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On the Statics of Latticed Shells

Sur la statique des voiles triangulés

Über die Statik der Gitterschalen

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Introduction

With increasing frequency in recent years, designers have been utilizing the inherent strength of doubly curved roof structures in latticed form instead of the continuous form classically associated with the term shell. Examples include structures for the 1964 and the two previous world fairs as well as earlier well known structures designed by P. Nervi and by B. Fuller. The preference for the latticed shape over the analogous continuum is rationally based in many cases. Some reasons for the preference, in addition to special functions and aesthetics, are: higher buckling strength with less material; better suited to prefabrication, assembly line techniques and use of materials in available form; more easily strengthened at highly loaded points; and easier to dismantle and reassemble or salvage.

As compared with tractable shapes in the continuum, the analysis of latticed shells is usually performed with less mathematical elegance. Instead of writing the stress components as functions of the coordinates on the shell surface, one of the following methods is used: 1. algebraic equations are solved simultaneously for the node deformations or member forces; 2. forces are predicted through model studies; or 3. an empirical analysis is performed by use of formulas for a similarly shaped continuous shell to approximate the member forces near the point. The purpose of this brief paper is to present and illustrate the elements of a field approach to the analysis of latticed shells; that is, the determination of lattice member forces as closed form functions of the discrete coordinates which label them. The objective is not an exhaustive treatment of this broad and opening subject but rather an elementary study of a restricted and, hopefully, heuristic class of latticed shells which are amenable to such an approach and show promise as practical civil engineering and architectural structures.

Many designers who have become interested in continuous shells are disinclined to become serious students of the esoteric mathematics required for continued work at the theoretical frontiers of the subject. Even so, they can gain insight into the manner in which these two-dimensional shaped structures carry loads by studying the relatively simple and widely published (for example, references [1, 2 and 3]) equations and formulas for the membrane analysis of shells by use of cartesian coordinates in a projected reference plane. The objective here is to obtain analogous equations and formulas for latticed shells.

In order to avoid possible confusion with other, remotely related, classes of problems which also require use of finite difference concepts, it should be emphasized that the object problems here are in the discrete field category. By writing *closed form* solutions to the governing difference equation, one finds *exact* formulas for the desired quantities throughout the *physical lattice* which may have an arbitrary number of nodes. This is as opposed, for example, to the description of an *open form* method which uses difference equations to *approximately* represent the mathematical model for a *physical continuum*.

Equilibrium Equations for the Latticed Shell

Consider a latticed surface whose nodes are connected by three sets of two-force members and are spaced so that their projections on a reference plane form an evenly spaced pattern; that is, the members (which are not necessarily regular) project as a number of identical parallelograms with diagonals (Fig. 1). The direction of the projection from the real node to its image is perpendicular to the reference plane; that is, parallel to the unit vector $\bar{k}$. The unit basis vectors in the plane, $\bar{i}$ and $\bar{j}$, may be, but are not necessarily, orthogonal; that is, $\varphi = \pi/2$. In general, the diagonal length for a typical element in the reference plane is related to the side lengths through the cosine law;

$$c^2 = a^2 + b^2 + 2\,a\,b\cos\varphi. \tag{1}$$

The position vectors for the surface nodes are given by:

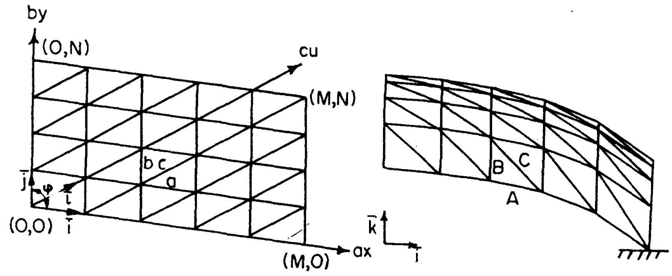
$$\bar{X}(x,y) = a\,x\,\bar{i} + b\,y\,\bar{j} + Z(x,y)\,\bar{k}, \tag{2 a}$$

where Z is the function, of the plane coordinates x and y, which must be given to specify the latticed surface geometry. Although not a theoretical necessity for this two-dimensional problem, a third coordinate in the plane will be introduced as a notation and conceptual aid. The third coordinate, u, is a factor in the distance measured parallel to the projected element diagonal; that is, in the direction of the unit vector $\bar{l}$, where

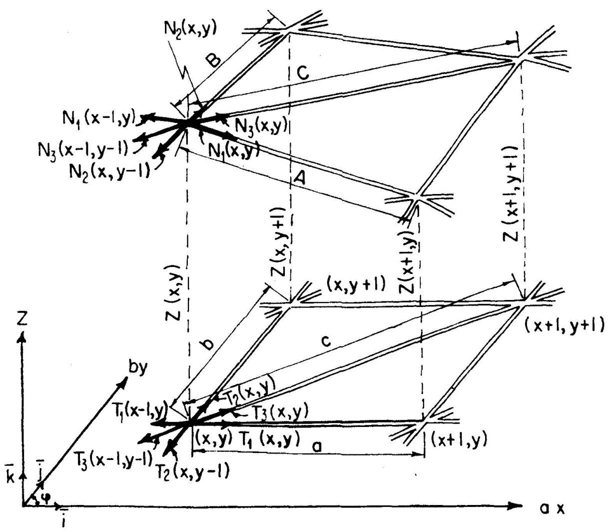
$$\bar{l} = \frac{a}{c}\bar{i} + \frac{b}{c}\bar{j}. \tag{3}$$

This coordinate could be used in lieu of either x or y in Eq. (2a);

$$\bar{X}(x', u) = a\,x'\,\bar{i} + c\,u\,\bar{l} + Z'(x', u)\,\bar{k}. \tag{2b}$$



a) Plan b) Elevation



c) Lattice element

Fig. 1. A latticed shell.

By use of Eq. 3, the relation between coordinates x', u and x, y is found to be:

$$\begin{Bmatrix} x' \\ u \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}, \tag{4a}$$

$$Z'(x', u) \equiv Z'(x-y, y) = Z(x, y). \tag{4b}$$

The member lengths are related to their projections as follows:

$$A^2 = a^2 + (\Delta_x Z)^2, \tag{5a}$$

$$B^2 = b^2 + (\Delta_y Z)^2, \tag{5b}$$

$$C^2 = c^2 + (\Delta_u Z)^2, \tag{5c}$$

where the Δ's denote first forward differences with respect to the indicated variables; that is,

$$\Delta_x Z = Z(x+1,y) - Z(x,y), \tag{6a}$$
$$\Delta_y Z = Z(x,y+1) - Z(x,y), \tag{6b}$$
$$\Delta_u Z = Z(x+1,y+1) - Z(x,y) \equiv (\Delta_x + \nabla_y)\, Z(x,y+1), \tag{6c, d}$$
$$\Delta_u = \Delta_x + \Delta_x \Delta_y + \Delta_y. \tag{6e}$$

The lattice is loaded at the nodes only and the components of the load vector are denoted $p_1(x,y)$, $p_2(x,y)$ and $p_z(x,y)$.

$$\bar{P}(x,y) = p_1 \bar{i} + p_2 \bar{j} + p_z \bar{k}. \tag{7}$$

Instead of working with the member forces, N_1, N_2, and N_3, the plane components of the member forces, T_1, T_2, and T_3, will be used. The components are related to the total forces by:

$$T_1 = \frac{a}{A} N_1; \quad T_2 = \frac{b}{B} N_2; \quad \text{and} \quad T_3 = \frac{c}{C} N_3. \tag{8a, b, c}$$

The out-of-plane components are as follows:

$$V_1 = \frac{1}{a}(\Delta_x Z)\, T_1; \quad V_2 = \frac{1}{b}(\Delta_y Z)\, T_2; \quad \text{and} \quad V_3 = \frac{1}{c}(\Delta_u Z)\, T_3. \tag{9a, b, c}$$

The three equilibrium equations can now be written by summing node forces in the $\bar{i}$, $\bar{j}$ and $\bar{k}$ directions, respectively.

$$\nabla_x T_1 + \frac{a}{c} \nabla_u T_3 + p_1 = 0, \tag{10a}$$

$$\nabla_y T_2 + \frac{b}{c} \nabla_u T_3 + p_2 = 0, \tag{10b}$$

$$\frac{1}{a} \nabla_x [(\Delta_x Z)\, T_1] + \frac{1}{b} \nabla_y [(\Delta_y Z)\, T_2] + \frac{1}{c} \nabla_u [(\Delta_u Z)\, T_3] + p_z = 0, \tag{10c}$$

where the ∇'s denote first backward differences with respect to the indicated variables; that is,

$$\nabla_x F(x,y) = F(x,y) - F(x-1,y), \tag{11a}$$
$$\nabla_y F(x,y) = F(x,y) - F(x,y-1), \tag{11b}$$
$$\nabla_u F(x,y) = F(x,y) - F(x-1,y-1) \equiv (\nabla_x + \Delta_y)\, F(x,y-1), \tag{11c, d}$$
$$\nabla_u = \nabla_x - \nabla_x \nabla_y + \nabla_y. \tag{11e}$$

The third of the equilibrium equations, as written by inspection, (10c), contains differences of products. If one carries out these indicated differences and substitutes Eqs. (10a) and (10b), the equation assumes a more convenient form which contains only the member force components and the first difference of T_3; that is,

$$\begin{aligned} &\frac{1}{a}(\triangle\!\!\!\!\nabla_x Z)\, T_1 + \frac{1}{b}(\triangle\!\!\!\!\nabla_y Z)\, T_2 + \frac{1}{c}(\triangle\!\!\!\!\nabla_u Z)\, T_3 - \frac{1}{c}(\nabla_x \nabla_y Z)\, \nabla_u T_3 = \\ &\qquad -p_z + \frac{1}{a}(\nabla_x Z)\, p_1 + \frac{1}{b}(\nabla_y z)\, p_2, \end{aligned} \tag{12}$$

where the $\Delta\nabla$'s denote second central differences (product of Δ and ∇) with respect to the indicated variables; that is,

$$\Delta\nabla_x Z = Z(x+1,y) \quad -2Z(x,y)+Z(x-1,y), \tag{13a}$$

$$\Delta\nabla_y Z = Z(x,y+1) \quad -2Z(x,y)+Z(x,y-1), \tag{13b}$$

$$\Delta\nabla_u Z = Z(x+1,y+1)-2Z(x,y)+Z(x-1,y-1). \tag{13c}$$

One can interpret the set, Eqs. (10a), (10b) and (12), as a three-component vector equation. As it contains only three scalar unknowns, a solution is possible, contingent upon the existence of suitable boundary conditions. In other words, the lattice with three sets of members is internally statically determinate. For some cases, the vector equation can be handled more easily if it is transformed to a scalar equation. This is done by selecting a new function of x and y, F, which is related to the components of the unknown vector, T_1, T_2 and T_3, so as to reduce Eqs. (10a) and (10b) to identities. The governing equation for F is then found by substituting these relations into Eq. (12). The results are:

$$T_1 = \frac{a}{c}\nabla_y\nabla_u F - \nabla_x^{-1} p_1; \qquad T_2 = \frac{b}{c}\nabla_x\nabla_u F - \nabla_y^{-1} p_2; \tag{14a, b}$$

$$T_3 = -\nabla_x\nabla_y F; \tag{14c}$$

and

$$(\Delta\nabla_x Z)\nabla_y\nabla_u F - (\Delta\nabla_u Z)\nabla_x\nabla_y F + (\Delta\nabla_y Z)\nabla_x\nabla_u F + (\nabla_x\nabla_y Z)\nabla_x\nabla_y\nabla_u F = c\,p_t, \tag{15}$$

$$p_t \equiv -p_z + \frac{1}{a}(\nabla_x Z)p_1 + \frac{1}{b}(\nabla_y Z)p_2 + \frac{1}{a}(\Delta\nabla_x Z)\nabla_x^{-1}p_1 + \frac{1}{b}(\Delta\nabla_y Z)\nabla_y^{-1}p_2. \tag{16}$$

A clearer understanding of the significance of the partial difference operators in Eq. (15) results from their presentation in the molecular form, as popularized by SALVADORI and BARON [4], that is, the operator is shown as a graphical array of coefficients of the function at adjoining nodes. For example,

$$F(x,y) - F(x-1,y) - F(x,y-1) + F(x-1,y-1) = \nabla_x\nabla_y F(x,y) \tag{17a}$$

would be represented as:

$$\begin{matrix} & (x-2) & (x-1) & (x) \\ (y-2) & \langle 0\rangle & \langle 0\rangle & \langle 0\rangle \\ (y-1) & \langle 0\rangle & \langle 1\rangle & \langle -1\rangle \\ (y) & \langle 0\rangle & \langle -1\rangle & \langle 1\rangle \end{matrix} \left.\right\} F. \tag{17b}$$

This useful representation of partial finite difference operators can be given a formal basis. Through use of the displacement operator, the coefficients in the molecules can be treated as elements of a matrix. A two-dimensional backward difference operator of order m in x and n in y can be representated as:

$$\Phi(E_x, E_y)\,F(x,y) = [E_y^i][C_{ij}]\{E_x^j\}; \tag{18a}$$

with $i = -n(1)0$ and $j = -m(1)0$.

For $m=n=2$,

$$\Phi(E_x, E_y)\, F(x,y) = [E_y^{-2}\, E_y^{-1}\, E_y^{0}] \begin{bmatrix} C_{-2,-2} & C_{-2,-1} & C_{-2,0} \\ C_{-1,-2} & C_{-1,-1} & C_{-1,0} \\ C_{0,-2} & C_{0,-1} & C_{0,0} \end{bmatrix} \begin{Bmatrix} E_x^{-2} \\ E_x^{-1} \\ E_x^{0} \end{Bmatrix} F(x,y), \quad (18\text{b})$$

where the E's are the standard displacement operators [5], defined so that:

$$E_x^j\, E_y^i\, F(x,y) = F(x+j, y+i). \quad (19)$$

In accordance with these definitions, the node coefficient matrices, $[C_{ij}]$, for the operators in Eq. (15) are:

$$\text{for } \nabla_y \nabla_u, \quad [C_{ij}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad (20\text{a})$$

$$\text{for } \nabla_x \nabla_u, \quad [C_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad (20\text{b})$$

$$\text{for } \nabla_x \nabla_y, \quad [C_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad (20\text{c})$$

$$\text{for } \nabla_x \nabla_y \nabla_u, \quad [C_{ij}] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \quad (20\text{d})$$

Eqs. (10a), (10b) and (12), or Eq. (15), can be solved in closed form or numerically to find the member forces for any loading on a given lattice configuration with suitable boundary conditions. Here, attention will be restricted to finding closed form, or function type, solutions.

The Hyperbolic Paraboloidal Lattice

As a first illustrative example of the application of Eqs. (10a, b) and (12) to the analysis of a latticed shell, consider the system shown in Figs. 2a, b. In order to appreciate the breadth of application for the solution to this case, one should recall that any hyperbolic paraboloidal lattice — which is formed by (1) connecting opposite sides of an arbitrary space quadrilateral with straight members spaced uniformly along the edge members and (2) adding diagonal members to brace the elemental quadrilaterals — projects a uniform pattern upon a plane parallel to the two directors (common normals) for the two sets of surface generators. The lattice, and indicated boundary force capacities, in Figs. (2a, b) may serve as the entire structure or it may be one of a number of such lattices comprising a more complex system. Examples of systems of hyperbolic paraboloidal lattices are shown in Figs. 2c, d, e, f. Many other, both practical and fanciful, combinations have been constructed as continuous shells.

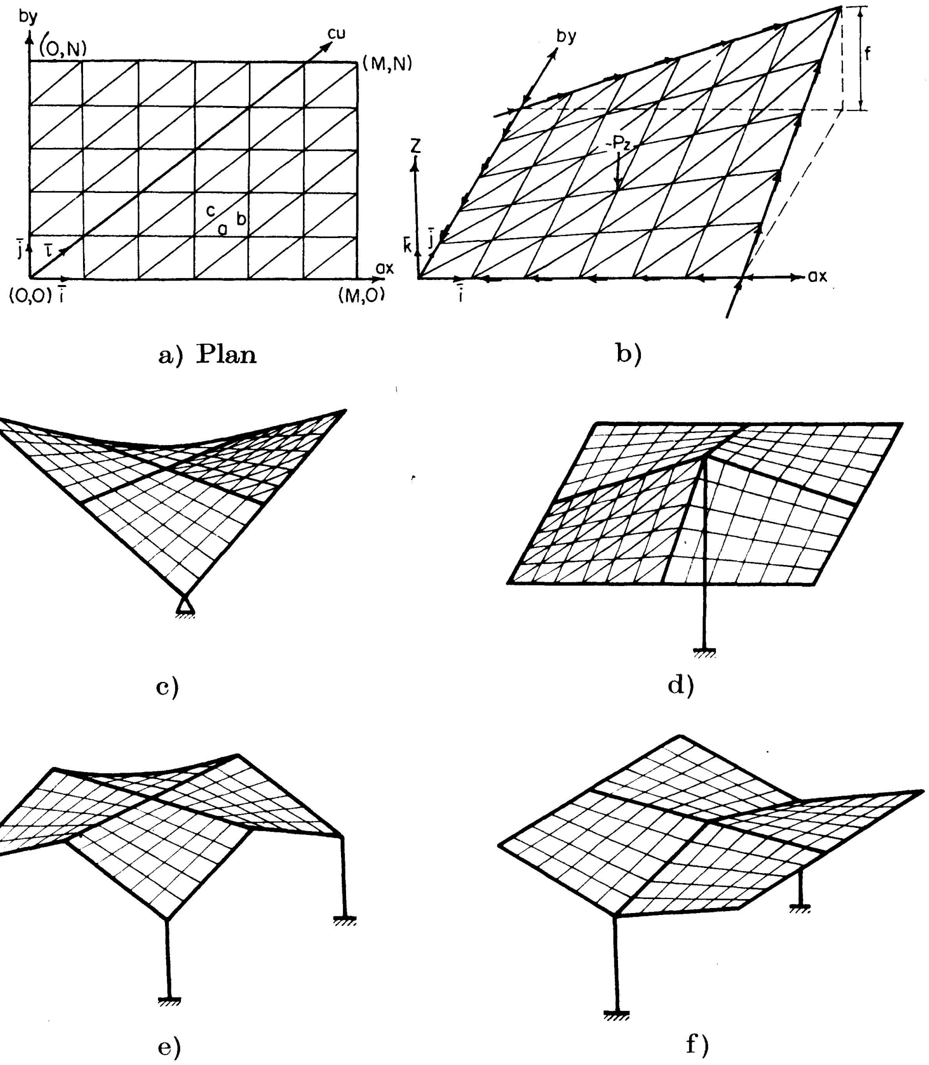


Fig. 2. Hyperbolic paraboloidal lattices.

The formula for the node heights of this lattice, and the various operations upon it, are:

$$Z = \frac{fxy}{MN}, \qquad \nabla_x Z = \frac{fy}{MN}, \qquad \nabla_y Z = \frac{fx}{MN}, \tag{21a, b, c}$$

$$\nabla_x \nabla_y Z = \frac{f}{MN}, \qquad \Delta\nabla_x Z = 0, \qquad \Delta\nabla_y Z = 0, \tag{21d, e, f}$$

$$\nabla_u Z = \frac{f}{MN}(x+y-1), \qquad \Delta\nabla_u Z = \frac{2f}{MN}. \tag{21g, h}$$

The joint loads, corresponding to a uniform snow load on the roof supported by the lattice, are taken as follows:

$$p_1 = p_2 = 0 \quad \text{for all } x, y, \tag{22a, b}$$

$$p_z = -p_0 \quad \text{for } 0<x<M \text{ and } 0<y<N, \tag{22c}$$

$$p_z = -\tfrac{1}{2} p_0 \quad \text{on edges, except, } p_z = 0 \text{ at } (M, 0) \text{ and } (0, N). \tag{22d}$$

$$\text{total load,} \quad W = M N p_0. \tag{22e}$$

Substitution of Eqs. (21) and (22) into Eq. (12), for the interior nodes, gives:

$$(2-\nabla_u) T_3 = \frac{c M N}{f} p_0. \tag{23}$$

The total solution for Eq. (23) can be written by inspection or through use of standard procedures [5] as:

$$T_3 = \frac{c M N}{2 f} p_0 [1 + (-1)^y G(x-y)]. \tag{24}$$

Substitution of Eq. (24) into Eqs. (10a, b) gives:

$$\nabla_x T_1 = -\frac{a M N}{f} p_0 (-1)^y G(x-y), \tag{25a}$$

$$\nabla_y T_2 = -\frac{b M N}{f} p_0 (-1)^y G(x-y). \tag{25b}$$

It is noted that the homogeneous solution alternates in sign but has a constant magnitude along a given set of diagonals. This fact, along with knowledge of the membrane stress variation in the continuous case, suggests that suitable boundary conditions may be met if $G(x-y)=0$. This trial solution will be tested. Substitution of $G(x-y)=0$ into Eqs. (24), (25a, b) gives:

$$T_3 = \frac{c M N}{2 f} p_0, \qquad T_1 = G_1(y) \quad \text{and} \quad T_2 = G_2(x). \tag{26a, b, c}$$

Reasonable and desirable boundary conditions are: 1. zero net $\bar{j}$ force component along $y=0$; and 2. zero net $\bar{i}$ force component along $x=0$; that is,

$$T_2(x, 0) + \frac{b}{c} T_3(x, 0) = 0 \quad \text{and} \quad T_1(0, y) + \frac{a}{c} T_3(0, y) = 0. \tag{27a, b}$$

Equations (27a, b) are satisfied provided:

$$T_1 = -\frac{a M N}{2 f} p_0 \quad \text{and} \quad T_2 = -\frac{b M N}{2 f} p_0. \tag{26d, e}$$

Eqs. (26a, d, e) comprise a solution to the problem outlined. It remains only to study the edge forces which must be supplied in order that this statics solution be consistent. Along the horizontal edge, $x=0$, the equilibrant force vectors are:

$$\bar{P}_e(0, y) = 0\bar{i} - \frac{b M N}{2 f} p_0 \bar{j} + 0\bar{k} \quad \text{for } y = 1\,(1)\,N, \tag{28a}$$

$$\bar{P}_e(0, 0) = 0\bar{i} + 0\bar{j} + 0\bar{k}. \tag{28b}$$

Along $y=0$,

$$\bar{P}_e(x,0) = -\frac{a\,M\,N}{2f}\,p_0\,\bar{i}+0\,\bar{j}+0\,\bar{k} \quad \text{for } x = 1\,(1)\,M. \quad (28\text{c})$$

In other words, only a strut or cable is required along the horizontal edges depending upon where the resultant force is to be resisted; that is, struts for reaction at $(M,0)$ and $(0,N)$ or cables for reaction at $(0,0)$. In either case, the required reactions for forces along $x=0$ and along $y=0$, respectively, are:

$$R_y = \frac{b\,M\,N^2}{2f}\,p_0 \quad \text{and} \quad R_x = \frac{a\,M^2\,N}{2f}\,p_0. \quad (29\text{a, b})$$

Along the sloping edge $x=M$, the equilibrant force vectors are:

$$\bar{P}_e(M,y) = 0\,\bar{i}+\frac{b\,M\,N}{2f}\,p_0\,\bar{j}+\frac{M}{2}\,p_0\,\bar{k} \quad \text{for } y = 0\,(1)\,N-1, \quad (30\text{a})$$

$$\bar{P}_e(M,N) = 0\,\bar{i}+0\,\bar{j}+0\,\bar{k}. \quad (30\text{b})$$

Along $y=N$:

$$\bar{P}_e(x,N) = \frac{a\,M\,N}{2f}\,p_0\,\bar{i}+0\,\bar{j}+\frac{N}{2}\,p_0\,\bar{k} \quad \text{for } x = 0\,(1)\,M-1. \quad (31)$$

Eqs. (30) and (31) represent force vectors which are parallel to the sloping edges so that there too simple struts or cables are sufficient to furnish the necessary edge forces. It should also be noted that the total of the vertical components of the edge forces is $M\,N\,p_0$, which balances the total vertical load, W, given in Eq. (22e).

The Elliptic Paraboloidal Lattice

For a second illustrative example of the application of projected plane shell lattice equations, consider the system shown in Figs. 3a, b. For this elliptic paraboloidal lattice the formula for the distance from the reference plane to the nodes, and the various operations upon it, are:

$$Z = \frac{d}{M^2}\,x^2+\frac{e}{N^2}\,y^2, \qquad \nabla_x Z = \frac{d}{M^2}\,(2x-1), \quad (32\text{a, b})$$

$$\nabla_y Z = \frac{e}{N^2}\,(2y-1), \qquad \Delta_x Z = \frac{2d}{M^2}, \quad (32\text{c, d})$$

$$\Delta_y Z = \frac{2e}{N^2}, \qquad \nabla_x \nabla_y Z = 0, \quad (32\text{e, f})$$

$$\nabla_u Z = \frac{d}{M^2}\,(2x-1)+\frac{e}{N^2}\,(2y-1), \qquad \Delta_u Z = \frac{2d}{M^2}+\frac{2e}{N^2}. \quad (32\text{g, h})$$

At this point, the loading is partially specified as having components in the $\bar{k}$ direction only; that is,

$$p_1 = p_2 = 0. \quad (33\text{a, b})$$

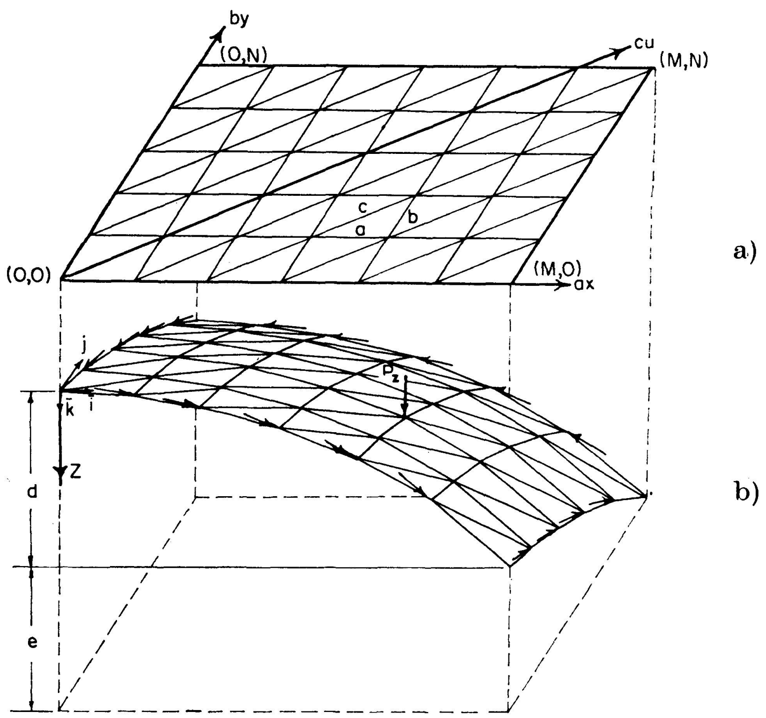
Substitution of Eqs. (32) and (33) into Eq. (12) gives:

$$\frac{2d}{a\,M^2}T_1+\frac{2}{c}\left(\frac{d}{M^2}+\frac{e}{N^2}\right)T_3+\frac{2e}{b\,N^2}T_2=-p_z. \tag{34}$$

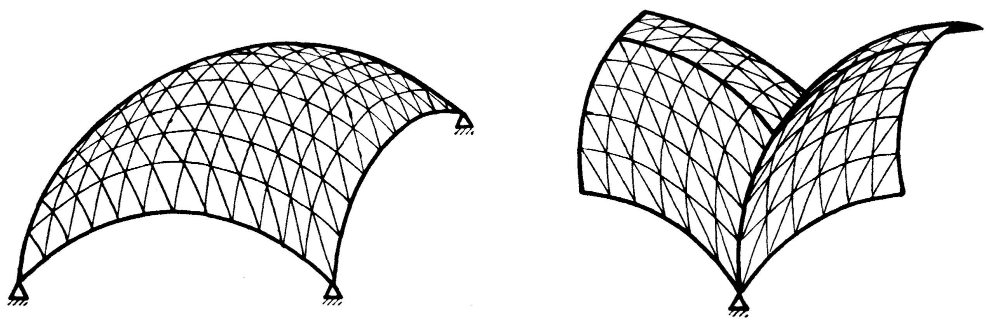
As Eq. (34) contains all three components of the vector unknown, the "Stress Function" approach is indicated. Substitution into (Eq. 15) and collection of terms yields:

$$\left(\frac{e}{N^2}\Delta_x+\frac{d}{M^2}\Delta_y\right)F(x,y)=-\frac{c}{2}p_z(x+1,y+1). \tag{35}$$

Eq. (35) is a second order partial difference equation which occurs frequently when a discrete field approach is used to problems in structural mechanics. Solutions, for certain boundary conditions, are available [6]. Here,



a), b) Quarter-Dome



c) Common Dome d) Butterfly Dome

Fig. 3. Elliptic paraboloidal lattices.

as in the first example, boundary conditions are considered which establish that all edge forces are in the plane of the boundary nodes; that is, 1. zero $\bar{j}$ components at $y=0, N$ and 2. zero $\bar{i}$ components at $x=0, M$. This means that the edge structures have only in-plane loads. If the Quarter-Dome Lattice, Fig. 3b, is not the entire structure but only a part of a Common Dome, Fig. 3c, edge members may not be needed along the two interior sides, where edge loads and boundary forces from adjoining sections serve as edge force equilibrants for the section in question. Quarter-Dome Lattices may be joined with diagonals oriented so as to form a full dome symmetric with respect to $x=0$ and $y=0$. In this case, different boundary conditions are needed to eliminate stiffeners in those planes; that is, zero $\bar{j}$ components at $x=0$ and zero $\bar{i}$ components at $y=0$. The Butterfly Dome, Fig. 3d, is another system which can be erected with Quarter-Domes. For this structure, one may require different boundary statements, depending upon the type of supports under study.

Eqs. (27a, b) are a partial statement of the boundary conditions for a Quarter-Dome with plane edge supports. At $y=N$ and $x=M$, the conditions are:

$$T_2(x,N)+\frac{b}{c}T_3(x,N)=0 \quad \text{and} \quad T_1(M,y)+\frac{a}{c}T_3(M,y)=0. \qquad (36\text{a, b})$$

Substitution of the definitions for T_1, T_2 and T_3, in terms of the "Stress Function", Eqs. (14), into the physical boundary statements, Eqs. (27) and (36), give the following conditions on F:

$$\varDelta_y F\left(\begin{matrix}-1\\ M-1\end{matrix}, y-1\right)=0, \qquad \varDelta_x F\left(x-1, \begin{matrix}-1\\ N-1\end{matrix}\right)=0. \qquad (37\text{a, b})$$

Eqs. (37a, b) require that F vary linearly along the boundary. As linear variations have no effect on the force components, T_1, T_2 and T_3, the functions may be taken as zero on the boundaries; that is, Eqs. (37a, b) are replaced by:

$$F\left(\begin{matrix}-1\\ M-1\end{matrix}, y-1\right)=0, \qquad F\left(x-1, \begin{matrix}-1\\ N-1\end{matrix}\right)=0. \qquad (38\text{a, b})$$

Solutions for Eq. (35) which satisfy Eqs. (38a, b) can be found by separation of variables [5, 6] provided the loading term can be appropriately expressed. Either a single series or a double series can be used; that is,

$$F(x,y)=\sum_{i=1}^{M-1}\sum_{j=1}^{N-1} A_{ij}\sin\frac{i\pi}{M}(x+1)\sin\frac{j\pi}{N}(y+1), \qquad (39)$$

$$F(x,y)=\sum_{i=1}^{M-1} Y_i \sin\frac{i\pi}{M}(x+1), \qquad (40)$$

where A_{ij} are a set of constants and Y_i are a set of functions of y which must be determined.

For use of the double series solution, the loading function, $p_z(x,y)$, must also be expressed as a double series; that is,

$$p_z(x,y) = \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} P_{ij} \sin\frac{i\pi}{M} x \sin\frac{j\pi}{N} y, \tag{41}$$

where the coefficients, P_{ij}, are found for a given loading, $p_z(x,y)$, from [6, 7]:

$$P_{ij} = \frac{4}{MN} \sum_{x=1}^{M-1} \sum_{y=1}^{N-1} p_z(x,y) \sin\frac{i\pi x}{M} \sin\frac{j\pi y}{N}, \tag{42a}$$

$$= \frac{4}{MN} \Delta_x^{-1} \Delta_y^{-1} p_z(x,y) \sin\frac{i\pi x}{M} \sin\frac{j\pi y}{N}\bigg|_{1,1}^{M,N}. \tag{42b}$$

In the case of a uniform load, $p_z(x,y) = p_0$, the coefficients in Eq. (41) are:

$$P_{ij} = \frac{4}{MN} p_0 \cot\frac{i\pi}{2M} \cot\frac{j\pi}{2N} \qquad i,j = 1,3,5,7,\ldots \tag{43}$$

For a single load, p_0, at joint $x=\alpha$, $y=\beta$, Kronecker Deltas serve as discrete functions for the loading; that is,

$$p_z(x,y) = p_0 \delta_x^\alpha \delta_y^\beta, \tag{44}$$

$$\delta_x^\alpha = \begin{Bmatrix} 0, & x \neq \alpha \\ 1, & x = \alpha \end{Bmatrix} \tag{45}$$

and the series coefficients for this kernel loading are:

$$P_{ij} = \frac{4}{MN} p_0 \sin\frac{i\pi\alpha}{M} \sin\frac{j\pi\beta}{N}. \tag{46}$$

Now, the coefficients for the double series solution, A_{ij}, are found by substituting Eqs. (41) and (39) into Eq. (35) and matching coefficients of like terms. The result is:

$$A_{ij} = \frac{c}{4} \frac{P_{ij}}{\frac{e}{N^2}\left(1-\cos\frac{i\pi}{M}\right) + \frac{d}{M^2}\left(1-\cos\frac{j\pi}{N}\right)}. \tag{47}$$

Eqs. (47) and (39) comprise a solution for F in Eq. (35) for an arbitrary loading function, p_z. The loading coefficients, P_{ij}, are given for two particular loading conditions, 1. the uniform load, Eq. (43), and 2. the single concentrated load, Eq. (46) (repeated use of this case for given p_0's at specified coordinates, α, β, enables one to find F for any and all loadings).

The double series solution, Eq. (39), is easily programmed on a computer, but may be inconvenient for manual computation if M and N are large. In view of the probability that the double series will be used primarily for automatic computation, separate formulas will not be given for T_1, T_2 and T_3. Instead, it is recommended that the program for Eq. (39) be used repeatedly, in accordance with the relations given in Eqs. (14) or their molecular equivalents, to find the member force components directly; for exemple,

$$T_3(x,y) = -\nabla_x \nabla_y F(x,y) = F(x-1,y) + F(x,y-1) - F(x,y) - F(x-1,y-1). \tag{48}$$

For manual computations, the single series solution, Eq. (40), is much preferred over the double series form. It arises more naturally through the separation of variables (one series in Eq. (39) is actually an expansion of non-trigonometric functions) and converges; that is, not all terms are needed for engineering accuracy. Of more importance than computational advantages, however, is the fact that the single series solution is valid for arbitrary conditions over part of the boundary; for example, the solution, Eq. (40), satisfies the boundary condition, Eq. (38a), but is not restricted to the condition, Eq. (38b). It is possible to adjust Y_i so as to satisfy a total of two conditions for given values of y, of which Eq. (38b) is a special case.

Again the loading function, $p_z(x,y)$, must be expressed in a series form:

$$p_z(x,y) = p_z'(y) \sum_{i=1}^{M-1} P_i \sin\frac{i\pi x}{M}. \tag{49}$$

In the case of a uniform load; $p_z(x,y) = p_0$,

$$p_z'(y) = 1, \qquad P_i = \frac{2}{M} p_0 \cot\frac{i\pi}{2M} \qquad i = 1,3,5,\ldots \tag{50a, b}$$

In the case of a single concentrated load at α, β, Eq. (44),

$$p_z'(y) = \delta_y^\beta, \qquad P_i = \frac{2}{M} p_0 \sin\frac{i\pi\alpha}{M}. \tag{51a, b}$$

The Euler Functions, $Y_i(y)$, in the single series solution, Eq. (40), can now be determined by substituting Eqs. (40) and (49) into Eq. (35) and matching coefficients of like terms. This results in an ordinary difference equation for Y_i; that is,

$$(\Delta_y - 2\gamma_i) Y_i = -\frac{M^2 c}{2d} P_i p_z'(y+1), \tag{52}$$

$$\gamma_i \equiv \frac{e}{d}\left(\frac{M}{N}\right)^2 \left(1 - \cos\frac{i\pi}{M}\right). \tag{53}$$

As the parameter, γ_i, is always positive, the solution to Eq. (52) can be written routinely [5, 6]:

$$Y_i = \frac{M^2 c}{2d} P_i [C_i \sinh\sigma_i y + D_i \cosh\sigma_i y + Y_{pi}], \tag{54}$$

where Y_{pi} are particular solutions to Eq. (52) and

$$\cosh\sigma_i = \gamma_i + 1. \tag{55}$$

For the single load (the most general case), use of Eq. (51a), gives:

$$Y_{pi} \quad = -\frac{\sinh\sigma_i\,(y-\beta+1)}{\sinh\sigma_i} \not{U}\,(y-\beta), \tag{56}$$

$$\not{U}\,(y-\beta) \equiv \begin{Bmatrix} 0, & y<\beta \\ 1, & y \geqq \beta \end{Bmatrix}. \tag{57}$$

In order to constrain Eqs. (54) and (56) to fit boundary conditions corresponding to in-plane-only forces at $y = 0, N$, Eq. (38b), the following formulas are used:

$$C_i = (\coth\sigma_i)\,D_i, \qquad D_i = \frac{\sinh\sigma_i\,(N-\beta)}{\sinh\sigma_i\,N}. \tag{58a, b}$$

Substitution of Eqs. (58) and (50) into Eq. (54), yields:

$$Y_i = \frac{M\,c}{d\sinh\sigma_i}\,p_0\sin\frac{i\,\pi\,\alpha}{M}\left[\frac{\sinh\sigma_i\,(N-\beta)}{\sinh\sigma_i\,N}\sinh\sigma_i\,(y+1) - \sinh\sigma_i\,(y-\beta+1)\,\not{U}\,(y-\beta)\right]. \tag{59}$$

Eqs. (59) and (40) comprise a single series solution for the stress function, F, related to the member forces in an Elliptic Paraboloidal Lattice which is loaded vertically at joint α, β and supported by in-plane forces on the boundary of a Quarter-Dome.

The solution for a uniform load, corresponding to Eq. (59) for the single load, is:

$$Y_i = \frac{M\,c}{2\,d\,\gamma_i}\,p_0\cot\frac{i\,\pi}{2\,M}\left[1 - \frac{\cosh\sigma_i\left(y-\frac{N}{2}+1\right)}{\cosh\sigma_i\,\frac{N}{2}}\right]. \qquad i = 1, 3, 5, \ldots \tag{60}$$

Operation upon the solution for F due to a uniform load, Eqs. (60) and (40), in accordance with formulas relating the stress function to the plane components of the member forces, Eqs. (14), gives the following results:

$$T_3 = \frac{M\,c}{d}\;p_0\sum_{i=1}^{M-1}\frac{\cos\frac{i\,\pi}{2\,M}\sinh\sigma_i\left(y-\frac{N}{2}+\frac{1}{2}\right)}{\cosh\frac{N}{2}\sigma_i\sinh\frac{1}{2}\sigma_i}\cos\frac{i\,\pi}{M}\left(x+\frac{1}{2}\right) \qquad i = 1, 3, 5, \ldots \tag{61}$$

$$T_1 = -\frac{M\,a}{2\,d}\;p_0\sum_{i=1}^{M-1}\frac{\cot\frac{i\,\pi}{2\,M}}{\sinh\frac{1}{2}\sigma_i\cosh\frac{N}{2}\,\sigma_i}\,\Delta_u\left[\sinh\sigma_i\left(y-\frac{N}{2}-\frac{1}{2}\right)\sin\frac{i\,\pi\,x}{M}\right] \tag{62}$$

$$i = 1, 3, 5, \ldots$$

$$T_2 = -\frac{2\,M\,b}{d}\;p_0\sum_{i=1}^{M-1}\frac{\cos\frac{i\,\pi}{2\,M}}{\gamma_i\cosh\frac{N}{2}\,\sigma_i}\,\Delta_u\left[\sinh\sigma_i\left(\frac{y}{2}\right)\sinh\sigma_i\left(\frac{y-N}{2}\right)\cos\frac{i\,\pi}{M}\left(x-\frac{1}{2}\right)\right] \tag{63}$$

$$i = 1, 3, 5, \ldots$$

These formulas complete the solutions for the Elliptic Paraboloidal Lattice for impulse and uniform vertical joint loads.

Conclusions

The object here is to encourage the use of discrete field techniques for the analysis and design of latticed shells. The paper is of an introductory nature and attention is restricted to the more elementary problems within the subject area. Difference equations are derived for the statically determinate member forces in single-layered triply-latticed systems whose joints project as evenly spaced points on a reference plane. New closed form solutions are found for the forces in a uniformly loaded Hyperbolic Paraboloidal Lattice and in the Elliptic Paraboloidal Lattice with an arbitrary variation of vertical joint loads.

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Appendix: Notation

The following symbols have been adopted for use in this paper:

A, B, C	=	Actual lengths of lattice members.
A_{ij}	=	Coefficients in double series.
a, b, c	=	Projected lengths of lattice members.
C_i, D_i	=	Constants in single series.
C_{ij}	=	Elements in finite difference operator matrix.
E_x, E_y	=	Finite difference displacement operators.
$F(x, y)$	=	Stress Function.
i, j	=	Exponents of displacement operators and indices denoting terms in series.
$\bar{i}, \bar{j}, \bar{k}, \bar{l}$	=	Unit basis vectors.
G, G_1, G_2	=	Arbitrary functions of discrete coordinates.
M, N	=	Discrete coordinates of lattice boundaries.
m, n	=	Orders of finite difference operators.
$\bar{P}, p_1, p_2, p_z$	=	Load vector and its components.
p_t, p_0	=	Linear combination of p_1, p_2, p_z and a reference load.
$\bar{P}_e, P_{ij}, P_i$	=	Edge load vector and coefficients of loading series.
N_1, N_2, N_3	=	Forces in lattice members.
T_1, T_2, T_3	=	Plane components of forces in lattice members.
d, e, f	=	Reference out-of-plane distances for lattice corners.
u, x, y	=	Discrete coordinates in the reference plane.

$\not{U}(y-\beta)$	=	Discrete step function.
V_1, V_2, V_3	=	Out-of-plane components of member forces.
$\overline{X}, Y_i$	=	Joint position vector and Euler functions in single series.
Z, W	=	Distance from reference plane to lattice node and total load on lattice.
α, β	=	Coordinates of load.
γ_i, σ_i	=	Related parameters in single series.
$\Delta, \nabla, \square$	=	Forward, backward and second central finite difference operators.
$\delta_x^\alpha, \delta_y^\beta$	=	Kronecker Delta load functions.
φ, Φ	=	Angle between lattice coordinates and function of displacement operators.

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Summary

The governing difference equations are derived for the member forces in a class of latticed shells constructed so that the node projections are evenly spaced on a plane. The approach is the discrete analog to the popular projected plane analysis of continuous surface shells. A "Stress Function" is used to convert the vector equation to scalar form. Solutions are written for the Latticed Hyperbolic Paraboloid and the Latticed Elliptic Paraboloid. The lattices are single layered and designed for two-force members.

Résumé

L'auteur établit les équations aux différences qui déterminent les efforts dans les barres de voiles triangulés dont les nœuds, en projection sur un plan, sont équidistants. Ce procédé est analogue, pour une structure discontinue, à celui bien connu pour l'étude en projection plane des contraintes des voiles classiques. Pour ramener l'équation vectorielle à une forme scalaire, on utilise une «fonction de contrainte». L'auteur donne des solutions pour les treillis en forme de paraboloïde hyperbolique ou elliptique. Toutes les barres du treillis se trouvent dans une même surface et résistent aussi bien à la compression qu'à la traction.

Zusammenfassung

Für die Bestimmung der Stabkräfte von Gitterschalen, deren Knoten in der Projektion auf eine Bezugsebene einen gleichen Abstand aufweisen, werden die maßgebenden Differenzengleichungen abgeleitet. Das Verfahren ist analog zur bekannten Berechnung kontinuierlicher Schalen in der Grundrißprojektion. Es wird eine Spannungsfunktion benutzt, um die Vektorgleichung in skalarer Form anschreiben zu können. Anschließend werden Lösungen angeschrieben für Gitterschalen mit der Form eines hyperbolischen oder elliptischen Paraboloids. Die Gitter sind in einlagiger Anordnung und entworfen für druck- und zugfeste Stabelemente.

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