Formulation of Equilibrium Equations for pin-jointed structures

Formulation of Equilibrium Equations for Pin-Jointed Structures

Sur la manière de formuler les conditions d'équilibre dans les systèmes triangulés

Die Formulierung der Gleichgewichtsbedingungen in Gelenkbolzenfachwerken

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1. Introduction

There are two methods of analyzing statically indeterminate structures; namely, the force method (Kraftgrößenverfahren) and the deformation method (Formänderungsverfahren). When we analyze the statically indeterminate pin-jointed structures, the force method is generally used. Taking the spandrel braced arch as an example, we take the horizontal reaction force as a redundant force, and the simple truss arch hinged at one end and supported on rollers at the other end is considered as a statically determinate structure. According to this method, the redundant force can be calculated by the following well known formula:

$$X_1 = H = -\frac{\delta_{10}}{\delta_{11}} = -\frac{\sum\left(S_1 S_0 \frac{l}{EA}\right)}{\sum\left(S_1^2 \frac{l}{EA}\right)}.$$

Two terms on the right hand side of the equation are calculated in the form of a table as shown in many reference books and only the numerator can be calculated by the elastic weight.

This traditional method does not seem suitable for the use of the digital computer. For effective use of a digital computer, analysis of the structure must yield simultaneous equations, involving only the properties and dimensions of the structures, i.e., Young's modulus, length of members and cross-sectional area, but not the axial force as in the ordinary calculation. One should be able to write these equations in concise and simple way, namely, by means of tabulation.

In analyzing the statically indeterminate pin-jointed structures using digital computers, it is better, therefore, to use the deformation method than the force method. Such a method has already been investigated by Prof. K. MISE, but his research has not been well known. Recently this method has become of interest and a few publications on this subject are found. The author has investigated this problem from the point of view of effective use of a digital computer in structural analysis, and wishes to propose the mechanical tabulation method to write the equilibrium equations in a short time.

2. Notation

A Cross-sectional area of a member.
E Modulus of elasticity.
P, Q Components of the external force applied to the pin joint in the direction of x- and y-axis.
S Axial force in a member.
k Extensibility-ratio $(=\rho/\rho_c)$.
u, v Displacements of the pin joint in the direction of x- and y-axis, or the production of displacement u and v and standard extensibility factor $(EA/l)_c$.
x, y Cartesian coordinates.
α, β Angles between the member and the x- and y-axis respectively.
ρ Extensibility-factor for a member (EA/l).

3. Equilibrium Equations at a Pin Joint

As shown by Prof. K. MISE and recently by Prof. H. C. MARTIN, the axial force of a member having pin joints at both ends can be expressed by the following well known equation, refering to Fig. 1:

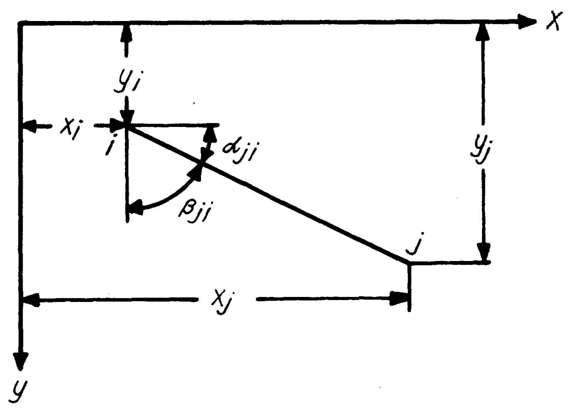

Fig. 1.

$$\begin{aligned} S_{ji} &= \frac{EA_{ji}}{l_{ji}}\left\{\frac{(u_j-u_i)(x_j-x_i)}{l_{ji}}+\frac{(v_j-v_i)(y_j-y_i)}{l_{ji}}\right\} \\ &= k_{ji}\{(u_j-u_i)\cos\alpha_{ji}+(v_j-v_i)\cos\beta_{ji}\}. \end{aligned} \tag{1}$$

Let us consider the pin joint 0, which is the point of intersection of members 01, 02, 03 and 04 (Fig. 2). The axial forces in each member can be expressed as follows:

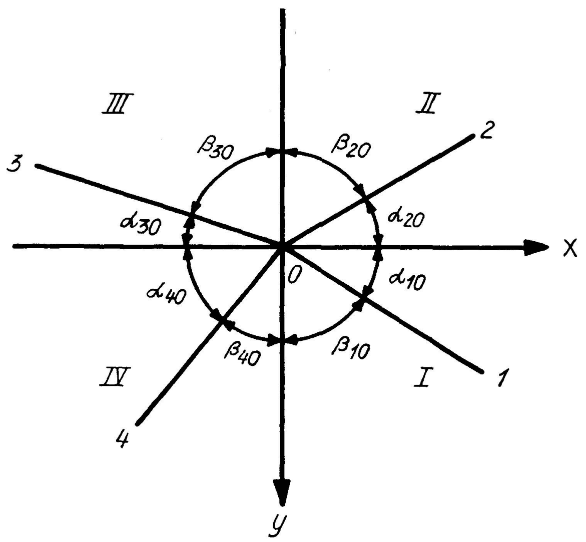

Fig. 2.

$$\begin{aligned} S_{10} &= k_{10}\{\ \ (u_1-u_0)\cos\alpha_{10}+(v_1-v_0)\cos\beta_{10}\}, \\ S_{20} &= k_{20}\{\ \ (u_2-u_0)\cos\alpha_{20}-(v_2-v_0)\cos\beta_{20}\}, \\ S_{30} &= k_{30}\{-(u_3-u_0)\cos\alpha_{30}-(v_3-v_0)\cos\beta_{30}\}, \\ S_{40} &= k_{40}\{-(u_4-u_0)\cos\alpha_{40}+(v_4-v_0)\cos\beta_{40}\}. \end{aligned}$$

The equilibrium equations for the pin joint 0 are as follows:

for x-direction,

$$\sum F_x = 0: \quad S_{10}\cos\alpha_{10}+S_{20}\cos\alpha_{20}-S_{30}\cos\alpha_{30}-S_{40}\cos\alpha_{40}+P_0 = 0 \tag{2$_1$}$$

for y-direction,

$$\sum F_y = 0: \quad S_{10}\cos\beta_{10}-S_{20}\cos\beta_{20}-S_{30}\cos\beta_{30}+S_{40}\cos\beta_{40}+Q_0 = 0 \tag{2$_2$}$$

Substituting the values of S_{10}, S_{20}, S_{30} and S_{40} into Eq. (2), the following equations can be obtained;

$$\begin{aligned} &[(\sum_{i=1}^{4} k_{i0}\cos^2\alpha_{i0})(u_0)-\sum_{i=1}^{4}(k_{i0}\cos^2\alpha_{i0})(u_i)]+[(\sum_{i=1,3} k_{i0}\cos\alpha_{i0}\cos\beta_{i0} \\ &\quad -\sum_{i=2,4} k_{i0}\cos\alpha_{i0}\cos\beta_{i0})(v_0)-\sum_{i=1,3}(k_{i0}\cos\alpha_{i0}\cos\beta_{i0})(v_i) \\ &\quad +\sum_{i=2,4}(k_{i0}\cos\alpha_{i0}\cos\beta_{i0})(v_i)] = P_0, \end{aligned} \tag{3$_1$}$$

$$[(\sum_{i=1,3} k_{i0}\cos\alpha_{i0}\cos\beta_{i0} - \sum_{i=2,4}(k_{i0}\cos\alpha_{i0}\cos\beta_{i0})(u_0)$$
$$-\sum_{i=1,3}(k_{i0}\cos\alpha_{i0}\cos\beta_{i0})(u_i) + \sum_{i=2,4}(k_{i0}\cos\alpha_{i0}\cos\beta_{i0})(u_i)] \qquad (3)_2$$
$$+[(\sum_{i=1}^{4} k_{i0}\cos^2\beta_{i0})(v_0) - \sum_{i=1}^{4}(k_{i0}\cos^2\beta_{i0})(v_i)] = Q_0.$$

From the above equations we can notice the following important points:

a) The coefficients of the term u in Eq. $(3)_1$ are the same as the coefficients of the term v in Eq. $(3)_2$, except the difference in $\cos^2\alpha_{i0}$ and $\cos^2\beta_{i0}$.

b) The coefficients of the term v in Eq. $(3)_1$ are the same as the coefficients of the term u in Eq. $(3)_2$.

c) The term $\sum_{i=0}^{4} k_{i0}\cos^2\alpha_{i0}$ or $k_{i0}\cos^2\beta_{i0}$ means the sum of the products of extensibility-ratios and the square of the direction cosine of the members around the pin joint 0. $\sum_{i=1,3} k_{i0}\cos\alpha_{i0}\cos\beta_{i0}$ is the sum of $k\cos\alpha\cos\beta$ of the two members which lie in quadrant I and III, and the coordinates of the other pin joint of these two members are both positive or negative ($x>0$, $y>0$ or $x<0$, $y<0$) (therefore, the product of the values of both ordinates are positive). $\sum_{i=2,4} k_{i0}\cos\alpha_{i0}\cos\beta_{i0}$ is the sum of $k\cos\alpha\cos\beta$ of the two members which lie in the quadrant II and IV, and the coordinates of the other pin joint of these two members are of opposite sign ($x>0$, $y<0$ or $x<0$, $y>0$) (therefore, their product is negative).

Eq. (3) facilitates the following simple, concise and mechanical tabulation method of equilibrium equations for each pin joint.

4. Mechanical Tabulation Method

To write the equilibrium equation of a pin joint by deformation method, the procedure given below can be followed:

1. From the given data, calculate $(k_{i0})_{\alpha\alpha} = k_{i0}\cos^2\alpha_{i0}$, $(k_{i0})_{\alpha\beta} = k_{i0}\cos\alpha_{i0}\cdot\cos\beta_{i0}$ and $(k_{i0})_{\beta\beta} = k_{i0}\cos^2\beta_{i0}$, and make three $k_{\alpha\alpha}$-, $k_{\alpha\beta}$- and $k_{\beta\beta}$-diagrams.
2. Calculate the values of $\sum k_{\alpha\alpha}$, $\sum_{xy>0} k_{\alpha\beta} - \sum_{xy<0} k_{\alpha\beta}$ and $\sum k_{\beta\beta}$ for all pin Joints.

The symbol $xy>0$ means that the member considered lies in the quadrant I and III ($x>0$, $y>0$ or $x<0$, $y<0$) and $xy<0$ that the member considered lies in the quadrant II and IV ($x>0$, $y<0$ or $x<0$, $y>0$), when the origin of coordinate is moved to the pin joint under consideration. The positive direction of axes may be chosen in any manner.

3. Number the pin joints of the given structures from left to right.
4. Prepar the row and column of the table.
5. Insert major column headings (u and v) and major row headings ($F_x=0$ and $F_y=0$), and introduce numerical row and column labels $(1,2,3,\ldots)$.
6. Write the coefficients of the terms u and v of the equilibrium equation $\sum F_x=0$, refering to $k_{\alpha\alpha}$- and $k_{\alpha\beta}$-diagrams.
7. Next, write the coefficients of term u and v of the equilibrium equation $\sum F_y=0$, refering to $k_{\alpha\beta}$- and $k_{\beta\beta}$-diagrams.

At steps 6. and 7., the coefficients must be written according to the following rules.

Rules of Coefficients of Term u and v

Equation	Term	Pin joint under consideration	Surrounding joints
$\sum F_x = 0$	u	$\sum k_{\alpha\alpha}$	$-k_{\alpha\alpha}$ for all members
	v	$\sum\limits_{xy>0} k_{\alpha\beta} - \sum\limits_{xy<0} k_{\alpha\beta}$	$-k_{\alpha\beta}$ for all members $(xy>0)$ $k_{\alpha\beta}$ for all members $(xy<0)$
$\sum F_y = 0$	u		
	v	$\sum k_{\beta\beta}$	$-k_{\beta\beta}$ for all members

8. The following checks can be made use of:

a) the coefficients must be symmetrical about the diagonal,
b) the sum of the coefficients must be zero in the quarter part, except for the equilibrium equation of the pin joints which contain the members connected to the supports.

9. It is better to write the terms of u and v and the equilibrium equations of both supports. After substituting the coefficients in the table, the coefficients of the term for the supports must be eliminated according to the following conditions:

a) for hinged support: $u=0$ and $v=0$ in column, $\sum F_x=0$ and $\sum F_y=0$ in row,
b) for roller support: $v=0$ in column, $\sum F_y=0$ in row.

The above systematic and mechanical tabulation method simplifies the completion of the equilibrium equations, and even in the case of very complicated pin-jointed structures, they can be written in a few minutes. An example will be shown in the next chapter.

5. Example

Let us analyze the spandrel braced arch (Fig. 3) which is shown as an example in the reference book "Analysis of Statically Indeterminate Structures" by J. I. PARCEL and R. B. B. MOORMAN, p. 505.

The calculation will be made in the following order:

1. Data of each member are shown in Table 1.
2. According to the last three columns of Table 1, make three k-diagrams (Fig. 4).

Table 1. Characteristics of Members

Member		A (in²)	l (in)	$(A/l)\cdot 10^2$	k	$k \cos\alpha$	$k \cos\beta$	$k \cos^2\alpha$	$k \cos\alpha\cdot\cos\beta$	$k \cos^2\beta$
U	24	16.80	360	4.66667	0.38251	0.38251	0	0.38251	0	0
	46	19.07	360	5.29722	0.43420	0.43420	0	0.43420	0	0
	68	16.80	360	4.66667	0.38251	0.38251	0	0.38251	0	0
V	12	21.70	552	3.93116	0.32223	0	0.32223	0	0	0.32223
	34	18.78	312	6.01923	0.49338	0	0.49338	0	0	0.49338
	56	17.58	168	10.98750	0.90061	0	0.90061	0	0	0.90061
	78	14.64	120	12.20000	1.00000	0	1.00000	0	0	1.00000
L	13	58.14	432.7	13.43656	1.10136	0.91632	0.61088	0.76236	0.50824	0.33882
	35	51.40	387.7	13.25767	1.08669	1.00905	0.40362	0.93695	0.37478	0.14991
	57	43.80	363.2	12.05947	0.98848	0.98077	0.13077	0.97114	0.12948	0.01726
D	23	12.06	476.4	2.53149	0.20750	0.15680	0.13589	0.11849	0.10269	0.08900
	45	12.06	397.3	3.03549	0.24881	0.22545	0.10521	0.20429	0.09533	0.04449
	67	17.58	379.5	4.68800	0.38361	0.36390	0.12130	0.34520	0.11507	0.03836

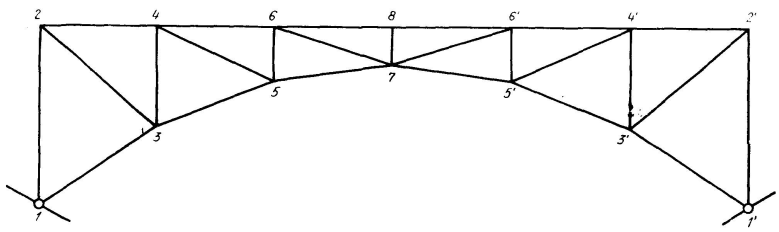

Fig. 3.

3. Make the rows and columns of Table 2, and insert the unknown terms ($u_1 \sim u_{1'}$, and $v_1 \sim v_{1'}$). In this case (u_1, v_1) and $(u_{1'}, v_{1'})$ are not necessary, but it is better to write them for providing a check. Write the number of joints at which the equilibrium equation must be written in the most left hand side of the column.

4. Write the coefficients, using three k-diagrams.

5. Check whether the sum of coefficients is zero or not.

6. Eliminate the columns of u_1, v_1, $u_{1'}$ and $v_{1'}$, and the rows of $\sum F_{x,1}=0$, $\sum F_{y,1}=0$, $\sum F_{x,1'}=0$ and $\sum F_{y,1'}=0$.

7. Check the symmetry of the coefficients about the diagonal.

If the above checks are satisfactory, the simultaneous equations are just completed. Table 2 shows the final form of the simultaneous equations.

8. Make the input tape or card for the digital computer.

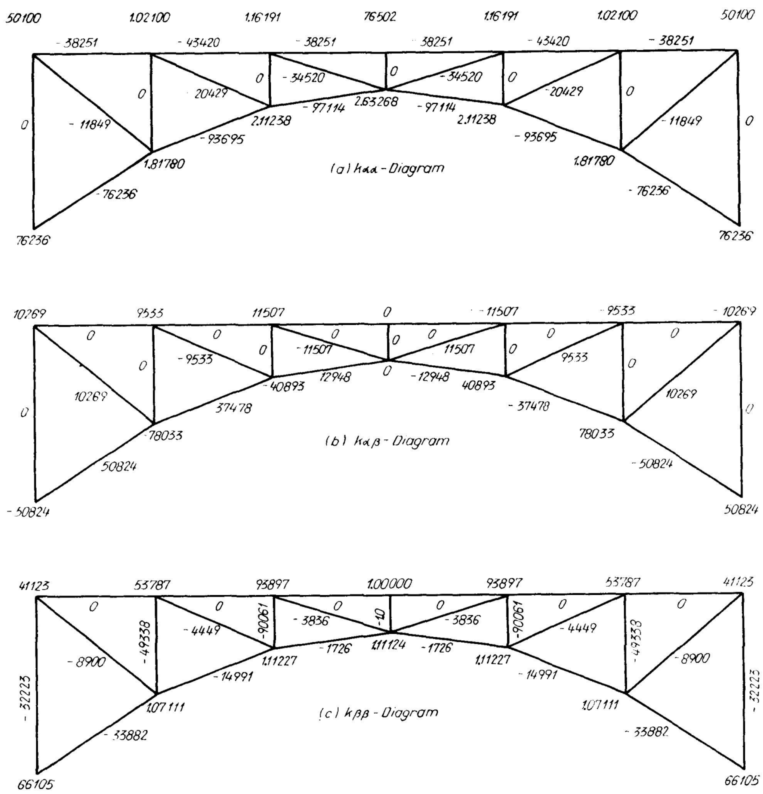

Fig. 4.

Such simultaneous equations can be solved easily in a very short time by a digital computer. If we wish to get the ordinate of the influence lines of axial force of each member, we solve seven sets of simultaneous equations; $(P=Q=0$ except $Q_2=1)$, $(P=Q=0$ except $Q_4=1)$, $(P=Q=0$ except $Q_6=1)$, $(P=Q=0$ except $Q_8=1)$, $(P=Q=0$ except $Q_{6'}=1)$, $(P=Q=0$ except $Q_{4'}=1)$ and $(P=Q=0$ except $Q_{2'}=1)$. The results of solution by ILLIAC, the University of Illinois high-speed digital computer, are shown in Table 3. These are the ordinates of influence lines of displacement of each pin joint multiplied by standard extensibility-factor $(EA/l)_c$.

Using Table 3, we can calculate the ordinates of influence lines of the axial force of each member. For example, the equations for axial force of several members near the hinged support 1 will be shown as follows and the results will be shown in Table 4.

Table 2. Equilibrium Equations at the Pin Joints

	Unknowns	u											
		2	3	4	5	6	7	8	6′	5′	4′	3′	2′
$\sum F_x = 0$	2	0.50100	-0.11849	-0.38251									
	3	-0.11849	1.81780	0.	-0.93695								
	4	-0.38251	0.	1.02100	-0.20429	-0.43420							
	5		-0.93695	-0.20429	2.11238	0.	-0.97114					$k_{\alpha\alpha}$	
	6			-0.43420	0.	1.16191	-0.34520	-0.38251					
	7				-0.97114	-0.34520	2.63268	0.	-0.34520	-0.97114			
	8					-0.38251	0.	0.76501	-0.38251	0.			
	6′						-0.34520	-0.38251	1.16191	0.	-0.43420		
	5′						-0.97114	0.	0.	2.11238	-0.20429	-0.93695	
	4′								-0.43420	-0.20429	1.02100	0.	-0.38251
	3′									-0.93695	0.	1.81780	-0.11849
	2′										-0.38251	-0.11849	0.50100
$\sum F_y = 0$	2	0.10269	-0.10269	0.									
	3	-0.10269	-0.78033	0.	0.37478								
	4	0.	0.	0.09533	-0.09533	0.							
	5		0.37478	-0.09533	-0.40893	0.	0.12948					$k_{\alpha\beta}$	
	6			0.	0.	0.11507	-0.11507	0.					
	7				0.12948	-0.11507	0.	0.	0.11507	-0.12948			
	8					0.	0.	0.	0.				
	6′						0.11507	0.	-0.11507	0.	0.		
	5′						-0.12948		0.	0.40893	0.09533	-0.37478	
	4′								0.	0.09533	-0.09533	0.	0.
	3′									-0.37478	0.	0.78033	0.10269
	2′										0.	0.10269	-0.10269

continued on p. 187

	v												Right Hand Side
	2	3	4	5	6	7	8	6′	5′	4′	3′	2′	
	0.10269	-0.10269	0.										P_2
	-0.10269	-0.78033	0.	0.37478									P_3
	0.	0.	0.09533	-0.09533	0.								P_4
		0.37478	-0.09533	-0.40893	0.	0.12948							P_5
			0.	0.	0.11507	-0.11507	0.				$k_{\alpha\beta}$		P_6
				0.12948	-0.11507	0.	0.	0.11507	-0.12948				P_7
					0.	0.	0.	0.					P_8
						0.11507	0.	-0.11507	0.	0.			P'_6
						-0.12948		0.	0.40893	0.09533	-0.37478		P'_5
								0.	0.09533	-0.09533	0.	0.	P'_4
									-0.37478	0.	0.78033	0.10269	P'_3
										0.	0.10269	-0.10269	P'_2
	0.41123	-0.08900	0.										Q_2
	-0.08900	1.07111	-0.49338	-0.14991									Q_3
	0.	-0.49338	0.53787	-0.04449	0.								Q_4
		-0.14991	-0.04449	1.11227	-0.90061	-0.01726					$k_{\beta\beta}$		Q_5
			0.	-0.90061	0.93897	-0.03836	0.						Q_6
				-0.01726	-0.03836	1.11124	-1.00000	-0.03836	-0.01726				Q_7
					0.	-1.00000	1.00000	0.					Q_8
						-0.03836	0.	0.93897	-0.90061	0.			Q'_6
						-0.01726		-0.90061	1.11227	-0.04449	-0.14991		Q'_5
								0.	-0.04449	0.53787	-0.49338	0.	Q'_4
									-0.14991	-0.49338	1.07111	-0.08900	Q'_3
										0.	-0.08900	0.41123	Q'_2

$$S_{31} = k_{31}(u_3 \cos\alpha_{31} - v_3 \cos\beta_{31}),$$
$$S_{42} = k_{42}(u_4 - u_2),$$
$$S_{12} = -k_{12} v_{12},$$
$$S_{32} = k_{32}\{(u_3 - u_2)\cos\alpha_{32} + (v_3 - v_2)\cos\beta_{32}\},$$
$$S_{43} = k_{43}(v_3 - v_4),$$
$$S_{53} = k_{53}\{(u_5 - u_3)\cos\alpha_{53} - (v_5 - v_3)\cos\beta_{53}\}.$$

Table 3

	Q_2	Q_4	Q_6	Q_8	Q'_6	Q'_4	Q'_2
u_2	0.04329	5.71197	6.59276	0.97543	-4.88688	-4.03118	-0.43964
u_3	1.13664	5.14941	2.97199	-1.35987	-3.82050	-2.68919	-0.24276
u_4	0.06715	4.04861	6.20063	1.54394	-4.27340	-3.68341	-0.41575
u_5	0.58658	4.77118	6.22139	-0.73845	-5.06270	-3.76157	-0.37906
u_6	0.12965	2.98468	3.85822	2.04515	-3.32557	-3.10214	-0.35321
u_7	0.41385	3.87148	5.01822	0	-5.01822	-3.87148	-0.41385
u_8	0.24143	3.04341	3.59190	0	-3.59190	-3.04341	-0.24143
v_2	3.07881	1.71130	0.40347	-0.58487	-0.63117	-0.35780	-0.02459
v_3	1.72834	8.55625	6.04196	-0.00790	-4.14677	-3.20166	-0.34076
v_4	1.71130	10.41820	6.86215	-0.00801	-4.31408	-3.31456	-0.35780
v_5	0.40925	7.04147	15.91327	4.88136	-4.47813	-4.39919	-0.62539
v_6	0.40347	6.86215	16.68485	5.25149	-4.28807	-4.31408	-0.63117
v_7	-0.58487	-0.00801	5.25149	20.07638	5.25149	-0.00801	-0.58487
v_8	-0.58487	-0.00801	5.25149	21.07638	5.25149	-0.00801	-0.58487

Table 4. Ordinates of Influence Lines of the Axial Force of Several Members

		Q_2	Q_4	Q_6	Q_8	Q'_6	Q'_4	Q'_2
(1)	S_{31}	-0.0143	-0.5083	-0.9676	-1.2413	-0.9676	-0.5083	-0.0143
(2)	S_{12}	-0.9921	-0.5514	-0.1300	0.1885	0.2034	0.1153	0.0079
(3)	S_{32}	-0.0121	0.8420	0.1985	-0.2878	-0.3105	-0.1760	-0.0121
(4)	S_{42}	0.0092	-0.6363	-0.1500	0.2175	0.2347	0.1330	0.0092
(5)	S_{13}	0.0084	-0.9187	-0.4047	0	0.0826	0.0557	0.0084
(6)	S_{53}	-0.0226	0.2298	-0.7055	-1.3464	-1.1197	-0.5988	-0.0226
(1′)	H	0.0119	0.4229	0.8051	1.0327	0.8051	0.4229	0.0119

According to the deformation method, the axial force of each member can be obtained directly, and there is no need to get the horizontal reaction in the beginning. In this case, it can be obtained as follows:

$$H + S_{31}\cos\alpha_{31} = 0, \qquad H = -S_{31}\cos\beta_{31}.$$

The values of ordinate of the influence line of horizontal reaction are the same as those obtained by the ordinary force method in the reference book by PARCEL and MOORMAN. The values of ordinates of the influence line of the axial force can not be compared with those shown in their book, because they are obtained as those for the preliminary design and not by the given data of each member. However, the result are very closely in agreement with each other.

6. Conclusion

Significant features of the application of the deformation method for the solution of statically indeterminate pin-jointed structures are the mechanical tabulation for writing the equilibrium equations of pin joints in terms of joint displacements and the convenience of the solution of these equations with the aid of a digital computer. The mechanical tabulation method and use of a digital computer can make the solution of complicated structures very easy. Only one example is shown, but this method can be effectively applied to the analysis of more complicated statically indeterminate pin-jointed structures such as: a) internally statically indeterminate trusses, b) externally statically indeterminate trusses (continuous truss), c) internally and externally statically indeterminate trusses, d) trussed tied arches, e) Langer trusses, f) fixed truss arches and g) balanced arches.

When we analyze the multi-story and multi-span rigid framed structures by slope deflection method, the slopes of a number of rigid joints and the lateral deflection of each story are the unknown terms. On the contrary, as can be seen from this paper, there are two unknown terms for each pin joint, that is, the displacements of the pin joint in the horizontal and vertical directions in the case of pin-jointed structures. If the number of pin joints increase, the unknown terms become numerous. This is the only weak point of this method. However, the recent development of digital computers made the solution of such equations feasible.

For example, the maximum capacity of ILLIAC in solving the linear simultaneous equations is 143 elements. Therefore, it can be said that ILLIAC can solve the statically indeterminate pin-jointed structures of about $143/4 \doteqdot 36$ panels, and so that a large digital computer such as ILLIAC can solve almost all type of statically indeterminate pin-jointed structures which we encounter in ordinary practice.

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Illinois as a postdoctoral fellow. He expresses his sincere thanks to staff members of Digital Computer Laboratory for their assistance and operation.

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Appendix 1

The author's method can be extended to the statically indeterminate space pin-jointed structures. For this case, the following notation is used:

P, Q, R: Components of the external force applied to pin joint.
u, v, w: Displacements of the pin joint corresponding to x-, y- and z-axis.
x, y, z: Cartesian coordinates.
$\cos\alpha$, $\cos\beta$, $\cos\gamma$: Direction cosines of the angle between the member and coordinate axis.

The equilibrium equations at the pin joint can be written according to following table as shown in Chapter 4, and the right hand side should be written as P, Q and R for $\sum F_x=0$, $\sum F_y=0$ and $\sum F_z=0$ respectively.

Rules of Coefficients of Term u, v and w

Row	Equation	Term	Pin joint under consideration	Surrounding pin joints
1		u	$\sum k_{\alpha\alpha}$	$-k_{\alpha\alpha}$ for all members
2	$\sum F_x=0$	v	$\sum_{xy>0} k_{\alpha\beta}-\sum_{xy<0} k_{\alpha\beta}$	$-k_{\alpha\beta}$ for all members $(xy>0)$ $k_{\alpha\beta}$ for all members $(xy<0)$
3		w	$\sum_{xz>0} k_{\alpha\gamma}-\sum_{xz<0} k_{\alpha\gamma}$	$-k_{\alpha\gamma}$ for all members $(xz>0)$ $k_{\alpha\gamma}$ for all members $(xz<0)$
4		u	same as row 2	
5	$\sum F_y=0$	v	$\sum k_{\beta\beta}$	$-k_{\beta\beta}$ for all members
6		w	$\sum_{yz>0} k_{\beta\gamma}-\sum_{yz<0} k_{\beta\gamma}$	$-k_{\beta\gamma}$ for all members $(yz>0)$ $k_{\beta\gamma}$ for all members $(yz<0)$
7		u	same as row 3	
8	$\sum F_z=0$	v	same as row 6	
9		w	$\sum k_{\gamma\gamma}$	$-k_{\gamma\gamma}$ for all members

Appendix 2

If the elongation of a member due to uniform temperature rise ($t>0$) is considered, the fundamental equation of the axial force (1) is changed into the following form; $S_{ji}=k_{ji}\{(u_j-u_i)\cos\alpha_{ji}+(v_j-v_i)\cos\beta_{ji}\}-k_{ji}l_{ji}(\rho_c\epsilon t)$.

In this case the right hand side of the simultaneous equations are written as follows:

Equilibrium Equation	Simultaneous equation	
	Left hand side	Right hand side (unit: $-\rho_c\epsilon t$)
$\sum F_x=0$	same as that written before	$\sum\limits_{x>0}(k_\alpha\cdot l)-\sum\limits_{x<0}(k_\alpha\cdot l)$
$\sum F_y=0$		$\sum\limits_{y>0}(k_\beta\cdot l)-\sum\limits_{y<0}(k_\beta\cdot l)$
$\sum F_z=0$		$\sum\limits_{z>0}(k_\gamma\cdot l)-\sum\limits_{z<0}(k_\gamma\cdot l)$

Note: $k_\alpha=k\cos\alpha$, $k_\beta=k\cos\beta$, $k_\gamma=k\cos\gamma$.

$x>0\,(y>0, z>0)$ means that the member under consideration lies in the plane, of which the coordinate value with regard to x- (y-, z-) axis is positive.

Summary

This paper deals with the solution of complicated pin-jointed structures by means of the deformation method (not of the force method). We can write the equilibrium equations of the pin joints and obtain simultaneous equations expressed in terms of displacements of the joints of each member. It is necessary to write mechanically, systematically and rapidly these simultaneous equations for each joint; for this purpose the mechanical tabulation method is very convenient. Example applied to the spandrel braced arch is shown.

Résumé

L'auteur expose l'application de la méthode aux déformations (par opposition à la méthode aux forces) au calcul des systèmes triangulés compliqués. En écrivant les conditions d'équilibre à chaque articulation, on obtient un système d'équations dont les inconnues sont les déplacement des nœuds, aux articulations des barres. Ces équations doivent être écrites de façon mécanique, systématique et rapide, en chaque nœud; il est donc indiqué de recourir à des tables que l'on remplit de façon mécanique. L'auteur donne pour terminer une application numérique: le calcul d'un arc triangulé à deux articulations.

Zusammenfassung

Der Autor befaßt sich mit der Berechnung komplizierter Gelenkbolzenfachwerke mit Hilfe der Deformationsmethode (nicht mit der Kräftemethode). Die Gleichgewichtsbedingungen werden für jeden Gelenkpunkt angeschrieben, wobei in den entstehenden Simultangleichungen als Unbekannte die Verschiebungen aller Gelenkpunkte auftreten. Diese Simultangleichungen müssen mechanisch, rasch und systematisch für jeden Gelenkpunkt aufgestellt werden; es empfiehlt sich hier tabellarisch vorzugehen. Zuletzt wird die Methode auf die Berechnung eines Zweigelenkfachwerkbogens angewendet.