Analysis of corner supported grillages

Autor(en): **Lee, S.L. / Makino, M.**

Objekttyp: **Article**

Zeitschrift: **IABSE publications = Mémoires AIPC = IVBH Abhandlungen**

Band (Jahr): **20 (1960)**

PDF erstellt am: **24.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-17560

Analysis of Corner-Supported Grillages

Recherches concernant les grilles portantes, appuyées aux angles

Untersuchung von Trägerrosten, die in Eckpunkten gelagert sind

S. L. LEE
Associate Professor of Civil Engineering,

M. MAKINO
Institute Fellow, Technological Institute,

Northwestern University, Evanston, Illinois, U.S.A.

1. Introduction

A grillage, or grid framework, is a structure composed of two systems of intersecting flexural members, the members in each system being parallel to one another (Fig. 1a). Most grillages encountered in practice are orthogonal ones in which the two systems intersect each other at right angle. The members of each system are continuous through the points of intersection. The interaction between the two systems at these points under loads normal to the plane of the grillage may be represented, in general, by a force in the z-direction and two moment components about the x- and y-axes in cases where the members are rigidly connected. If the torsional stiffness of the members is negligible or when the connections are incapable of transmitting bending and torsional moments between the two systems, the interaction consists only of a vertical reaction.

The analysis of grillages supported on all four sides was discussed by TIMOSHENKO [1] [1]), KLITCHIEFF [2, 3], HOLMAN [4], and HENDRY and JAEGER [5]. The last reference also dealt with grillages supported on two and three sides. The interconnected bridge girders, grillages supported on two sides, were analyzed by PIPPARD and DE WAELE [6], HETÉNYI [7], COVINGTON [8], JAEGER and HENDRY [9, 10, 11] and BEER and REISINGER [12]. The load distribution in continuous interconnected bridge girders was discussed in [9]. A more complete listing of earlier references was given in [8] where a historical

[1]) Numbers in bracket [ ] refer to the listing of references at the end of the paper.

review of the subject, including brief discussion of the different methods of analysis used and the assumptions made, can be found.

The treatment of grillages with more complicated boundary conditions by moment distribution method was discussed by EWELL, OKUBO and ABRAMS [13]. More recently, SBAROUNIS and GAUS [14] analyzed a two-way orthogonal truss system by relaxation method. Another solution of the same problem by the use of influence coefficients was also indicated in the last reference in which only the results of the analysis were given.

2. Notation

A_{jx}, C_{iy} Fourier coefficients for the j-th member in the x-direction and the i-th member in the y-direction defined by (2a, c) respectively.

a, b Side dimensions of grillage.

a_i, b_j Distances from origin to the i-th member along x-axis and to the j-th member along y-axis respectively.

B_{ij}, D_{ij} Fourier coefficients for point ij defined by (2b, d) respectively.

$(EI)_{jx}, (EI)_{iy}$ Flexural rigidity of the j-th member in the x-direction and the i-th member in the y-direction respectively.

$F(y)$ Function of y.

ij, rs, uv Subscripts denoting points of intersection on grillage, the first subscript referring to x-axis and the second one to y-axis.

k $\frac{b^3}{(EI)_y}\frac{(EI)_x}{a^3}$.

k', k'', k''' Functions of k defined by (c), (f) and (g) respectively.

k_1, k_2, k_3, k_4 Functions of k defined by (e).

M_{2y} Bending moment at section y in the second member in the y-direction.

n Integer.

P Concentrated load.

Q'_{jx}, Q''_{jx} End reactions on the left end and the right end of the j-th member in the x-direction defined by (5a, b) respectively.

Q'_{iy}, Q''_{iy} End reactions on the left end and the right end of the i-th member in the y-direction defined by (5c, d) respectively.

p Number of members in the y-direction.

q Number of members in the x-direction.

R Reaction at intermediate supports.

R_{ij}, R_{rs}, R_{uv} Reactions at points ij, rs and uv respectively.

t Length.

U	Total strain energy in grillage.
U'	Total strain energy in grillage of irregular shape.
U_{jx}, U_{iy}	Strain energy in the j-th member in the x-direction and the i-th member in the y-direction respectively.
w	Uniformly distributed load.
$w_j(x), w_i(y)$	Arbitrary loads on the j-th member in the x-direction and the i-th member in the y-direction respectively.
x, y, z	Coordinate axes.
z_{ij}	Deflection of point ij.
z_{jx}, z_{iy}	Deflections of the j-th member in the x-direction and the i-th member in the y-direction respectively.
ρ_{ij}^{rs}	Relative flexibility influence coefficient defined by (9), the relative deflection of point rs due to reaction at ij.
λ^{rs}	Relative deflection at point rs due to applied loads, defined by (10).
$\varphi_{ij}^{rs}, \psi_{ij}^{rs}$	Terms in ρ_{ij}^{rs} defined by (9b, c, d, e).

3. Assumptions

In the following discussion, the assumption is made that the interaction between intersecting members consists only of a vertical force. This assumption is valid in the case of a two-way orthogonal truss system where the torsional rigidity of the trusses is negligible in comparison with their flexural rigidity. It should also yield fairly good approximation in the design of orthogonal grillages composed of relatively deep members of I-section for the same reason, even when the members are rigidly connected. The analysis is applicable to both orthogonal and skew grillages in which the connections are incapable of transmitting bending and torsional moment. A case in point is a grillage composed of one system of beams laid on top of another with simple connections at the points of intersection.

It is also assumed that the cross sections of all the members are uniform along their lengths. However, individual member may have different flexural rigidity. The spacings of the members in either direction are arbitrary.

4. Corner-Supported Grillages

Deflection Functions and Strain Energy

The grillage shown in Fig. 1a is composed of p members in the y-direction and q members in the x-direction and simply supported at the corners designated by 11, $1p$, $1q$ and pq. The grillage may be in the shape of either a rectangle or a parallelogram. It can be shown [2]) that the deflections of the j-th member in the x-direction and the i-th member in the y-direction my be expressed respectively by

$$
\begin{aligned}
z_{jx} &= z_{1j} + (z_{pj} - z_{1j})\frac{x}{a} + \frac{a^4}{(EI)_{jx}\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}(A_{jx} - \sum_{i=2}^{p-1} B_{ij})\sin\frac{n\pi x}{a}, \\
z_{iy} &= z_{i1} + (z_{iq} - z_{i1})\frac{y}{b} + \frac{b^4}{(EI)_{iy}\pi^4}\sum_{n-1}^{\infty}\frac{1}{n^4}(C_{iy} + \sum_{j=2}^{q-1} D_{ij})\sin\frac{n\pi y}{b},
\end{aligned}
\tag{1}
$$

in which

$$
\begin{aligned}
A_{jx} &= \frac{2}{a}\int_0^a w_j(x)\sin\frac{n\pi x}{a}\,dx, \\
B_{ij} &= \frac{2}{a}\lim_{t\to 0}\int_{a_i-t}^{a_i+t}\frac{R_{ij}}{2t}\sin\frac{n\pi x}{a}\,dx = \frac{2R_{ij}}{a}\sin\frac{n\pi a_i}{a}, \\
D_{iy} &= \frac{2}{b}\int_0^b w_i(y)\sin\frac{n\pi y}{b}\,dy, \\
D_{ij} &= \frac{2}{b}\lim_{t\to 0}\int_{b_j-t}^{b_j+t}\frac{R_{ij}}{2t}\sin\frac{n\pi y}{b}\,dy = \frac{2R_{ij}}{b}\sin\frac{n\pi b_j}{b}.
\end{aligned}
\tag{2}
$$

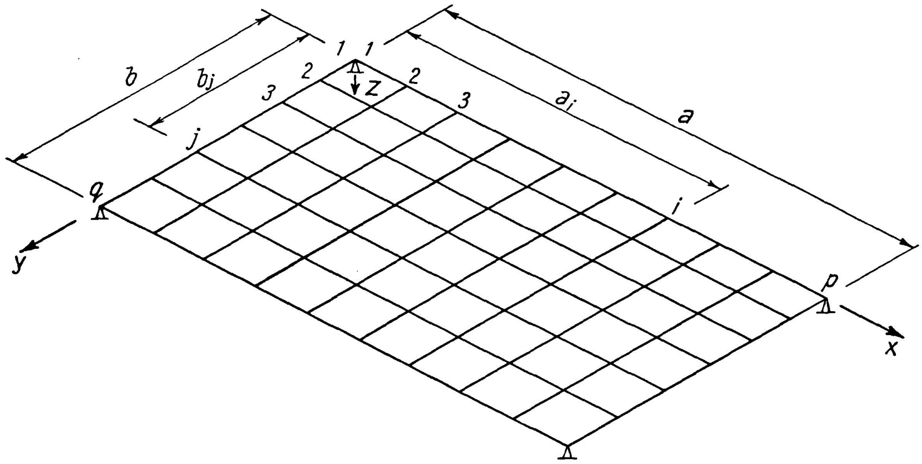

Fig. 1a. Grillage Supported at the Corners.

In these equations, $z_{1j}, z_{pj} \ldots$ denote respectively the deflection at points $1j, pj \ldots$; $(EI)_{jx}$ and $(EI)_{iy}$ the flexural rigidity of the j-th member in the x-direction and the i-th member in the y-direction respectively; $w_j(x)$ and $w_i(y)$ the arbitrary loads on the j-th member and the i-th member respectively; a and b are the overall dimensions of the grillage and R_{ij} represents the reaction between the members at points ij. The positive direction of R_{ij} is shown in

[2]) See for instance [15].

Fig. 1b, c. The corresponding strain energy in the members are given respectively by

$$U_{jx} = \frac{a^5}{4(EI)_{jx}\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}(A_{jx} - \sum_{i=2}^{p-1} B_{ij})^2,$$
$$U_{iy} = \frac{b^5}{4(EI)_{iy}\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}(C_{iy} + \sum_{j=2}^{q-1} D_{ij})^2. \qquad (3)$$

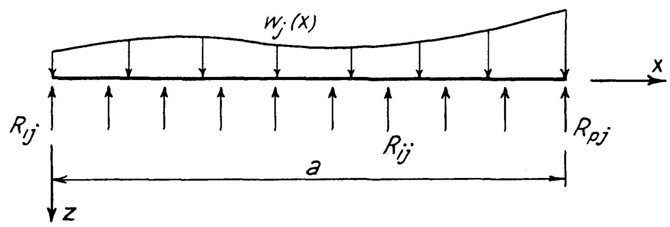

Fig. 1b. Positive Direction of R_{ij} on Member j in x-Direction.

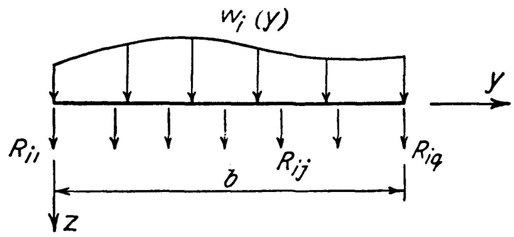

Fig. 1c. Positive Direction of R_{ij} on Member i in y-Direction.

Equations (1), (2) and (3) are applicable to all the members in the grillage. When dealing with the four edge members, however, it should be noted that $z_{11} = z_{p1} = z_{1q} = z_{pq} = 0$. For these four members, R_{ij} in (2b, d) are respectively the end reactions on the j-th member and the i-th member shown respectively in Fig. 1b, c which may be expressed in terms of the intermediate reactions. Thus

$$R_{1j} = Q'_{jx} - \sum_{i=2}^{p-1}\frac{a-a_i}{a}R_{ij}, \qquad R_{pj} = Q''_{jx} - \sum_{i=2}^{p-1}\frac{a_i}{a}R_{ij},$$
$$R_{i1} = -Q'_{iy} - \sum_{j=2}^{q-1}\frac{b-b_j}{b}R_{ij}, \qquad R_{iq} = -Q''_{iy} - \sum_{j=2}^{q-1}\frac{b_j}{b}R_{ij}, \qquad (4)$$

in which

$$Q'_{jx} = \frac{1}{a}\int_0^a (a-x)\,w_j(x)\,dx, \qquad Q''_{jx} = \frac{1}{a}\int_0^a x\,w_j(x)\,dx,$$
$$Q'_{iy} = \frac{1}{b}\int_0^b (b-y)\,w_i(y)\,dy, \qquad Q''_{iy} = \frac{1}{b}\int_0^b y\,w_i(y)\,dy. \qquad (5)$$

It follows, therefore, that the grillage presently under consideration is statically indeterminate to the $(p-2)(q-2)$ degree and that the intermediate reactions R_{ij}, for $i=2,3\ldots(p-1)$ and $j=2,3\ldots(q-1)$, may be conveniently taken as the redundant quantities.

The total strain energy in the grillage is obtained by taking the summation of (3) over the entire structure which leads to

$$U = \sum_{j=1}^{q} U_{jx} + \sum_{i=1}^{p} U_{iy} \tag{6}$$
$$= \frac{a^5}{4\pi^4}\sum_{j=1}^{q}\sum_{n=1}^{\infty}\frac{1}{(EI)_{jx}n^4}(A_{jx}-\sum_{i=2}^{p-1}B_{ij})^2 + \frac{b^5}{4\pi^4}\sum_{i=1}^{p}\sum_{n=1}^{\infty}\frac{1}{(EI)_{iy}n^4}(C_{iy}+\sum_{j=2}^{q-1}D_{ij})^2.$$

Redundant Reactions

Differentiating (6) with respect to the reaction at any intermediate point of interaction, rs, and setting the result equal to zero in accordance with the principle of least work yield

$$\begin{aligned}\frac{\partial U}{\partial R_{rs}} = &-\frac{a^4}{(EI)_{sx}\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}(A_{sx}-\sum_{i=2}^{p-1}B_{is})\sin\frac{n\pi a_r}{a}\\
&+\frac{b^4}{(EI)_{ry}\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}(C_{ry}+\sum_{j=2}^{q-1}D_{rj})\sin\frac{n\pi b_s}{b}\\
&+\frac{a^4(b-b_s)}{(EI)_{1x}\pi^4 b}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(A_{1x}-\frac{2}{a}\sum_{i=2}^{p-1}R_{i1}\sin\frac{n\pi a_i}{a}\right)\sin\frac{n\pi a_r}{a}\\
&+\frac{a^4 b_s}{(EI)_{qx}\pi^4 b}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(A_{qx}-\frac{2}{a}\sum_{i=2}^{p-1}R_{iq}\sin\frac{n\pi a_i}{a}\right)\sin\frac{n\pi a_r}{a}\\
&-\frac{b^4(a-a_r)}{(EI)_{1y}\pi^4 a}\sum_{n-1}^{\infty}\frac{1}{n^4}\left(C_{1y}+\frac{2}{b}\sum_{j=2}^{q-1}R_{1j}\sin\frac{n\pi b_j}{b}\right)\sin\frac{n\pi b_s}{b}\\
&-\frac{b^4 a_r}{(EI)_{py}\pi^4 a}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(C_{py}+\frac{2}{b}\sum_{j=2}^{q-1}R_{pj}\sin\frac{n\pi b_j}{a}\right)\sin\frac{n\pi b_s}{b} = 0.\end{aligned} \tag{7}$$

It should be observed that the first and second terms of (7) concern only the s-th member in the x-direction and the r-th member in the y-direction respectively, the third and fourth terms involve only the edge members in the x-direction and the last two terms the edge members in the y-direction. Expanding (7) and rearranging terms lead to

$$\sum_{i=2}^{p-1}\sum_{j=2}^{q-1}\rho_{ij}^{rs}R_{ij} = \lambda^{rs}, \tag{8}$$

in which

$$\begin{aligned}\rho_{ij}^{rs} &= \varphi_{ij}^{rs}+\psi_{ij}^{rs}\\ &+\frac{2a^3(b-b_s)(b-b_j)}{(EI)_{1x}b^2}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi a_i}{a}\sin\frac{n\pi a_r}{a}\\ &+\frac{2a^3 b_s b_j}{(EI)_{qx}b^2}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi a_i}{a}\sin\frac{n\pi a_r}{a}\\ &+\frac{2b^3(a-a_r)(a-a_i)}{(EI)_{1y}a^2}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi b_j}{b}\sin\frac{n\pi b_s}{b}\\ &+\frac{2b^3 a_r a_i}{(EI)_{py}a^2}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi b_j}{b}\sin\frac{n\pi b_s}{b},\end{aligned} \tag{9}$$

$$\varphi_{ij}^{rs}=\frac{2a^3}{(EI)_{sx}}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi a_i}{a}\sin\frac{n\pi a_r}{a},\qquad \text{when } j=s,$$

$$\varphi_{ij}^{rs}=0,\qquad \text{when } j\neq s,$$

$$\psi_{ij}^{rs}=\frac{2b^3}{(EI)_{ry}}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi b_j}{b}\sin\frac{n\pi b_s}{b},\qquad \text{when } i=r,$$

$$\psi_{ij}^{rs}=0,\qquad \text{when } i\neq r.$$

$$\begin{aligned}\lambda^{rs} &= \frac{a^4}{(EI)_{sx}}\sum_{n=1}^{\infty}\frac{1}{n^4}A_{sx}\sin\frac{n\pi a_r}{a}-\frac{b^4}{(EI)_{ry}}\sum\frac{1}{n^4}C_{ry}\sin\frac{n\pi b_s}{b}\\ &-\frac{a^4(b-b_s)}{(EI)_{1x}b}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(A_{1x}+\frac{2}{a}\sum_{i=2}^{p-1}Q'_{iy}\sin\frac{n\pi a_i}{a}\right)\sin\frac{n\pi a_r}{a}\\ &-\frac{a^4 b_s}{(EI)_{qx}b}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(A_{qx}+\frac{2}{a}\sum_{i=2}^{p-1}Q''_{iy}\sin\frac{n\pi a_i}{a}\right)\sin\frac{n\pi a_r}{a}\\ &+\frac{b^4(a-a_r)}{(EI)_{1y}a}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(C_{1y}+\frac{2}{b}\sum_{j=2}^{q-1}Q'_{jx}\sin\frac{n\pi b_j}{b}\right)\sin\frac{n\pi b_s}{b}\\ &+\frac{b^4 a_r}{(EI)_{py}a}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(C_{py}+\frac{2}{b}\sum_{j=2}^{q-1}Q''_{jx}\sin\frac{n\pi b_j}{b}\right)\sin\frac{n\pi b_s}{b}.\end{aligned} \tag{10}$$

Eq. (8) expresses the compatibility of deformation at points rs and (9) gives, in general form, the relative flexibility influence coefficients. Setting $r=2, 3 \ldots (p-1)$ and $s=2, 3 \ldots (q-1)$ in (8), the combination leads to $(p-2)\times(q-2)$ simultaneous linear equations solving which yields the $(p-2)(q-2)$ redundant intermediate reactions. With the redundant quantities known, the deformation of and the internal forces acting in the structure are readily determined.

The procedure of analysis is best illustrated by means of the following numerical examples.

Example 1

Consider the corner-supported grillage loaded as shown in Fig. 2a. For this case (8) becomes

$$\rho_{22}^{22} R_{22} + \rho_{32}^{22} R_{32} = \lambda^{22}, \qquad \rho_{22}^{32} R_{22} + \rho_{32}^{32} R_{32} = \lambda^{32}. \tag{a}$$

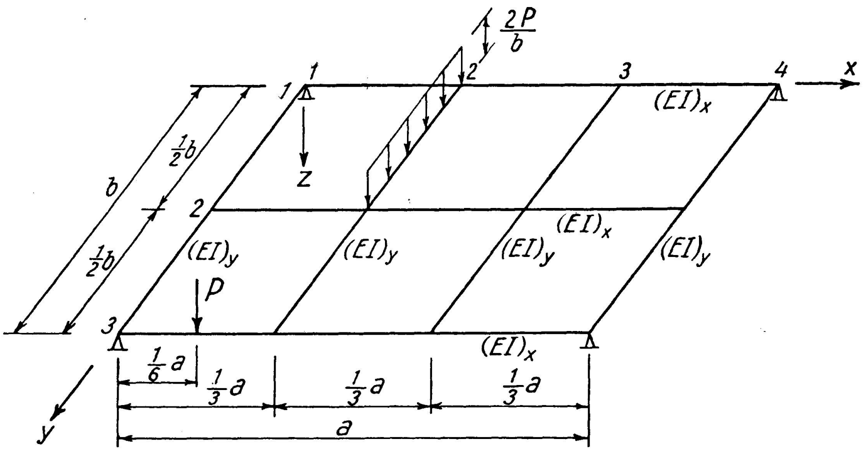

Fig. 2a. Partially Loaded Grillage Supported at the Corners.

Applying (9) yields

$$\rho_{22}^{22} = \rho_{32}^{32} = (2+\tfrac{1}{2}+\tfrac{1}{2}) \frac{a^3}{(E\,I)_x} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin^2 \frac{n\pi}{3} + \left(2+\frac{8}{9}+\frac{2}{9}\right) \frac{b^3}{(E\,I)_y} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin^2 \frac{n\pi}{2}$$

$$= 2.405 \frac{a^3}{(E\,I)_x} + 3.157 \frac{b^3}{(E\,I)_y},$$

$$\rho_{32}^{22} = \rho_{22}^{32} = (2+\tfrac{1}{2}+\tfrac{1}{2}) \frac{a^3}{(E\,I)_x} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin \frac{n\pi}{3} \sin \frac{2\,n\pi}{3} + \left(\frac{4}{9}+\frac{4}{9}\right) \frac{b^3}{(E\,I)_y} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin^2 \frac{n\pi}{2}$$

$$= 2.105 \frac{a^3}{(E\,I)_x} + 0.9019 \frac{b^3}{(E\,I)_y}.$$

For the third member in the x-direction and the second member in the y-direction, (2a, c) yield respectively

$$A_{3x} = \frac{2}{a} \lim_{t \to 0} \int_{a/6-t}^{a/6+t} \frac{P}{2t} \sin \frac{n\pi x}{a} dx = \frac{2P}{a} \sin \frac{n\pi}{6},$$

$$C_{2y} = \frac{2}{b} \int_{0}^{b/2} \frac{2P}{b} \sin \frac{n\pi y}{b} dy = \frac{4P}{n\pi b} \left(1 - \cos \frac{n\pi}{2}\right).$$

Substituting these values in (10) leads to

$$\lambda^{22} = -\left(\frac{4P}{\pi b}\right)\frac{b^4}{(EI)_y}\sum_{n=1}^{\infty}\frac{1}{n^5}\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}$$

$$-\left(\frac{2P}{a}\right)\frac{a^4}{2(EI)_x}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(\frac{3}{4}\sin\frac{n\pi}{3}+\sin\frac{n\pi}{6}+\frac{1}{4}\sin\frac{n\pi}{3}\right)\sin\frac{n\pi}{3}$$

$$= -\left[1.278\frac{a^3}{(EI)_x}+1.268\frac{b^3}{(EI)_y}\right]P,$$

$$\lambda^{32} = -\left(\frac{2P}{a}\right)\frac{a^4}{2(EI)_x}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(\frac{3}{4}\sin\frac{n\pi}{3}+\sin\frac{n\pi}{6}+\frac{1}{4}\sin\frac{n\pi}{3}\right)\sin\frac{2n\pi}{3}$$

$$= -1.090\frac{a^3}{(EI)_x}P.$$

Substituting the values of ρ_{ij}^{rs} and λ^{rs} in (a) and introducing

$$k = \frac{b^3}{(EI)_y}\frac{(EI)_x}{a^3} \tag{b}$$

lead to the simultaneous equations

$$(2.405+3.157\,k)\,R_{22}+(2.105+0.9019\,k)\,R_{32} = -(1.278+1.268\,k)\,P,$$

$$(2.105+0.9019\,k)\,R_{22}+(2.405+3.157\,k)\,R_{32} = -1.090\,P,$$

whose solution yields

$$R_{22} = -k'(0.7791+6.101\,k+4.003\,k^2)\,P,$$

$$R_{32} = \quad k'(0.0687+0.3806\,k+1.144\,k^2)\,P,$$

in which

$$k' = \frac{1}{1.353+11.39\,k+9.153\,k^2}. \tag{c}$$

For $k=1$, $R_{22}=-0.497\,P$ and $R_{32}=0.0727\,P$.

With the redundant quantities known, the deflection functions are readily determined. For instance, for the edge members in the x-direction and the second member in the y-direction, (1) yields respectively

$$z_{1x} = \frac{2a^3}{(EI)_x\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(\frac{3}{4}p\sin\frac{n\pi}{3}+\frac{1}{2}R_{22}\sin\frac{n\pi}{3}+\frac{1}{2}R_{22}\sin\frac{2n\pi}{3}\right)\sin\frac{n\pi x}{a},$$

$$z_{3x} = \frac{2a^3}{(EI)_x\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(p\sin\frac{n\pi}{6}+\frac{1}{4}p\sin\frac{n\pi}{3}+\frac{1}{2}R_{22}\sin\frac{n\pi}{3}\right.$$

$$\left.+\frac{1}{2}R_{32}\sin\frac{2n\pi}{3}\right)\sin\frac{n\pi x}{a},$$

$$z_{2y} = z_{21} + (z_{23} - z_{21})\frac{y}{b} + \frac{b^3}{(EI)_y \pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4}\left[\frac{P}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) + 2 R_{22} \sin\frac{n\pi}{2}\right] \sin\frac{n\pi y}{b}.$$

The values of z_{21} and z_{23} in the last expression are obtained from the first two by substituting $x = a/3$. For $k = 1$,

$$Z_{2y} = \frac{p b^3}{(EI)_y \pi^4}\left\{0.856 + 0.1501\frac{y}{b} + \sum_{n=1}^{\infty} \frac{1}{n^4}\left[\frac{1}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) - 0.994 \sin\frac{n\pi}{2}\right] \sin\frac{n\pi y}{b}\right\}.$$

It should be noted that while the deflection and slope functions converge rapidly, the speed of convergence of successive derivatives decreases with each differentiation. Once the redundant quantities are known, however, the bending moments and shearing forces are easily determined by statics. For the grillage under consideration, the variation of these functions are shown respectively in Fig. 2b, c.

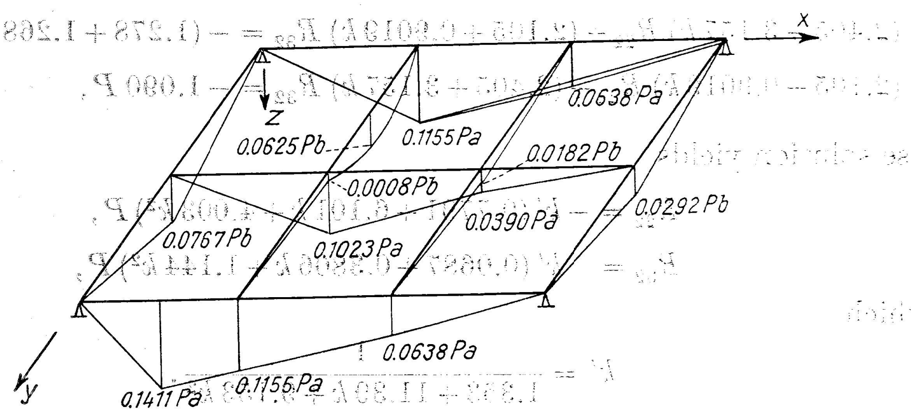

Fig. 2b. Bending Moment Plotted on Tension Side.

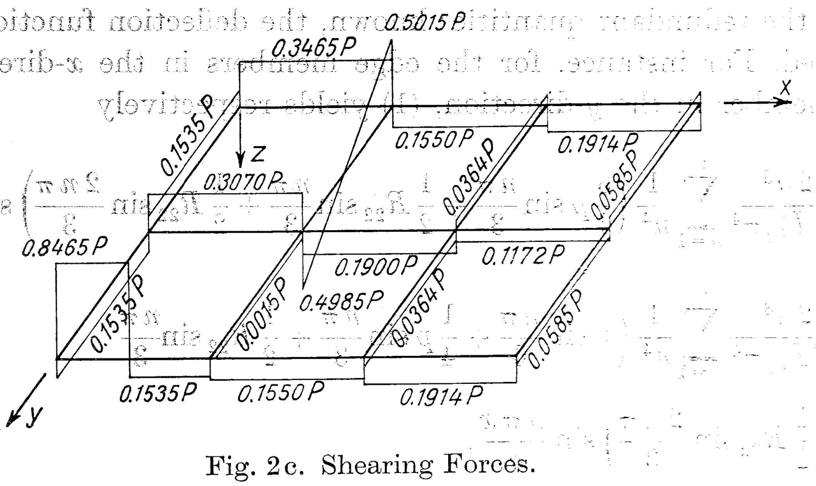

Fig. 2c. Shearing Forces.

Influence Lines

To obtain the influence lines for R_{uv}, the reaction at any point of intersection uv, the values of the redundant quantities are determined from the simultaneous equations

$$\frac{\partial U}{\partial R_{rs}} = -1 \qquad \text{for } r = u \text{ and } s = v,$$

$$\frac{\partial U}{\partial R_{rs}} = 0 \qquad \text{for other combinations of } r \text{ and } s.$$

The negative sign in the first equation is due to the sign conventions that downward w is positive and that R_{uv} is positive if acting upward on the member in the x-direction and downward on the member in the y-direction. Since $w_j(x)$ and $w_i(y)$ are zero, the equations take the form

$$\begin{aligned} \sum_{i=2}^{p-1}\sum_{j=2}^{q-1} \rho_{ij}^{rs} R_{ij} &= -\pi^4 \qquad \text{for } r = u \text{ and } s = v, \\ \sum_{i=2}^{p-1}\sum_{j=2}^{q-1} \rho_{ij}^{rs} R_{ij} &= 0 \qquad \text{for other combinations of } r \text{ and } s. \end{aligned} \tag{11}$$

Once the redundant reactions are determined, the influence lines for R_{uv} are obtained by substituting their values in (1).

The influence lines for other functions may be expressed in terms of those for the redundant reactions. The following example will illustrate this point.

Example 2

Considering again the grillage shown in Fig. 2a, for the influence lines for R_{22}, (11) becomes

$$(2.405 + 3.157\,k)\,R_{22} + (2.105 + 0.9019\,k)\,R_{32} = -\frac{\pi^4 (E\,I)_x}{a^3},$$

$$(2.105 + 0.9019\,k)\,R_{22} + (2.405 + 3.157\,k)\,R_{32} = 0,$$

whose solution yields

$$R_{22} = -k'\,(2.405 + 3.157\,k)\frac{\pi^4 (E\,I)_x}{a^3},$$

$$R_{32} = k'\,(2.105 + 0.9019\,k)\frac{\pi^4 (E\,I)_x}{a^3}.$$

The factor k' is given in Example 1. Substituting these values in (1) leads to the influence lines for R_{22} as follows:

$$z_{1y} = k'\,k\,(1.803 + 3.608\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{b},$$

$$z_{4y} = k'\,k\,(-1.203 + 0.9014\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{b},$$

$$z_{1x} = z_{3x} = -k'(2.405+3.157\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{a}$$

$$+k'(2.105+0.9019\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{2n\pi}{3}\sin\frac{n\pi x}{a},$$

$$z_{2y} = -k'(2.405+3.157\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{3}$$

$$+k'(2.105+0.9019\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{3}\sin\frac{2n\pi}{3}$$

$$-k'k(4.810+6.314\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{b},$$

$$z_{3y} = -k'(2.405+3.157\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{3}\sin\frac{2n\pi}{3}$$

$$+k'(2.105+0.9019\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{2n\pi}{3}$$

$$+k'k(4.210+1.804\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{b},$$

$$z_{2x} = k'k(1.803+3.608\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{2}$$

$$-k'k(3.006+2.707\,k)\frac{x}{a}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{2}$$

$$+k'k(4.810+6.314\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{a}$$

$$-k'k(4.210+1.804\,k)\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{2n\pi}{3}\sin\frac{n\pi x}{a}.$$

For the particular case where $k=1$, these influence lines are plotted in Fig. 2d. For example:

$$(z_{3x})_{k=1} = -0.254\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{a}+0.137\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{2n\pi}{3}\sin\frac{n\pi x}{a},$$

$$(z_{2y})_{k=1} = -0.107-0.508\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{b}.$$

The value of R_{22} due to the loading condition shown in Fig. 2a may be obtained by

$$R_{22} = \frac{P}{2b} \int_0^{b/2} (z_{2y})_{k=1} \, dy + P (z_{3x})_{\substack{k=1 \\ x=a/6}} = -0.497\,P,$$

which is identical to the result obtained in Example 1.

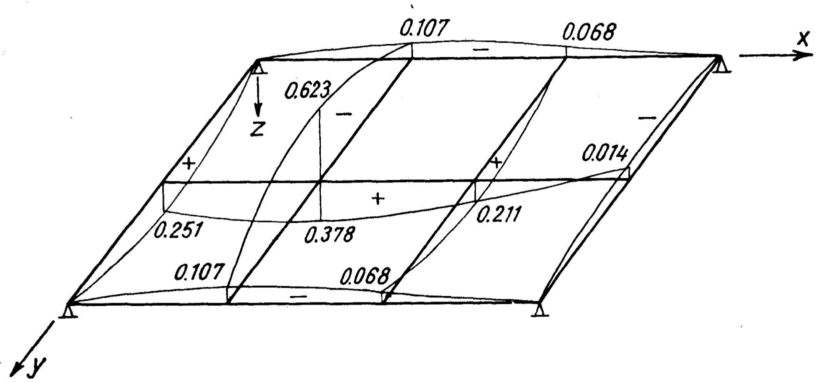

Fig. 2d. Influence Lines for R_{22} when $k=1$, Positive if Acting Upward on Members in x-Direction.

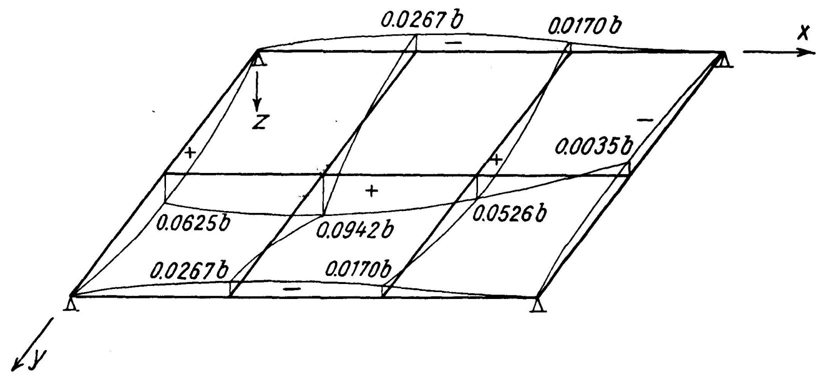

Fig. 2e. Influence Lines for $(M_{2y})_{y=b/2}$, Positive if Causing Tension in Bottom Fibre of Members.

The influence lines for R_{32} are the same as those of R_{22} by symmetry. The influence lines for other functions may be expressed in terms of those for the redundant reactions. For instance, the influence lines for the bending moment in the second member in the y-direction at midspan is given by

$$(M_{2y})_{y=b/2} = F(y) + \tfrac{1}{4} R_{22}\, b,$$

in which

$F(y) = y/2$ when the unit load is between points 21 and 22,

$F(y) = b/2 - y/2$ when it is between points 22 and 23, and

$F(y) = 0$ when it is elsewhere on the structure.

The bending moment is positive if it causes tension in the bottom fibre of the member. For $k = 1$, the influence lines for $(M_{2y})_{y=b/2}$ are plotted in Fig. 2e.

5. Treatment of Intermediate Supports

For corner-supported grillages which are also supported at intermediate points, such as the one shown in Fig. 3a, the solution may be obtained by the procedure just discussed with only minor modification. A particular case will be discussed in the following example.

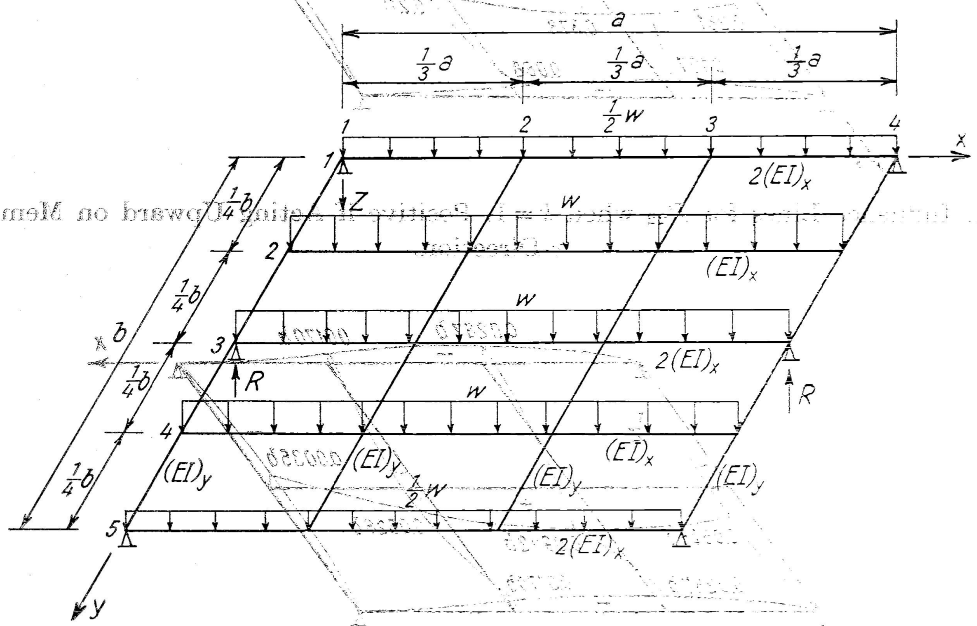

Fig. 3a. Grillage with Intermediate Supports.

Example 3

In the grillages loaded as shown in Fig. 3a, the reactions at the intermediate supported, R are treated as external loads on the edge members in the y-direction. Due to symmetry

$$R_{22} = R_{32} = R_{24} = R_{34},$$

$$R_{23} = R_{33}$$

and applying (8) to points 22 and 23 yields respectively

$$(\rho_{22}^{22} + \rho_{24}^{22} + \rho_{32}^{22} + \rho_{34}^{22})\, R_{22} + (\rho_{23}^{22} + \rho_{33}^{22})\, R_{23} = \lambda^{22},$$

$$(\rho_{22}^{23} + \rho_{24}^{23} + \rho_{32}^{23} + \rho_{34}^{23})\, R_{22} + (\rho_{23}^{23} + \rho_{33}^{23})\, R_{23} = \lambda^{23}.$$

Applying (9) and (10) leads to

$$\left[\frac{3a^3}{(EI)_x}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{3}+\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{3}\sin\frac{2n\pi}{3}\right)\right.$$
$$\left.+\frac{4b^3}{(EI)_y}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{4}+\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{4}\sin\frac{3n\pi}{4}\right)\right]R_{22}$$
$$+\left[\frac{a^3}{2(EI)_x}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{3}+\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{3}\sin\frac{2n\pi}{3}\right)\right.$$
$$\left.+\frac{4b^3}{(EI)_y}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{4}\sin\frac{n\pi}{2}\right]R_{23}$$
$$=\frac{3wa^4}{2\pi(EI)_x}\sum_{n=1}^{\infty}\frac{1}{n^5}(1-\cos n\pi)\sin\frac{n\pi}{3}$$
$$+\frac{b^3}{(EI)_y}\left[wa\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{4}\left(\sin\frac{n\pi}{4}+\sin\frac{n\pi}{2}+\sin\frac{3n\pi}{4}\right)\right.$$
$$\left.-2R\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{2}\sin\frac{n\pi}{4}\right],$$

$$\left[\frac{a^3}{(EI)_x}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{3}+\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{3}\sin\frac{2n\pi}{3}\right)\right.$$
$$\left.+\frac{4b^3}{(EI)_y}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{4}\sin\frac{n\pi}{2}+\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{3n\pi}{4}\sin\frac{n\pi}{2}\right)\right]R_{22}$$
$$+\left[\frac{3a^3}{2(EI)_x}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{3}+\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{3}\sin\frac{2n\pi}{3}\right)\right.$$
$$\left.+\frac{4b^3}{(EI)_y}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{2}\right]R_{23}$$
$$=\frac{wa^4}{2\pi(EI)_x}\sum_{n=1}^{\infty}\frac{1}{n^5}(1-\cos n\pi)\sin\frac{n\pi}{3}$$
$$+\frac{b^3}{(EI)_y}\left[wa\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{2}\left(\sin\frac{n\pi}{4}+\sin\frac{n\pi}{2}+\sin\frac{3n\pi}{4}\right)-2R\sum_{n=1}^{\infty}\frac{1}{n^4}\sin^2\frac{n\pi}{2}\right],$$

the solution of which yields

$$R_{22}=k_1\,wa+k_2\,R;\qquad R_{23}=k_3\,wa+k_4\,R, \qquad \text{(d)}$$

in which

$$\begin{aligned} k_1 &= \quad (1.657+4.636\,k+0.224\,k^2)\,k'', \\ k_2 &= -\,1.621\,k\,k'', \\ k_3 &= \quad (4.799+0.224\,k)\,k\,k'', \\ k_4 &= -\,(7.055+0.448)\,k\,k'', \end{aligned} \tag{e}$$

$$k'' = \frac{1}{9.041+19.065\,k+0.896\,k^2}. \tag{f}$$

The value of R is determined by means of the boundary condition that the deflection at the points of intermediate supports is zero, i. e.,

$$(z_{1y})_{y=b/2} = \frac{2\,b^3}{(E\,I)_y\,\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\left[\left(\frac{w\,a}{2}-R_{22}\right)\sin\frac{n\,\pi}{4}\right.$$

$$\left.+\left(\frac{w\,a}{2}-R_{23}-R\right)\sin\frac{n\,\pi}{2}+\left(\frac{w\,a}{2}-R_{22}\right)\sin\frac{3\,n\,\pi}{4}\right]\sin\frac{n\,\pi}{2}=0.$$

Solving this equation for R and substituting (d) and (e) yield

$$R = k'''\,(8.582+11.634\,k+0.540\,k^2)\,w\,a,$$

in which

$$k''' = \frac{1}{9.173+9.924\,k+0.455\,k^2}. \tag{g}$$

Substituting the value of R just obtained in (d) leads to

$$\begin{aligned} R_{22} &= k''\,k'''\,(15.20+45.06\,k+29.96\,k^2+3.457\,k^3+0.1019\,k^4)\,w\,a, \\ R_{23} &= -\,k''\,k'''\,k\,(16.53+36.24\,k+4.615\,k^2+0.1400\,k^3)\,w\,a, \end{aligned}$$

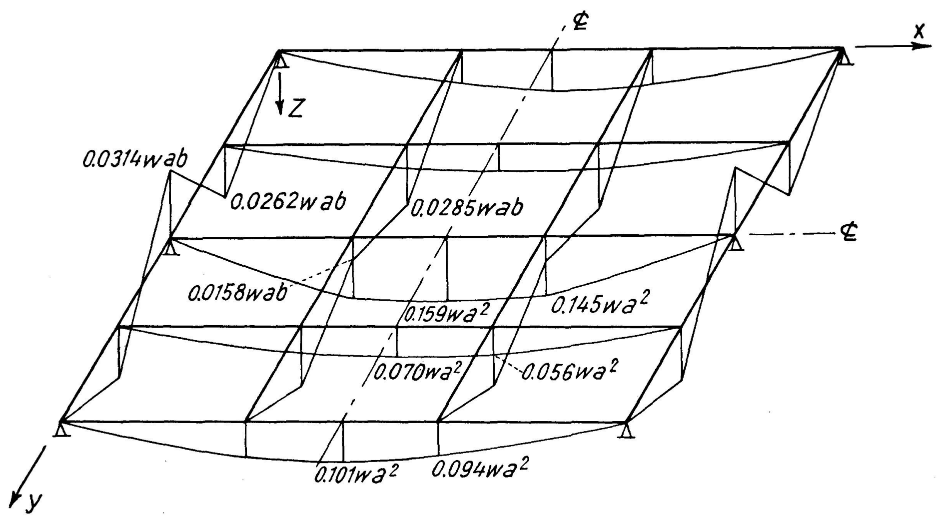

Fig. 3b. Bending Moment Plotted on Tension Side.

For the particular case where $k=1$, $R=1.0615\,w\,a$, $R_{22}=0.1654\,w\,a$ and $R_{23}=-0.1015\,w\,a$. The corresponding bending moments are plotted in Fig. 3b.

6. Discussion and Conclusion

The general equations derived in the foregoing may be applied to the analysis of simply supported interconnected bridge girders in which there are relatively few crossbeams that are spaced too far apart to be approximated by a continuous elastic system of equivalent stiffness. In this application, it is only necessary to assume that the two edge members parallel to the crossbeams are infinitely rigid.

Although the general equations no longer apply in dealing with corner-supported grillages of irregular shape such as the L-shape one shown in Fig. 4,

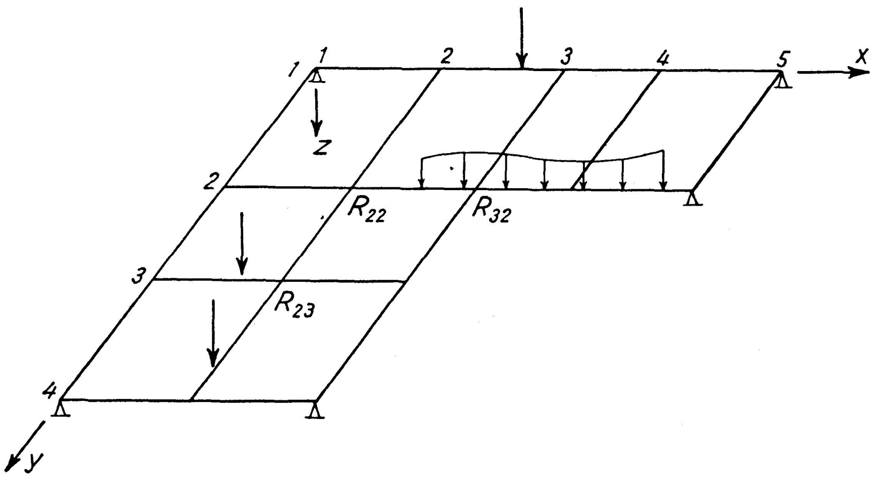

Fig. 4. Corner-Supported Grillage of Irregular Shape.

the latter can be treated in similar manner. It is convenient, in this case, to take R_{22}, R_{32} and R_{23} as the redundant quantities. As before the total strain energy U' may be expressed in terms of these three reactions which are then determined by the solution of the simultaneous equations

$$\frac{\partial U'}{\partial R_{22}}=0, \qquad \frac{\partial U'}{\partial R_{32}}=0, \qquad \frac{\partial U'}{\partial R_{23}}=0. \tag{h}$$

Eq. (h) takes essentially the same form as (7) with minor changes due to the difference in the lengths of the members.

The use of trigonometric series in the analysis of corner-supported grillages facilitates the formulation of the general equations which lend themselves to a systematic solution of the problem. The series contained in the flexibility influence coefficients in (8) and (11) converge rapidly. In grillages where the spacings of the members are uniform, such as those discussed in the examples, the repeated occurance of the same series in these coefficients helps to reduce the numerical work required to obtain a solution. The relative ease with which the influence lines are obtained once the redundant reactions are known makes this method of analysis a useful tool in the treatment of grillages subjected to moving loads.

7. References

1. S. Timoshenko, "Über die Biegung von Trägerrosten". Zeitschrift für angewandte Mathematik und Mechanik, Band 13, Heft 2, 1933, pp. 153–159.
2. J. M. Klitchieff, "Beams on Elastic Supports and on Cross-Girders". Aeronautical Quarterly, Vol. 2, Part 3, Nov. 1950, pp. 157–166.
3. J. M. Klitchieff, "Beams on Cross-Girders with Clamped Ends". Aeronautical Quarterly, Vol. 3, Part 3, Nov. 1951, pp. 230–237.
4. D. F. Holman, "A Finite Series Solution for Grillages under Normal Loading". Aeronautical Quarterly, Vol. 8, Part 1, Feb. 1957, pp. 49–57.
5. A. W. Hendry and L. G. Jaeger, "A General Method for the Analysis of Grid Frameworks". Proc. I.C.E., Vol. 4, No. 3, 1955, p. 939.
6. A. J. S. Pippard and J. P. A. de Waele, "The Loading of Interconnected Bridge Girders". Journal I.C.E., Vol. 10, No. 1, 1938, pp. 97–114.
7. M. Hetényi, "A Method of Calculating Grillage Beams". Stephen Timoshenko 60th Anniversary Volume, Macmillan, New York, 1938, pp. 60–72.
8. B. J. Covington, "Analysis of Interconnected Girders and Grillage Beams". Ph. D. Dissertation, Northwestern University, Evanston, Illinois, April, 1955.
9. A. W. Hendry and L. G. Jaeger, "The Load Distribution in Interconnected Bridge Girders with Special Reference to Continuous Beams". Publ. Int. Assoc. Bridge Struct. Eng., Vol. 15, 1955, p. 95–116.
10. L. G. Jaeger and A. W. Hendry, "The Analysis of Interconnected Bridge Girders by the Distribution of Harmonics". Struct. Engr., Vol. 34, No. 7, 1956, pp. 241–266.
11. A. W. Hendry and L. G. Jaeger, "The Analysis of Certain Interconnected Skew-Bridge Girders". Proc. I. C. E., Vol. 6, 1957, pp. 79–106.
12. H. Beer and F. Reisinger, "Genaue Berechnung schiefer Trägerrostbrücken mit Einflußlinien". Bauingenieur, Band 30, Heft 12, 1955, pp. 425–431.
13. W. W. Ewell, S. Okubo and J. I. Abrams, "Deflections in Gridworks and Slabs". Trans. A. S. C. E., Vol. 117, 1952, p. 869.
14. J. A. Sbarounis and M. P. Gaus, "Analysis of Two Way Truss System". Proc. A. S. C. E., Vol. 85, No. ST 2, Feb., 1959, pp. 45–69.
15. S. L. Lee, "Analysis of Continuous Beams by Fourier Series". Proc. A. S. C. E., Journal Eng. Mech. Div., Vol. 83, No. EM 4, Oct., 1957; Trans. A. S. C. E., Vol. 124, 1959, p. 164.

Summary

An analysis of orthogonal or skew grillages supported at the corners is presented. The assumption is made that the interaction between the members at the points of intersection consists only of a vertical force. The members may have different flexural rigidity and their spacings in either direction are arbitrary.

The deflection functions are represented by trigonometric series and the reactions at intermediate points of intersection are taken as the redundant quantities. The latter are determined by the solution of simultaneous linear equations which express the compatibility of deformation at the points of intersection.

General equations are derived for rectangular or skew grillages supported only at the corners and for the construction of the influence lines. The treatment of grillages supported also at intermediate points is illustrated by means of a numerical example. The procedure for analyzing certain interconnected bridge girders and grillages of irregular shape is also discussed.

Résumé

Les auteurs exposent leurs recherches concernant les grilles portantes orthogonales ou biaises qui sont appuyées dans leurs angles. Il est admis à titre d'hypothèse de base que l'interaction entre les éléments dans les points de croisement ne se manifeste que sous forme d'un effort vertical. Les éléments porteurs peuvent présenter des rigidités de flexion différentes et leurs écartements dans les deux directions sont quelconques.

Les fonctions de déformation sont représentées par des séries trigonométriques et les réactions aux points intermédiaires de croisement sont prises comme inconnues hyperstatiques. Ces dernières grandeurs sont définies par la résolution d'équations linéaires simultanées qui expriment la compatibilité des déformations en ces points de croisement.

Les auteurs établissent des équations générales pour les grilles portantes rectangulaires ou biaises qui ne sont appuyées qu'aux angles, ainsi que pour le tracé des lignes correspondantes d'influence. Il traite sur un exemple numérique le cas des grilles portantes qui sont également appuyées en des points intermédiaires; il montre de plus comment on peut étudier certains ponts à poutres multiples entretoisées, ainsi que les grilles portantes de formes irrégulières.

Zusammenfassung

In dieser Arbeit wird eine Untersuchung von orthogonalen oder schiefen Trägerrosten, die in den Eckpunkten unterstützt werden, dargestellt. Die Annahme wird zu Grunde gelegt, daß die gegenseitige Einwirkung in den Kreuzungspunkten nur in einer vertikalen Kraft bestehe. Die Tragglieder können verschiedene Biegesteifigkeiten aufweisen, und ihre Abstände in beiden Richtungen sind willkürlich.

Die Durchbiegungsfunktionen werden durch trigonometrische Reihen dargestellt, und die Reaktionen in den Zwischenkreuzungspunkten werden als

überzählige Größen eingeführt. Letztere werden mit der Lösung von simultanen linearen Gleichungen bestimmt, die die Verträglichkeit der Durchbiegungen an diesen Kreuzungspunkten ausdrücken.

Für rechteckige oder schiefe Trägerroste, die nur in den Eckpunkten gelagert sind, und für die Konstruktion der entsprechenden Einflußlinien werden allgemeine Gleichungen abgeleitet. Die Behandlung von Trägerrosten, die auch in Zwischenpunkten unterstützt sind, wird durch ein numerisches Beispiel veranschaulicht. Ebenso wird das Verfahren zur Untersuchung von gewissen zusammenhängenden Brückenträgern und von Trägerrosten mit unregelmäßiger Form behandelt.