), \quad a_3 = a(a^2 - 3b^2), \quad a_4 = a_2^2 - b_2^2, \\ &a_5 = a(a^4 - 10a^2b^2 + 5b^4), \quad a_6 = a_3^2 - b_3^2, \quad a_7 = a[a^6 - 21a^4b^2 + 35a^2b^4 - 7b^6], \\ &a_9 = a(a^8 - 36a^6b^2 + 126a^4b^4 - 84a^2b^6 + 9b^8), \end{aligned}$$

$$\begin{aligned} &b_0 = 0, \quad b_1 = 1, \quad b_2 = 2ab, \quad b_3 = b(3a^2 - b^2), \quad b_4 = 2a_2b_2, \\ &b_5 = b(5a^4 - 10a^2b^2 + b^4), \quad b_6 = 2a_3b_3, \quad b_7 = b[7a^6 - 35a^4b^2 + 21a^2b^4 - b^6], \\ &b_9 = b(9a^8 - 84a^6b^2 + 126a^4b^4 - 36a^2b^6 + b^8). \end{aligned}$$

The p_n and q_n coefficients are obtained by replacing a by p and b by q in the above equations.

$$\begin{aligned} e_n &= -a_{6+n} + c_4 a_{4+n} + c_2 a_{2+n} + c_0 a_n, & r_n &= -p_{6+n} + c_4 p_{4+n} + c_2 p_{2+n} + c_0 p_n, \\ f_n &= -b_{6+n} + c_4 b_{4+n} + c_2 b_{2+n} + c_0 b_n, & t_n &= -q_{6+n} + c_4 q_{4+n} + c_2 q_{2+n} + c_0 q_n. \end{aligned}$$

The relations for M_x and M_z may now be written as

$$\begin{aligned} \frac{M_x}{\beta D} &= (XS + YT)\left(\frac{e_0}{g} - \frac{a_2}{R}\right) + (YS - XT)\left(\frac{f_0}{g} - \frac{b_2}{R}\right) \\ &\quad + (ZK + WL)\left(\frac{r_0}{g} - \frac{p_2}{R}\right) + (WK - ZL)\left(\frac{t_0}{g} - \frac{q_2}{R}\right), \end{aligned} \tag{20}$$

$$\frac{M_z}{D} = (X\,V + Y\,U)\left[\alpha\left(\frac{e_1}{g} + \frac{a_1}{R}\right) - \left(\frac{e_3}{g} + \frac{a_3}{R}\right)\right] + (Y\,V - X\,U)\left[\alpha\left(\frac{f_1}{g} + \frac{b_1}{R}\right)\right.$$
$$\left. - \left(\frac{f_3}{g} + \frac{b_3}{R}\right)\right] + (Z\,N + W\,M)\left[\alpha\left(\frac{r_1}{g} + \frac{p_1}{R}\right) - \left(\frac{r_3}{g} + \frac{p_3}{R}\right)\right] \tag{21}$$
$$+ (W\,N - Z\,M)\left[\alpha\left(\frac{t_1}{g} + \frac{q_1}{R}\right) - \left(\frac{t_3}{g} + \frac{q_3}{R}\right)\right].$$

For a concentrated load Q and a concentrated torque M_0[3]) at $\phi = \pi$ the constants of integration are derived from the boundary conditions that at $\phi = \pi$

$$v' = \psi' = 0, \qquad M_z = \frac{M_0}{2}, \qquad Q_y = \frac{Q}{2}. \tag{22}$$

The last two of Eqs. (22) may be written as

$$v''' = \frac{(M_0 - Q\,R)\,R}{2\,D\,\beta}, \qquad \psi''' = \frac{Q\,R - (1+\beta)\,M_0}{2\,D\,\beta}.$$

These constants of integration will be of the form

$$X = Q\,X_1 + M_0\,X_2, \quad Y = Q\,Y_1 + M_0\,Y_2, \quad Z = Q\,Z_1 + M_0\,Z_2$$
$$\text{and} \quad W = Q\,W_1 + M_0\,W_2,$$

where $X_1\,X_2\,Y_1 \ldots W_1\,W_2$ are influence coefficients.

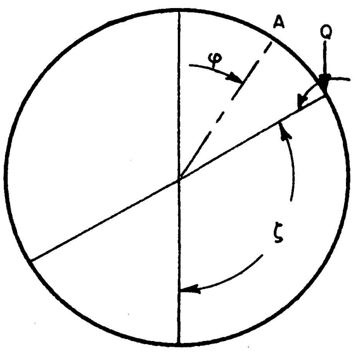

Fig. 4.

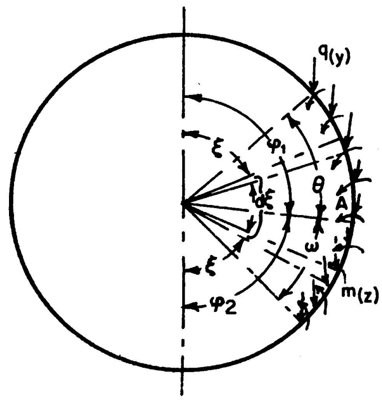

Fig. 5.

For a concentrated load and torque at any angle ξ, the deflection at any point such as A (Fig. 4) will be given by (Eq. (19) becomes)

$$\begin{aligned} v_A = {} & (Q\,X_1 + M_0\,X_2)\,S_{(\xi+\phi)} + (Q\,Y_1 + M_0\,Y_2)\,T_{(\xi+\phi)} \\ & + (Q\,Z_1 + M_0\,Z_2)\,K_{(\xi+\phi)} + (Q\,W_1 + M_0\,W_2)\,L_{(\xi+\phi)}. \end{aligned} \tag{23}$$

For any distributed load such as that shown in Fig. 5 let $Q = q_{(y)}\,R\,d\xi$ and $M_0 = m_{(z)}\,R\,d\xi$ in Eq. (23) to obtain

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[3]) To solve the case of a ring subjected to a concentrated bending moment let X, Y, Z and W be zero in (19).

$$\frac{v_A}{R} = \int\limits_{\pi-\phi_1-\omega}^{\pi-\phi_1} [q_{(y)} X_1 + m_z X_2) S_{\xi+\phi_1} + (q_y Y_1 + m_z Y_2) T_{\xi+\phi_1} + (q_y Z_1 + m_z Z_2) K_{\xi+\phi_1}$$
$$+ (q_y W_1 + m_z W_2) L_{\xi+\phi_1}] d\xi + \int\limits_{\pi-\phi_2-\theta}^{\pi-\phi_2} [(q_y X_1 + m_z X_2) S_{\xi+\phi_2}$$
$$+ (q_y Y_1 + m_z Y_2) T_{\xi+\phi_2} + (q_y Z_1 + m_z Z_2) K_{\xi+\phi_2} + (q_y W_1 + m_z W_2) L_{\xi+\phi_2}] d\xi ,$$

if $m_z = m_0$ (constant) and $q_{(y)} = q_{(0)}$ (constant) then

$$\frac{v_A}{R} = (2 V_\pi - V_{\pi-\theta} - V_{\pi-\omega}) \left[\frac{m_0 (a X_2 - b Y_2) + q_0 (a X_1 - b Y_1)}{a^2 + b^2}\right]$$
$$+ (2 U_\pi - U_{\pi-\theta} - U_{\pi-\omega}) \left[\frac{m_0 (b X_2 + a Y_2) + q_0 (b X_1 + a Y_1)}{a^2 + b^2}\right]$$
$$+ (2 N_\pi - N_{\pi-\theta} - N_{\pi-\omega}) \left[\frac{m_0 (p Z_2 - q W_2) + q_0 (p Z_1 - q W_1)}{p^2 + q^2}\right]$$
$$+ (2 M_\pi - M_{\pi-\theta} - M_{\pi-\omega}) \left[\frac{m_0 (q Z_2 + p W_2) + q_0 (q Z_1 + p W_1)}{p^2 + q^2}\right].$$

Using Eqs. (19)—(23) the deflection at any point not under the load and relations for ψ, M_x and M_z may be derived in a similar manner.

The problem may also be solved by assuming a trigonometric series for v and ψ as

$$v = Y_0 + \sum_{n=1}^{\infty} Y_n \cos n (\xi - \phi), \tag{24}$$

$$\psi = \psi_0 + \sum_{n=1}^{\infty} \psi_n \cos n (\xi - \phi), \tag{25}$$

where ξ defines the section where the concentrated load and moment are applied. The Galerkin equations for the curved beam are:

$$\int\limits_0^{2\pi} \left[(\alpha+\beta)\psi'' - \psi^{\mathrm{IV}} + \frac{1}{R}(\alpha v'' - (\beta+1) v^{\mathrm{IV}} - \rho\beta v) + \frac{R^2 q_y}{D}\right] \delta v \, d\phi = 0, \tag{26}$$

$$\int\limits_0^{2\pi} \left[\beta(1+\gamma)\psi - \alpha\psi'' + \psi^{\mathrm{IV}} + \frac{1}{R}\{v^{\mathrm{IV}} - (\alpha+\beta) v''\} - \frac{R m_z}{D}\right] \delta\psi \, d\phi = 0 \tag{27}$$

substituting (24) and (25) into (26) and (27) gives

$$Y_0 = \frac{Q R^2}{2\pi\rho D \beta}, \tag{28}$$

$$\frac{Y_n}{R} = \frac{R Q [\beta(1+\gamma) + \alpha n^2 + n^4] - M_0 [n^4 + (\alpha+\beta) n^2]}{\beta\pi D [n^8 + (\alpha-2) n^6 + (\rho+1+\gamma(\beta+1) - 2\alpha) n^4 + \alpha(\rho+\gamma+1) n^2 + \rho\beta(1+\gamma)]}, \tag{29}$$

$$\psi_0 = \frac{M_0}{2\pi D \beta (1+\gamma)}, \tag{30}$$

$$\psi_n = \frac{\frac{M_0}{\pi D} - [n^4 + (\alpha+\beta)\, n^2]\,\frac{Y_n}{R}}{[\beta\,(1+\gamma) + \alpha\, n^2 + n^4]}, \tag{31}$$

$$M_x = D\beta \left[\psi_0 + \sum_{n=1}^{\infty} \left(\psi_n + \frac{n^2\, Y_n}{R}\right) \cos n\,(\xi - \phi)\right], \tag{32}$$

$$M_z = D \sum_{n=1}^{\infty} n\,(n^2+\alpha) \left(\psi_n + \frac{Y_n}{R}\right) \sin n\,(\xi - \phi). \tag{33}$$

To solve the problem of a ring subjected to a distributed load and moment, substitute $Q = q_{(y)}\, R\, d\xi$ and $M_0 = m_z\, R\, d\xi$ in Eqs. (24)—(33) and integrate within the proper limits. Knowing v and ψ, the bending moment in the flanges can be computed from

$$M_f = \frac{E\, t\, b^3\, h}{24\, R^2} \left[\psi'' + \frac{v''}{R}\right].$$

The total bending stress is then equal to the algebraic sum due to M_x and M_f.

Numerical Examples

Consider bending of the ring shown in Fig. 6. The ring is simply supported at two points and subjected to a concentrated moment M_0. From statics we have

$$M_x = \frac{M_0}{4} \sin\phi + \kappa \cos\phi, \qquad M_z = -\frac{M_0}{4}(1 + \cos\phi) + \kappa \sin\phi.$$

If $H = \frac{M_0 R^2}{4 E I_x}$ equation (15) becomes

$$v^{\mathrm{V}} + (1-\alpha)\, v''' - \alpha\, v' = H \left[\beta + (\alpha+\beta+1)\cos\phi - (\alpha+\beta+1)\frac{4\kappa}{M_0}\sin\phi\right],$$

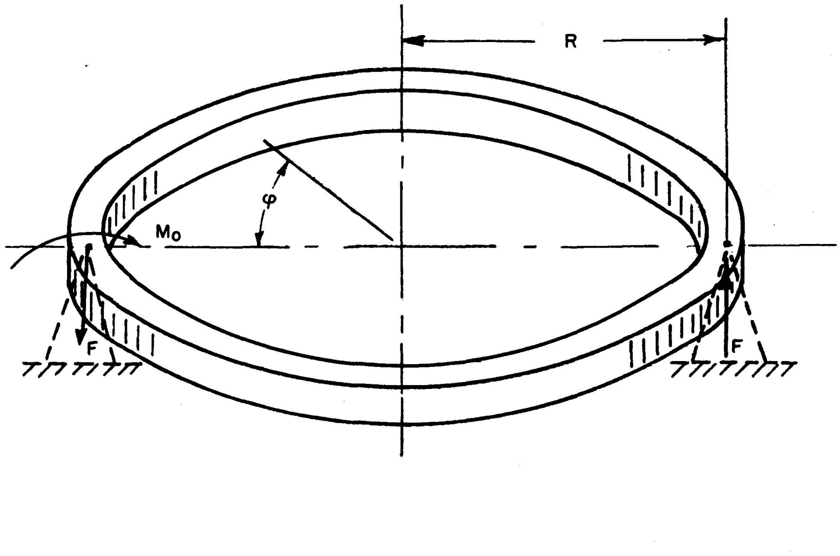

Fig. 6.

whose solution is

$$v = A\cos\phi + B\sin\phi + C\cosh\sqrt{\alpha}\,\phi + E\sinh\sqrt{\alpha}\,\phi + F$$
$$-H\left[\frac{2\kappa}{M_0}\frac{(\alpha+\beta+1)}{(\alpha+1)}\phi\sin\phi - \frac{(\alpha+\beta+1)}{2(\alpha+1)}\phi\cos\phi + \frac{\beta\phi}{\alpha}\right],$$

Eq. (12) now gives

$$R\psi = -A\sin\phi - B\cos\phi + C\alpha\cosh\sqrt{\alpha}\,\phi + E\alpha\sinh\sqrt{\alpha}\,\phi + H\sin\phi$$
$$+\frac{4\kappa H}{M_0}\cos\phi - H\left[\frac{(\alpha+\beta+1)}{2(\alpha+1)}(\phi\cos\phi + 2\sin\phi)\right.$$
$$\left.+\frac{2\kappa}{M_0}\frac{(\alpha+\beta+1)}{(\alpha+1)}(-\phi\sin\phi + 2\cos\phi)\right].$$

The boundary conditions

$$v = v' = \psi' = 0 \quad \text{when} \quad \phi = 0 \quad \text{and} \quad \phi = \pi$$

give the constants as

$$A = H\left[\frac{\beta(\alpha+1)(\cosh\sqrt{\alpha}\,\pi - 1)}{\alpha\sqrt{\alpha}(\alpha+1)^2\sinh\sqrt{\alpha}\,\pi} - \frac{(\alpha^2+3\alpha\beta+\alpha+\beta)}{4\alpha(\alpha+1)}\pi\right],$$

$$B = H\frac{(\alpha\beta-\alpha^2-2\alpha-\beta-1)}{2(\alpha+1)^2}, \qquad C = H\beta\frac{[1-(1+2\alpha)\cosh\sqrt{\alpha}\,\pi]}{\alpha\sqrt{\alpha}(\alpha+1)^2\sinh\sqrt{\alpha}\,\pi},$$

$$E = \frac{H\beta(1+2\alpha)}{\alpha\sqrt{\alpha}(1+\alpha)^2}, \qquad \kappa = \frac{M_0\beta}{\pi(\alpha+\beta+1)},$$

and
$$F = H\left[\frac{\beta(1+\cosh\sqrt{\alpha}\,\pi)}{\sqrt{\alpha}(1+\alpha)^2\sinh\sqrt{\alpha}\,\pi} + \frac{(\alpha^2+3\alpha\beta+\alpha\beta)}{4\alpha(\alpha+1)}\right].$$

Note that this problem is statically indeterminate and hence the moment κ must be determined from the displacement boundary conditions.

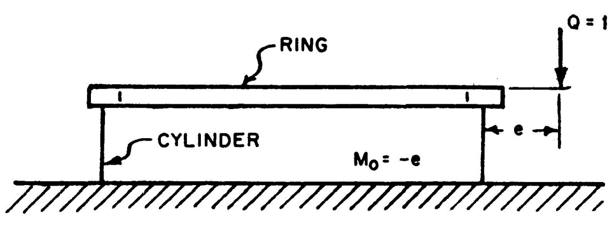

Fig. 7a.

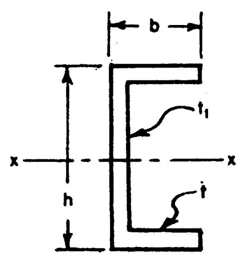

Fig. 7b.

Consider a ring supported by a cylinder whose K_y is 75,000 lb./in.^{2} and K_z is 400 lb.-in./in./rad. The cross section of the ring, whose radius is 24 inches, is a channel (Fig. 7b) whose properties are $E = 30\times10^6$ psi, $G = 11{,}54\times10^6$ psi, $I_x = 2.1$ in.^{4}, $I_T = 0.0285$, $t = t_1 = 0.25$ in., $h = 3$ in., $b = 1.5$ in. and hence

$$D = \frac{E\,t\,b^3h^2}{24R^3}\left(1 + \frac{t\,h^3}{4I_x}\right) = 716.675\,\text{lb.-in.}$$

Let the ring be subjected to an eccentric load of one lb. (Fig. 7a). For $Q=1$ and $M_0=-1$ equations (19)—(20) for $\phi=\pi$ give

$$v = 1.617\times 10^{-6}\,\text{in.}, \qquad \psi = -1.848\times 10^{-6}\,\text{rad.},$$
$$M_x = -0.285\,\text{lb.-in.}, \qquad M_f = 0.285\,\text{lb.-in.}$$

when $D=0$

$$v^{4)} = 2.276\times 10^{-6}\,\text{in.}, \quad \psi = 5.424\times 10^{-6}\,\text{rad.}, \quad M_x = -0.819\,\text{lb.-in.}, \quad M_f = 0.$$

Note that the in-plane bending stiffens the ring. This is also true in the previous example. The sum of Y_0 and seven terms of Eq. (29) give the deflection at $\phi=\pi$ as $v=1.470\times 10^{-6}$ in. The sum of Y_0 and twelve terms of (29) give $v=1.584\times 10^{-6}$ in. The convergence of Eq. (29) is rather slow, that for M_x is probably worse. Eq. (33) gives zero for the value of M_z at $\phi=\xi$. This is an average value for M_z at this section.

Assuming a series for v and ψ and using Eqs. (26) and (27), solutions for a portion of a ring on an elastic foundation may be obtained. However, convergence is slow as was the case for a complete ring. A particular integral of Eq. (17) corresponding to a concentrated force and a concentrated torque at $\phi=\xi$ may be found by La Place Transforms so that exact solutions for a portion of a ring are possible but algebraically complex. Hence, for a complete ring with several elastic or other supports in addition to an elastic foundation or for a portion of a ring with elastic or other supports, the solution using the methods of the form of Runge-Kutta is recommended.

It should be indicated that the solution here obtained is still not right for warping of the cross sections is not included.

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Summary

The problem treated is that of a circular ring of open section (I-beams, channels etc.) subjected to any distributed load and any distributed twisting moment. The ring is supported vertically by an elastic foundation and any cross section is restrained from rotating by a torsional foundation. Equations for the ring without the foundations are also included. A solution for a concentrated load and a concentrated twisting moment is obtained and a solution for any distributed load and a distributed twisting moment is then developed using the principle of superposition.

Résumé

L'auteur étudie le problème d'un anneau circulaire à section ouverte (double-té, section en U, etc.) soumis à des charges et à des moments de torsion répartis de façon quelconque. L'anneau repose verticalement sur une fondation élastique et chaque section est encastrée élastiquement à la rotation. On indique également les équations pour un anneau libre. On développe les solutions pour une force concentrée et un moment de torsion concentré et, en utilisant le principe de superposition, on en déduit la solution pour des charges et des moments de torsion répartis de façon quelconque.

Zusammenfassung

Der Verfasser untersucht Kreisringe mit offenem Querschnitt (I-Träger, [-Profile usw.), die durch beliebige, verteilte Belastungen und verteilte Torsionsmomente beansprucht werden. Der Ring ist vertikal auf einer elastischen Bettung gelagert und jeder Querschnitt ist zudem elastisch drehbehindert. Die Gleichungen für den Ring ohne elastische Bettungen werden ebenfalls angegeben. Die Lösungen für eine Einzellast und ein einzelnes Torsionsmoment werden angeschrieben und unter Benützung des Superpositionsprinzipes wird eine Lösung entwickelt für eine beliebige, verteilte Last und ein verteiltes Torsionsmoment.

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