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Objekttyp: **Article**

Zeitschrift: **IABSE publications = Mémoires AIPC = IVBH Abhandlungen**

Band (Jahr): **18 (1958)**

PDF erstellt am: **21.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-16507

On the Bending of a Sectorial Plate

Flexion des dalles en forme de secteur

Über die Biegung von Sektorplatten

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I. Introduction

Recently attention has been paid by aeronautical engineers to the problem of bending of sectorial plates in conjunction with an investigation of the stress distribution in the neighborhood of the wing fuselage connection. As a matter of fact, it has been discovered analytically as well as experimentally that for certain values of the included angle, the bending stress in the corner reaches extremely high values. Specifically, it was found that within the limitations of the ordinary plate theory, the stress tends to infinity as the included angle exceeds 90°, with the strength of the singularity increasing with the angle.

M. L. WILLIAMS, Jr. [1] has investigated these stress singularities, discussing the problem only qualitatively. S. WOINOWSKY-KRIEGER [2] derived a general method of solution using the Fourier Integral. However, a numerical application requires laborious computations. Fortunately, simpler solutions can be obtained in two different ways for the case of a sectorial plate whose radial edges are simply supported. As an example, Green's function for the deflection of a sectorial plate with a clamped circumferential edge will be derived in double (Fourier-Bessel series) and in single series form. From the latter solution, Green's function for bending and twisting moments of a wedge-shaped plate are obtained in closed form.

With this solution a general discussion of the stress singularity can be successfully made and influence surfaces as well as moment surfaces can be developed. The latter solutions will have direct application in the calculation of influence surfaces for skewed plates which in turn should be useful in the design of skewed bridge slabs.

II. Method of Solution

Considering a sectorial plate with radius a and included angle α under a concentrated load $P=1$ at (ρ, φ), the classical bending theory of plates of uniform thickness requires the integration of the following differential equation:

$$D \Delta \Delta W(r, \theta) = 0 \tag{1}$$

with the exception of the loading point (ρ, φ). The notation is as follows:

$W(r, \theta)$ = deflection of point (r, θ),

$\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$ (Laplacian operator),

$D = \dfrac{E h^3}{12(1-\nu^2)}$ (Bending Stiffness of Plate),

h = uniform plate thickness.

The corresponding boundary conditions are along the radial edges:

$$\theta = 0;\quad W = 0;\quad M_\theta = -D\left[\nu\left(\frac{\partial^2 W}{\partial r^2}\right) + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right] = 0;$$

$$\theta = \alpha;\quad W = 0;\quad M_\theta = 0 \tag{2}$$

and along the circumferential edge:

$$r = a;\quad W = 0;\quad \frac{\partial W}{\partial r} = 0.$$

A. *Single Series Solution*[1]

The method of solution parallels the one by CLEBSCH used to solve the problem of a circular plate under a concentrated load (see for example [4], p. 266). Dividing the plate into two parts by a cylindrical section of radius ρ, as shown in fig. 1 by the dotted line, the following product solution can be assumed:

$$W(r, \theta) = \sum_{n=1}^{\infty} R_n(r) \sin\frac{n\pi\theta}{\alpha} \tag{3}$$

$$(0 \leqq \alpha \leqq 2\pi,\ \alpha \neq \pi).$$

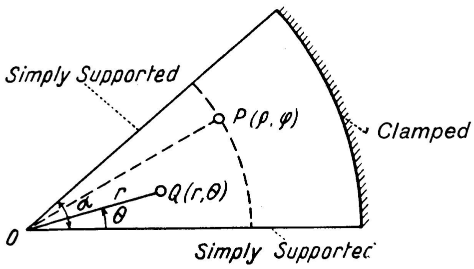

Fig. 1. Coordinates of a Sectorial Plate.

Evidently, eq. (3) satisfies the boundary conditions imposed on the radial edges. Substituting eq. (3) into eq. (1), the following equation is obtained:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\left(\frac{n\pi}{\alpha}\right)^2}{r^2}\right)^2 R_n = 0.$$

[1]) A recent literature review disclosed that NOWACKI and MOSSAKOWSKI [3] treated the problem on the bending of a wedge-shaped plate using a similar method.

The general solution of this equation is:

$$R_n(r) = A_n r^{\frac{n\pi}{\alpha}} + B_n r^{-\frac{n\pi}{\alpha}} + C_n r^{\frac{n\pi}{\alpha}+2} + D_n r^{-\frac{n\pi}{\alpha}+2} \quad (\text{for } \rho \leqq r \leqq a),$$
$$R'_n(r) = A'_n r^{\frac{n\pi}{\alpha}} + C'_n r^{\frac{n\pi}{\alpha}+2} \quad (\text{for } 0 \leqq r \leqq \rho). \tag{4}$$

Hence for each term of the series four constants for the outer portion of the plate and two for the inner portion must be determined. The six necessary equations can be obtained from the boundary conditions at the circumferential edge of the plate and from the continuity conditions along the section of radius ρ. Denoting the deflection of the inner portion of the plate by W', those six boundary conditions are written as follows:

$r = \rho$:

$$W' = W, \tag{a}$$
$$\frac{\partial W'}{\partial r} = \frac{\partial W}{\partial r}, \tag{b}$$
$$\frac{\partial^2 W'}{\partial r^2} = \frac{\partial^2 W}{\partial r^2}, \tag{c}$$
$$D\frac{\partial}{\partial r}(\Delta W) - D\frac{\partial}{\partial r}(\Delta W') = \frac{2}{\rho\alpha}\sum_{n=1}^{\infty}\sin\frac{n\pi\varphi}{\alpha}\sin\frac{n\pi\theta}{\alpha}, \tag{d}$$

$r = a$:

$$W = 0, \tag{e}$$
$$\frac{\partial W}{\partial r} = 0. \tag{f}$$

(5)

Eq. (d) equates the difference in shear of the outer and inner portion to the applied load $P=1$ at (ρ, φ) the latter being developed into a Fourier series. Using eq. (5) the six constants A_n, B_n, C_n, D_n, A'_n, and C'_n are determined:

$$A'_n = \frac{a^{-\frac{2n\pi}{\alpha}}\rho^{\frac{n\pi}{\alpha}}}{4\alpha D}\left\{\frac{\rho^2}{\frac{n\pi}{\alpha}} - \frac{a^2}{\frac{n\pi}{\alpha}-1} + \frac{\rho^2\left(\frac{a}{\rho}\right)^{\frac{2n\pi}{\alpha}}}{\frac{n\pi}{\alpha}\left(\frac{n\pi}{\alpha}-1\right)}\right\}\sin\frac{n\pi\varphi}{\alpha},$$

$$C'_n = \frac{a^{-\frac{2n\pi}{\alpha}}\rho^{\frac{n\pi}{\alpha}}}{4\alpha D}\left\{\frac{1}{\frac{n\pi}{\alpha}} - \frac{\left(\frac{\rho}{a}\right)^2}{\frac{n\pi}{\alpha}+1} - \frac{\left(\frac{a}{\rho}\right)^{\frac{2n\pi}{\alpha}}}{\frac{n\pi}{\alpha}\left(\frac{n\pi}{\alpha}-1\right)}\right\}\sin\frac{n\pi\varphi}{\alpha},$$

$$A_n = \frac{a^{-\frac{2n\pi}{\alpha}}\rho^{\frac{n\pi}{\alpha}}}{4\alpha D}\left(\frac{\rho^2}{\frac{n\pi}{\alpha}} - \frac{a^2}{\frac{n\pi}{\alpha}-1}\right)\sin\frac{n\pi\varphi}{\alpha}, \tag{6}$$

$$B_n = -\frac{\rho^{\frac{n\pi}{\alpha}+2}}{4n\pi D\left(\frac{n\pi}{\alpha}+1\right)}\sin\frac{n\pi\varphi}{\alpha},$$

$$C_n = \frac{a^{-\frac{2n\pi}{\alpha}} \rho^{\frac{n\pi}{\alpha}}}{4\alpha D}\left(\frac{1}{\frac{n\pi}{\alpha}} - \frac{\left(\frac{\rho}{a}\right)^2}{\frac{n\pi}{\alpha}+1}\right)\sin\frac{n\pi\varphi}{\alpha},$$

$$D_n = \frac{\rho^{\frac{n\pi}{\alpha}}}{4n\pi D\left(\frac{n\pi}{\alpha}-1\right)}\sin\frac{n\pi\varphi}{\alpha}. \qquad (6)$$

By substitution the following single series solution is obtained.

$$W(r,\theta;\rho,\varphi) = \frac{1}{4\alpha D}\sum_{n=1}^{\infty}\left(\frac{\rho r}{a^2}\right)^{\frac{n\pi}{\alpha}}\left\{\left[\frac{\rho^2}{\frac{n\pi}{\alpha}} - \frac{a^2}{\frac{n\pi}{\alpha}-1} + \frac{\rho^2\left(\frac{a}{\rho}\right)^{\frac{2n\pi}{\alpha}}}{\frac{n\pi}{\alpha}\left(\frac{n\alpha}{\alpha}-1\right)}\right]\right.$$

$$\left. + r^2\left[\frac{1}{\frac{n\pi}{\alpha}} - \frac{\left(\frac{\rho}{a}\right)^2}{\frac{n\pi}{\alpha}+1} - \frac{\left(\frac{a}{\rho}\right)^{\frac{2n\pi}{\alpha}}}{\frac{n\pi}{\alpha}\left(\frac{n\pi}{\alpha}+1\right)}\right]\right\}\sin\frac{n\pi\varphi}{\alpha}\sin\frac{n\pi\theta}{\alpha} \qquad (0\leqq r\leqq\rho),$$

$$= \frac{1}{4\alpha D}\sum_{n=1}^{\infty}\left\{\left(\frac{\rho r}{a^2}\right)^{\frac{n\pi}{\alpha}}\left[\left(\frac{\rho^2}{\frac{n\pi}{\alpha}} - \frac{a^2}{\frac{n\pi}{\alpha}-1}\right) + r^2\left(\frac{1}{\frac{n\pi}{\alpha}} - \frac{\left(\frac{\rho}{a}\right)^2}{\frac{n\pi}{\alpha}+1}\right)\right]\right. \qquad (7)$$

$$\left. + \left(\frac{\rho}{r}\right)^{\frac{n\pi}{\alpha}}\left[\frac{r^2}{\frac{n\pi}{\alpha}\left(\frac{n\pi}{\alpha}-1\right)} - \frac{\rho^2}{\frac{n\pi}{\alpha}\left(\frac{n\pi}{\alpha}+1\right)}\right]\right\}\sin\frac{n\pi\varphi}{\alpha}\sin\frac{n\pi\theta}{\alpha}$$

$$(\rho\leqq r\leqq a),$$

where $0\leqq\alpha\leqq 2\pi, \quad \alpha\neq\pi.$

B. *Double Series Solution* (Fourier-Bessel Series)

The second solution in form of a Fourier-Bessel series can be derived in the following way. In a first step the natural frequencies of the plate are determined, leading to an orthogonal function system whose elements represent the modes of vibration corresponding to the specific natural frequencies. The concentrated load can then be expanded in terms of these orthogonal functions. Assuming that $W(r,\theta;\rho,\varphi)$ can be also expanded into such a series, the unknown coefficients are determined by substitution into the original differential eq. (1).

1. Set of Eigen Functions for the Given Plate (fig. 1): The differential equation of free vibration of a plate is:

$$D\Delta\Delta W + \frac{\gamma h}{g}\frac{\partial^2 W}{\partial t^2} = 0, \qquad (8)$$

where γ = the weight of the plate per unit volume,
h = uniform thickness of the plate,
g = gravitational acceleration,
t = time.

Assuming $W(r,\theta;t) = F_n(r)\sin\frac{n\pi\theta}{\alpha}e^{ipt}$ and substituting it into eq. (8), an equation for $F_n(r)$ is obtained:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{\left(\frac{n\pi}{\alpha}\right)^2}{r^2}\right)^2 F_n = k^4 F_n, \tag{9}$$

where
$$k^4 = \frac{\gamma h p^2}{gD}.$$

The general solution of eq. (9) is:

$$F_n(r) = A_n J_{\frac{n\pi}{\alpha}}(kr) + B_n J_{-\frac{n\pi}{\alpha}}(kr) + C_n I_{\frac{n\pi}{\alpha}}(kr) + D_n I_{-\frac{n\pi}{\alpha}}(kr), \tag{10}$$

where $J_{\pm\frac{n\pi}{\alpha}}(kr)$: Bessel's Function of $\pm\frac{n\pi}{\alpha}$ order,

$I_{\pm\frac{n\pi}{\alpha}}(kr)$: Modified Bessel's Function of $\pm\frac{n\pi}{\alpha}$ order.

Since the deflection at the center is finite the coefficients B_n and D_n must be equal to zero, or

$$F_n(r) = A_n J_{\frac{n\pi}{\alpha}}(kr) + C_n I_{\frac{n\pi}{\alpha}}(kr). \tag{11}$$

The boundary conditions along the circumferential edge,

$$(W)_{r=a} = 0, \qquad \left(\frac{\partial W}{\partial r}\right)_{r=a} = 0,$$

yield the following secular equation:

$$J_{\frac{n\pi}{\alpha}}(ka)\, I'_{\frac{n\pi}{\alpha}}(ka) - J'_{\frac{n\pi}{\alpha}}(ka)\, I_{\frac{n\pi}{\alpha}}(ka) = 0. \tag{12}$$

This transcendental eq. (12) has infinite numbers of eigen values. Ordering these values the following infinite set of eigen functions is constructed:

$$\Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)\sin\frac{n\pi\theta}{\alpha} = \left[J_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)\, I'_{\frac{n\pi}{\alpha}}(k_s^{(n)}a) - J'_{\frac{n\pi}{\alpha}}(k_s^{(n)}a)\, I_{\frac{n\pi}{\alpha}}(k_s^{(n)}r)\right]\sin\frac{n\pi\theta}{\alpha},$$

$$n = 1, 2, 3\ldots, \qquad s = 1, 2, 3\ldots, \tag{13}$$

where $k_s^{(n)}$ is the s^{th} root of the eq. (12).

2. *Orthogonality of the Eigen Functions:* The orthogonality condition takes the form:

$$\int_0^a\int_0^\alpha \left[\Delta_{\frac{m\pi}{\alpha}}(k_i^{(m)}r)\sin\frac{m\pi\theta}{\alpha}\right]\left[\Delta_{\frac{n\pi}{\alpha}}(k_j^{(n)}r)\sin\frac{n\pi\theta}{\alpha}\right] r\,dr\,d\theta = 0 \quad (m \neq n,\ i \neq j)$$

or
$$\int_0^a \Delta_{\frac{m\pi}{\alpha}}(k_i^{(m)}r)\,\Delta_{\frac{n\pi}{\alpha}}(k_j^{(n)}r)\, r\,dr \int_0^\alpha \sin\frac{m\pi\theta}{\alpha}\sin\frac{n\pi\theta}{\alpha}\,d\theta = 0. \tag{14}$$

The second integral takes the following values

$$\int_0^{\alpha} \sin\frac{m\pi\theta}{\alpha}\sin\frac{n\pi\theta}{\alpha}\,d\theta \begin{cases} = 0 & (m \neq n), \\ = \dfrac{\alpha}{2} & (m = n). \end{cases}$$

The value of the first integral depends on i and j:

$$\int_0^{a} \Delta_{\frac{n\pi}{\alpha}}(k_i^{(n)} r)\,\Delta_{\frac{n\pi}{\alpha}}(k_j^{(n)} r)\, r\,dr \begin{cases} = 0 & (i \neq j), \\ \neq 0 & (i = j). \end{cases}$$

The case $(i \neq j)$ can be derived by making use of the following Lommel integrals of Bessel functions:

$$\begin{aligned} \int_0^a r J_N(\lambda r) J_N(\mu r)\,dr &= \frac{a}{\lambda^2-\mu^2}[\mu J_N(\lambda a) J'_N(\mu a) - \lambda J'_N(\lambda a) J_N(\mu a)], \\ \int_0^a r J_N(\lambda r) I_N(\mu r)\,dr &= \frac{a}{\lambda^2+\mu^2}[\mu J_N(\lambda a) I'_N(\mu a) - \lambda J'_N(\lambda a) I_N(\mu a)], \\ \int_0^a r I_N(\lambda r) I_N(\mu r)\,dr &= -\frac{a}{\lambda^2+\mu^2}[\mu I_N(\lambda a) I'_N(\mu a) - \lambda I'_N(\lambda a) I_N(\mu a)], \end{aligned} \tag{15}$$

$$\text{where} \quad N = \frac{n\pi}{\alpha}, \quad \lambda = k_i^{(n)}, \quad \mu = k_j^{(n)}.$$

Using the additional relations:

$$\begin{aligned} \int_0^a r[J_N(\lambda r)]^2 dr &= \frac{a^2}{2}\left\{[J'_N(\lambda a)]^2 + \left(1-\frac{N^2}{\lambda^2 a^2}\right)[J_N(\lambda a)]^2\right\}, \\ \int_0^a r J_N(\lambda r) I_N(\lambda r)\,dr &= \frac{a}{2\lambda}[J_N(\lambda a) I'_N(\lambda a) - J'_N(\lambda a) I_N(\lambda a)], \\ \int_0^a r[I_N(\lambda r)]^2 dr &= -\frac{a^2}{2}\left\{[I'_N(\lambda a)]^2 - \left(1+\frac{N^2}{\lambda^2 a^2}\right)[I_N(\lambda a)]^2\right\}. \end{aligned} \tag{16}$$

the second case $(i = j)$ becomes:

$$\begin{aligned} \int_0^a r[\Delta_N(\lambda r)]^2 dr &= a^2 [J_N(\lambda a)]^2 [I'_N(\lambda a)]^2 = a^2 [J'_N(\lambda a)]^2 [I_N(\lambda a)]^2, \\ &= a^2 J_N(\lambda a) I_N(\lambda a) J'_N(\lambda a) I'_N(\lambda a). \end{aligned}$$

Observing in eq. (14) that $m \neq n$ and $i \neq j$ the orthogonality relation is proven.

3. Fourier-Bessel Expansion of an Arbitrary Function $f(r,\theta)$: Assuming

$$f(r,\theta) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} A_{ns}\,\Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)} r)\sin\frac{n\pi\theta}{\alpha} \tag{17}$$

and multiplying both sides of eq. (17) by

$$r\,\Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)} r)\sin\frac{n\pi\theta}{\alpha}$$

and integrating over the whole area of the plate, A_{ns} can be determined as follows[2]:

$$A_{ns} = \frac{2\int_0^a\int_0^\alpha f(r,\theta)\, \Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)} r) \sin\frac{n\pi\theta}{\alpha}\, r\, dr\, d\theta}{\alpha a^2 J_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, I_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, J'_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, I'_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)}. \tag{18}$$

4. Green's Function for the Deflection of the Sectorial Plate: Making use of eq. (18) the distributed load $q(\rho,\varphi)$ over the infinitesimal area $\rho\, d\rho\, d\varphi$, as

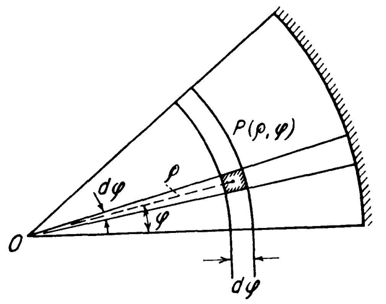

Fig. 2. Mathematical Formulation of a Concentrated Unit Load Acting at P (ρ, φ).

shown in fig. 2, can be developed into a Fourier-Bessel series. Taking only the limiting case

$$\lim_{\substack{d\varphi\to 0\\ d\rho\to 0}} q\rho\, d\varphi\, d\rho = 1,$$

corresponding to a concentrated load $P=1$ at (ρ,φ), the coefficient A_{ns} takes the following value:

$$\begin{aligned} A_{ns} &= \frac{2\lim\limits_{\substack{d\varphi\to 0\\ d\rho\to 0}} \sin\frac{n\pi\varphi}{\alpha}\, d\varphi\, q\rho\, \Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)}\rho)\, d\rho}{\alpha a^2 J_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, I_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, J'_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, I'_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)}, \\ &= \frac{2\Delta_{\frac{n\pi}{\alpha}}(k_s^{(n)}\rho) \sin\frac{n\pi\varphi}{\alpha}}{\alpha a^2 J_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, I_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, J'_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, I'_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)}. \end{aligned} \tag{19}$$

Using eqs. (17) and (19) the concentrated load $P=1$ at (ρ,φ) can be expressed in the following series:

$$P(\rho,\varphi) = 1 = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \psi_{\frac{n\pi}{\alpha}}(k_s^{(n)}\rho,\varphi)\, \psi_{\frac{n\pi}{\alpha}}(k_s^{(n)} r,\theta), \tag{20}$$

where $$\psi_{\frac{n\pi}{\alpha}}(k_s^{(n)} r,\theta) = \frac{\left[J_{\frac{n\pi}{\alpha}}(k_s^{(n)} r)\, I'_{\frac{n\pi}{\alpha}}(k_s^{(n)} a) - J'_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, I_{\frac{n\pi}{\alpha}}(k_s^{(n)} r)\right] \sin\frac{n\pi\theta}{\alpha}}{\sqrt{\frac{\alpha}{2} a^2 J_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, I_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, J'_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)\, I'_{\frac{n\pi}{\alpha}}(k_s^{(n)} a)}},$$

[2]) The mathematical discussion of the completeness of the set of eigen functions, etc. are beyond the scope of this paper.

assuming
$$W(r,\theta) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} B_{ns}\psi_{ns}(k_s^{(n)}r,\theta) \tag{21}$$

and substituting into the original differential eq. (1) the deflection W by eq. (21) and the right hand side by eq. (20) — the latter being zero except at the loading point (ρ,φ) — B_{ns} is determined.

$$B_{ns} = \frac{\psi_{\frac{n\pi}{\alpha}}(k_s^{(n)}\rho,\varphi)}{(k_s^{(n)})^4 D}. \tag{22}$$

Hence the deflection becomes:

$$W(r,\theta;\rho,\varphi) = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\frac{\psi_{\frac{n\pi}{\alpha}}(k_s^{(n)}\rho,\varphi)\,\psi_{\frac{n\pi}{\alpha}}(k_s^{(n)}r,\theta)}{\lambda_{ns}^2}, \tag{23}$$

where
$$\lambda_{ns}^2 = (k_s^{(n)})^4 D.$$

This solution is suitable for solving eigen value problems such as buckling or vibration.

III. Green's Functions for Bending Moments M_r, M_θ and Twisting Moment $M_{r\theta}$ of a Wedge-Shaped Plate

Taking the limit $a \to \infty$ in eq. (7), Green's function for the deflection of a wedge-shaped plate is derived in single series form,

$$W(r,\theta;\rho,\varphi) = \frac{\alpha}{4\pi^2 D}\sum_{n=1}^{\infty}\frac{\left(\frac{r}{\rho}\right)^{\frac{n\pi}{\alpha}}}{n\left(n^2-\frac{\alpha^2}{\pi^2}\right)}\left[\left(n+\frac{\alpha}{\pi}\right)\rho^2-\left(n-\frac{\alpha}{\pi}\right)r^2\right]\sin\frac{n\pi\varphi}{\alpha}\sin\frac{n\pi\theta}{\alpha}$$
$$(0 \leqq r \leqq \rho), \tag{24}$$
$$\frac{\alpha}{4\pi^2 D}\sum_{n=1}^{\infty}\frac{\left(\frac{\rho}{r}\right)^{\frac{n\pi}{\alpha}}}{n\left(n^2-\frac{\alpha^2}{\pi^2}\right)}\left[-\left(n-\frac{\alpha}{\pi}\right)\rho^2+\left(n+\frac{\alpha}{\pi}\right)r^2\right]\sin\frac{n\pi\varphi}{\alpha}\sin\frac{n\pi\theta}{\alpha}$$
$$(\rho \leqq r < \infty).$$

Green's functions for the moments M_r, M_θ and $M_{r\theta}$ are obtained through differentiation:

$$M_r(r,\theta;\rho,\varphi) = \left\{-D\left[\frac{\partial^2 W}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}\right)\right]\right\},$$
$$M_\theta(r,\theta;\rho,\varphi) = \left\{-D\left[\nu\frac{\partial^2 W}{\partial r^2}+\left(\frac{1}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}\right)\right]\right\},$$
$$M_{r\theta}(r,\theta;\rho,\varphi) = \left[-D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial\theta}\right)\right].$$

The resulting series expressions can be summed observing the following summation formulæ:

$$\sum_{n=1}^{\infty}\frac{r^n}{n}\cos n x = -\frac{1}{2}\log(1-2r\cos x+r^2),$$

$$\sum_{n=1}^{\infty} r^n \cos n x = \frac{1-r\cos x}{1-2r\cos x+r^2}-1 = \frac{1}{2}\left(\frac{1-r^2}{1-2r\cos x+r^2}-1\right), \quad \text{for } |r|<1. \tag{25}$$

The final equations in closed form are:

$$M_r(r,\theta;\rho,\varphi) = \frac{1}{8\pi}\left[(1+\nu)\log\frac{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)-\cos\frac{\pi}{\alpha}(\theta+\varphi)}{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)-\cos\frac{\pi}{\alpha}(\theta-\varphi)} - \frac{(1-\nu)\pi}{2\alpha}\left(1-\frac{\rho^2}{r^2}\right)\right.$$
$$\left.\cdot\left(\frac{\sinh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)}{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)-\cos\frac{\pi}{\alpha}(\theta-\varphi)} - \frac{\sinh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)}{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)-\cos\frac{\pi}{\alpha}(\theta+\varphi)}\right)\right],$$

$$M_\theta(r,\theta;\rho,\varphi) = \frac{1}{8\pi}\left[(1+\nu)\log\frac{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)-\cos\frac{\pi}{\alpha}(\theta+\varphi)}{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)-\cos\frac{\pi}{\alpha}(\theta-\varphi)} + \frac{(1-\nu)\pi}{2\alpha}\left(1-\frac{\rho^2}{r^2}\right)\right.$$
$$\left.\cdot\left(\frac{\sinh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)}{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)-\cos\frac{\pi}{\alpha}(\theta-\varphi)} - \frac{\sinh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)}{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)-\cos\frac{\pi}{\alpha}(\theta+\varphi)}\right)\right], \tag{26}$$

$$M_{r\theta}(r,\theta;\rho,\varphi) = \frac{(1-\nu)}{16\alpha}\left(1-\frac{\rho^2}{r^2}\right)$$
$$\cdot\left(\frac{\sin\frac{\pi}{\alpha}(\theta-\varphi)}{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)-\cos\frac{\pi}{\alpha}(\theta-\varphi)} - \frac{\sin\frac{\pi}{\alpha}(\theta+\varphi)}{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right)-\cos\frac{\pi}{\alpha}(\theta+\varphi)}\right).$$

Eqs. (26) are valid for the entire domain of the plate. If the loading point $P(\rho,\varphi)$ is fixed, the functions represent the moment surfaces $M_r(r,\theta)$, $M_\theta(r,\theta)$ and $M_{r\theta}(r,\theta)$. On the other hand if the point $Q(r,\theta)$ is fixed the functions represent the influence surfaces $m_r(\rho,\varphi)$, $m_\theta(\rho,\varphi)$ and $m_{r\theta}(\rho,\varphi)$ for the influence point (r,θ). The following figs. 3, 4, and 5, are examples of influence surfaces for the opening angles $\alpha=60°$.

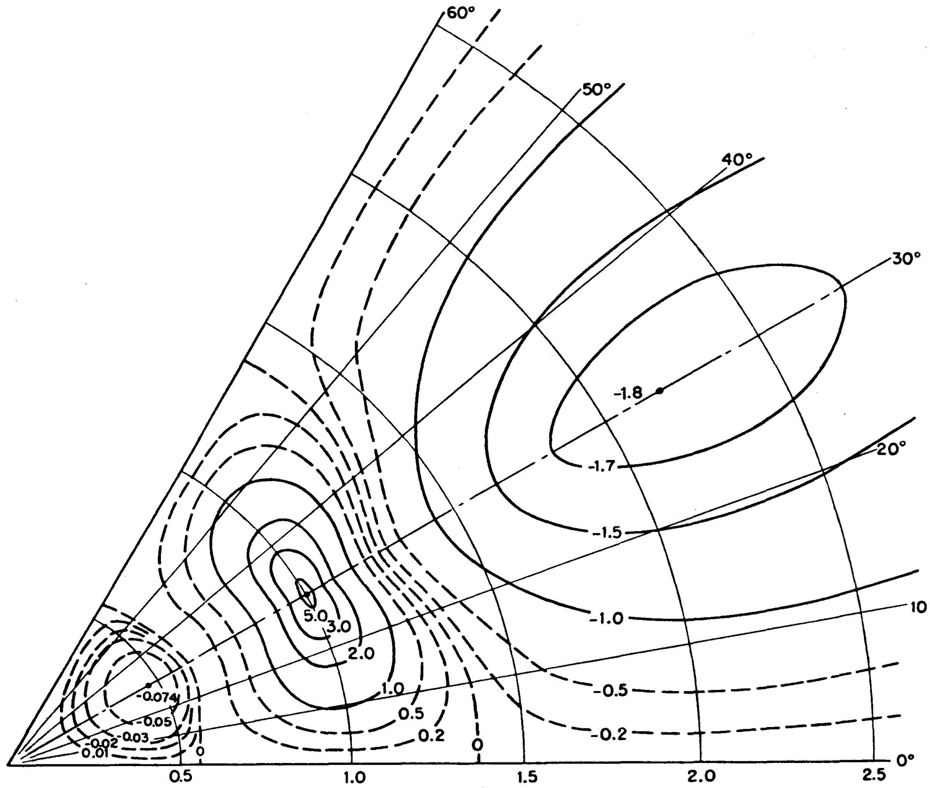

Fig. 3a. $8\pi m_r$ — Influence Surface for $0 \leqq \frac{\rho}{r} \leqq 2.5$. Influence Point $\left(\frac{\rho}{r}=1,\ \theta=30°\right)$.
$\alpha=60°$; $\nu=0$.

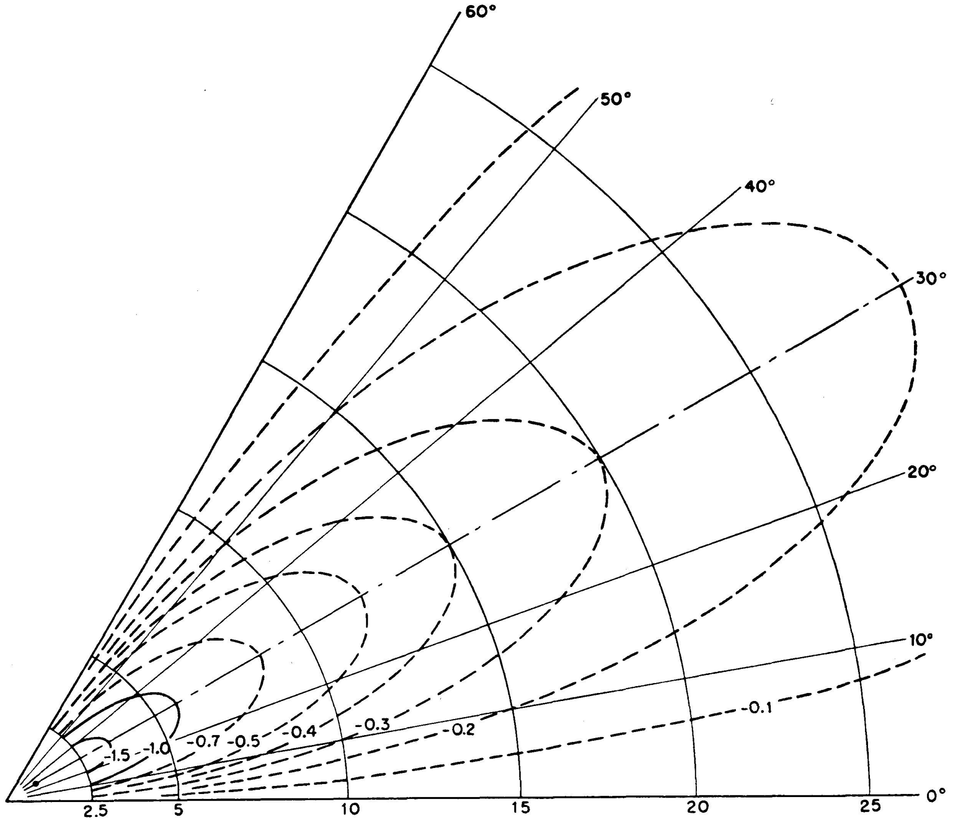

Fig. 3b. $8\pi m_r$ — Influence Surface for $\frac{\rho}{r} > 2.5$. Influence Point $\left(\frac{\rho}{r}=1,\ \theta=30°\right)$.
$\alpha=60°$; $\nu=0$.

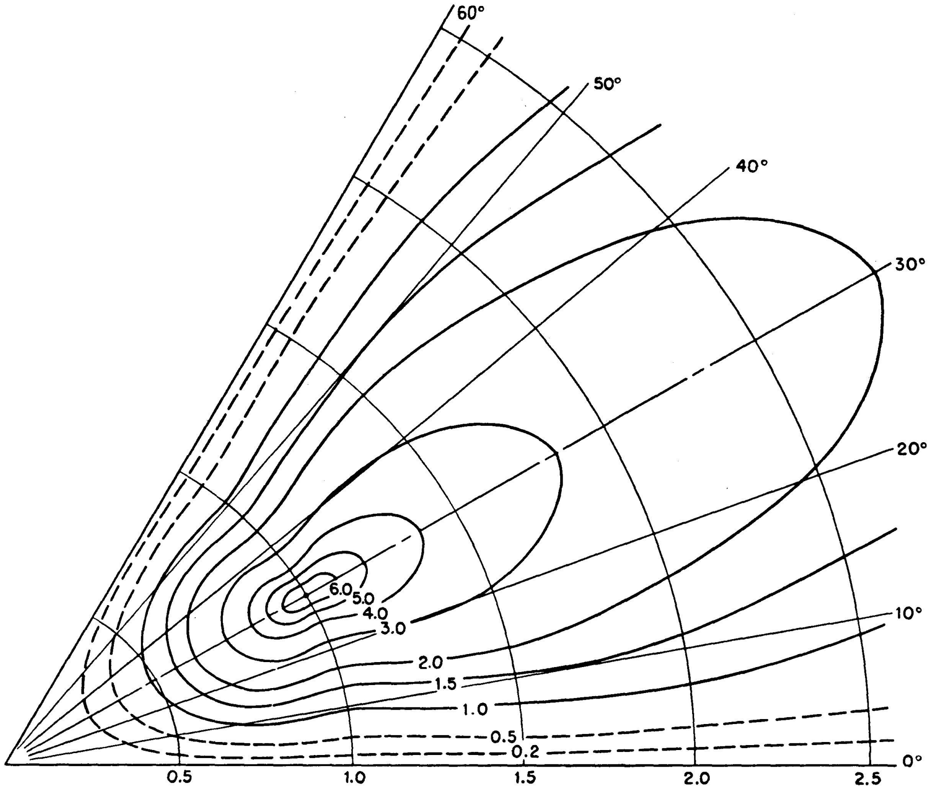

Fig. 4a. $8\pi m_\theta$ — Influence Surface for $0 \leqq \frac{\rho}{r} \leqq 2.5$. Influence Point $\left(\frac{\rho}{r}=1,\ \theta=30^\circ\right)$.
$\alpha=60^\circ;\ \nu=0.$

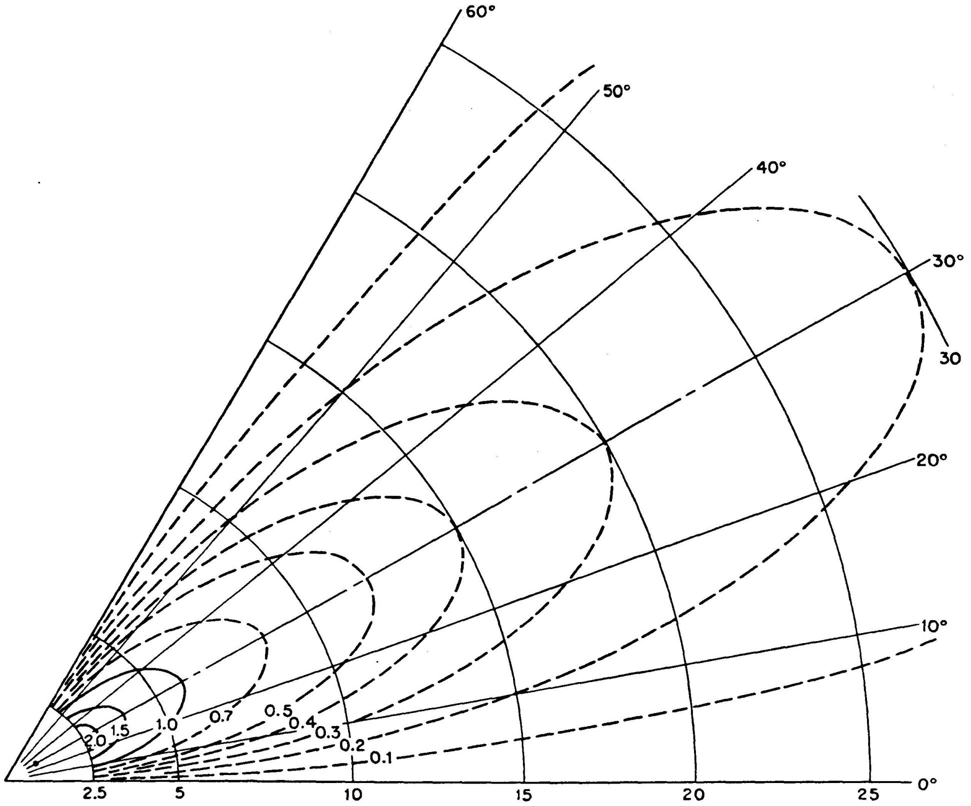

Fig. 4b. $8\pi m_\theta$ — Influence Surface for $\frac{\rho}{r} > 2.5$. Influence Point $\left(\frac{\rho}{r}=1,\ \theta=30^\circ\right)$.
$\alpha=60^\circ;\ \nu=0.$

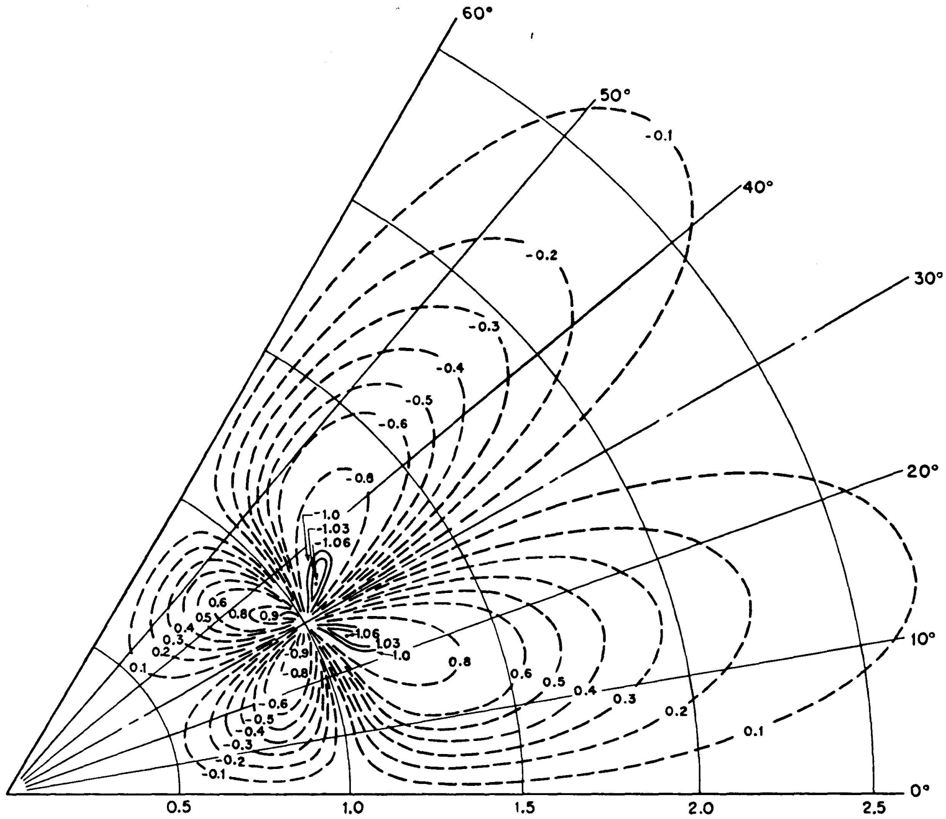

Fig. 5. $8\pi m_{r\theta}$ — Influence Surface for $0 \leqq \frac{\rho}{r} \leqq 2.5$. Influence Point $\left(\frac{\rho}{r}=1;\ \theta=30°\right)$. $\alpha=60°$; $\nu=0$.

IV. Stress Singularities at the Corner

In the vicinity of the corner 0, fig. 1, r approaches zero ($r \to 0$). However, ρ being finite it follows that

$$\log\frac{r}{\rho} \to -\infty.$$

Therefore

$$\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) \sim \frac{1}{2}\left(\frac{\rho}{r}\right)^{\pi/\alpha},$$

$$\sinh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) \sim -\frac{1}{2}\left(\frac{\rho}{r}\right)^{\pi/\alpha}$$

and approximate expressions of eq. (26) are obtained.

$$\begin{aligned} M_r(r,\theta;\rho,\varphi) &\sim -\frac{(1-\nu)}{4\alpha}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2}\sin\frac{\pi\varphi}{\alpha}\sin\frac{\pi\theta}{\alpha}, \\ M_\theta(r,\theta;\rho,\varphi) &\sim \frac{(1-\nu)}{4\alpha}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2}\sin\frac{\pi\varphi}{\alpha}\sin\frac{\pi\theta}{\alpha}, \\ M_{r\theta}(r,\theta;\rho,\varphi) &\sim -\frac{(1-\nu)}{4\alpha}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2}\sin\frac{\pi\varphi}{\alpha}\cos\frac{\pi\theta}{\alpha}. \end{aligned} \tag{27}$$

The corner reaction $r(\rho, \varphi)$ acting downward at the corner to prevent uplifting can be computed as follows

$$r(\rho,\varphi) = M_{r\theta}(0,0;\,\rho,\varphi) + M_{\theta r}(0,\alpha;\,\rho,\varphi)$$
$$= 2\,M_{r\theta}(0,0;\,\rho,\varphi),$$
$$r(\rho,\varphi) \sim -\frac{(1-\nu)}{2\alpha}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2}\sin\frac{\pi\varphi}{\alpha}. \tag{28}$$

From eqs. (27) and (28) it is evident that M_r, M_θ, $M_{r\theta}$, and r have a singularity at the corner governed by the term $\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2}$. Several cases, depending on the value of α, should be distinguished.

a) $0<\alpha<\frac{\pi}{2}$, $\quad \frac{\pi}{\alpha}>2$, $\quad \lim_{r\to 0}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} = 0$,

$$\therefore\; M_r(0,\theta;\,\rho,\varphi) = M_\theta(0,\theta;\,\rho,\varphi) = M_{r\theta}(0,\theta;\,\rho,\varphi) = r(\rho,\varphi) = 0.$$

b) $\alpha = \frac{\pi}{2}$, $\quad \lim_{r\to 0}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} = 1$,

$$M_r(0,\theta;\,\rho,\varphi) = -\frac{(1-\nu)}{2\pi}\sin 2\theta\sin 2\varphi,$$
$$M_\theta(0,\theta;\,\rho,\varphi) = \frac{(1-\nu)}{2\pi}\sin 2\theta\sin 2\varphi,$$
$$M_{r\theta}(0,\theta;\,\rho,\varphi) = -\frac{(1-\nu)}{2\pi}\cos 2\theta\sin 2\varphi,$$
$$r(\rho,\varphi) = -\frac{(1-\nu)}{\pi}\sin 2\varphi.$$

c) $\frac{\pi}{2}<\alpha<\pi$, $\quad 1<\frac{\pi}{\alpha}<2$ $\quad \lim_{r\to 0}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} = +\infty$,

$$M_r(0,\theta;\,\rho,\varphi) = -\infty, \qquad M_\theta(0,\theta;\,\rho,\varphi) = +\infty,$$
$$M_{r\theta}(0,\theta;\,\rho,\varphi) = \pm\infty, \qquad r(\rho,\varphi) = -\infty.$$

d) $\alpha = \pi$.

Although eqs. (27) and (28) are not valid for this case, simple physical considerations will show the vanishing of the moments:

$$M_r = M_\theta = M_{r\theta} = 0.$$

e) $\pi<\alpha<2\pi$, $\quad 0<\frac{\pi}{\alpha}<1$, $\quad \lim_{r\to 0}\left(\frac{r}{\rho}\right)^{\frac{\pi}{\alpha}-2} = +\infty$,

$$M_r = -\infty, \qquad M_\theta = +\infty,$$
$$M_{r\theta} = \pm\infty, \qquad r = -\infty.$$

It is interesting to note that in the case of the two dimensional flow of an ideal fluid around a wedge the velocity q is governed by the term $r^{\frac{\pi}{\alpha}-1}$.

Finally, the moment $M_r(r,\theta)$ along the bisecting line $\alpha/2$ is plotted in fig. 6 for different values of the opening angle α, illustrating the behavior at the corner as discussed under (a) to (e).

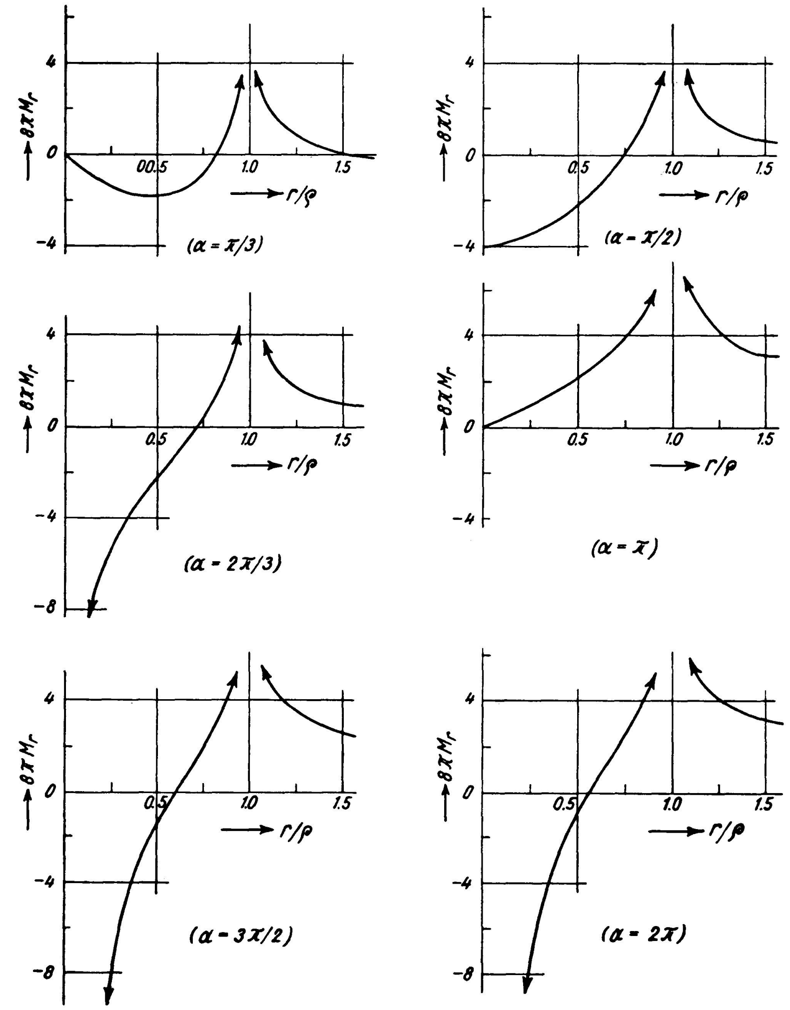

Fig. 6. $M_r\left(r,\frac{\alpha}{2}\right)$ for Different Values of Opening Angle α ($\nu=0$).

V. Alternative Solutions

A. *Cases Where the Opening Angles* $\alpha=\frac{\pi}{n}$ $(n=1,2,3\ldots)$

The Green's function for the deflections of a circular plate with a clamped edge was developed by J. H. Michell [5].

$$W(r,\theta;\rho,\varphi)=\frac{1}{16\pi a^4 D}\left\{a^2\left[r^2-2\rho r\cos(\theta-\varphi)+\rho^2\right]\cdot\log\frac{a^2\left[r^2-2\rho r\cos(\theta-\varphi)+\rho^2\right]}{a^4-2a^2\rho r\cos(\theta-\varphi)+\rho^2 r^2}+(a^2-\rho^2)(a^2-r^2)\right\}. \tag{29}$$

Solutions for sectorial plates can be derived by using the "Method of Images" (see for example [4], p. 174). As an example the case of a plate with an opening angle $\alpha = \frac{\pi}{2}$ will be treated. The loading points P_1 to P_4, shown in fig. 7, are symmetrical with respect to the diameters AC and BD respectively. By applying downward unit loads at P_1 and P_4 and upwards unit loads at P_2 and P_3 the symmetry conditions will effectively produce conditions along the diameters AC and BD corresponding to the ones of a simply supported edge. Therefore the solution of the sectorial plate OAB is obtained by superposition of the effects of the four loads at P_1 to P_4, or:

$$\begin{aligned} W(r,\theta;\rho,\varphi) &= \\ = \frac{1}{16\pi a^4 D}\Big\{ & a^2[r^2-2\rho r\cos(\theta-\varphi)+\rho^2]\log\frac{a^2[r^2-2\rho r\cos(\theta-\varphi)+\rho^2]}{a^4-2a^2\rho r\cos(\theta-\varphi)+\rho^2 r^2} \\ & - a^2[r^2-2\rho r\cos(\theta+\varphi-\pi)+\rho^2]\log\frac{a^2[r^2-2\rho r\cos(\theta+\varphi-\pi)+\rho^2]}{a^4-2a^2\rho r\cos(\theta+\varphi-\pi)+\rho^2 r^2} \\ & + a^2[r^2-2\rho r\cos(\theta-\varphi-\pi)+\rho^2]\log\frac{a^2[r^2-2\rho r\cos(\theta-\varphi-\pi)+\rho^2]}{a^4-2a^2\rho r\cos(\theta-\varphi-\pi)+\rho^2 r^2} \\ & - a^2[r^2-2\rho r\cos(\theta+\varphi-2\pi)+\rho^2]\log\frac{a^2[r^2-2\rho r\cos(\theta+\varphi-2\pi)+\rho^2]}{a^4-2a^2\rho r\cos(\theta+\varphi-2\pi)+\rho^2 r^2}\Big\}. \end{aligned} \tag{30}$$

It should be noted that the solution is obtained in closed form. However this method applies only to cases $\alpha = \frac{\pi}{n}$, with $n = 1, 2 \ldots.$

B. *Application of Conformal Mapping*

Considering the moment sum

$$M = M_r + M_\theta = -D(1+\nu)\Delta W \tag{31}$$

substitution into eq. (1) furnishes the following differential equation for M

$$\Delta M = 0 \tag{32}$$

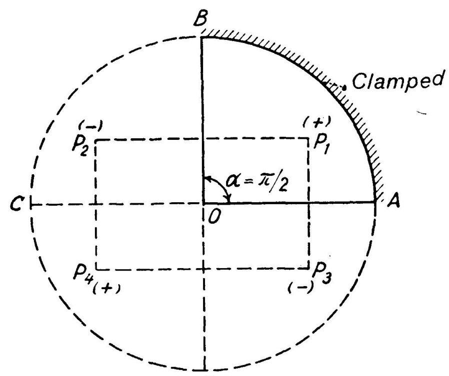

Fig. 7. Sectorial Plate with Opening Angle $\pi/2$.

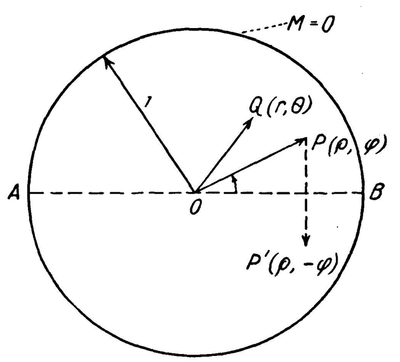

Fig. 8. Unit Circle for Conformal Mapping.

with the exception of the loading point (ρ, φ). The boundary condition for the plate of fig. 8 with radius $r=1$ is taken as $(M)_{r=1}=0$.

Since eq. (32) is similar to the equation of a membrane it can be concluded that M is proportional to the Green's function of circular membrane with unit radius, that is,

$$M(r, \theta; \rho, \varphi) = \frac{(1+\nu)}{4\pi} \log \frac{1-2\rho r \cos(\theta-\varphi)+\rho^2 r^2}{r^2-2\rho r \cos(\theta-\varphi)+\rho^2}. \tag{33}$$

Taking the image of the load with respect to the diameter AB, fig. 8, the Green's function for M of a semi-unit circle is derived.

$$M = \frac{(1+\nu)}{4\pi} \log \frac{[1-2\rho r \cos(\theta-\varphi)+\rho^2 r^2][r^2-2\rho r \cos(\theta+\varphi)+\rho^2]}{[r^2-2\rho r \cos(\theta-\varphi)+\rho^2][1-2\rho r \cos(\theta+\varphi)+\rho^2 r^2]}. \tag{34}$$

From eq. (34), M for a semi-infinite circular plate is derived, making $r \ll 1$ and $\rho \ll 1$. The variables are written with a bar in order to distinguish them from the final variables after mapping:

$$M(\bar{r}, \bar{\theta}; \bar{\rho}, \bar{\varphi}) = \frac{(1+\nu)}{4\pi} \log \frac{\bar{r}^2-2\bar{r}\bar{\rho}\cos(\bar{\theta}+\bar{\varphi})+\bar{\rho}^2}{\bar{r}^2-2\bar{r}\bar{\rho}\cos(\bar{\theta}-\bar{\varphi})+\bar{\rho}^2}. \tag{35}$$

By conformal mapping, M of a wedge-shaped plate with an opening angle α can be obtained applying the following mapping function:

$$w = z^{\pi/\alpha},$$

where

$$w = \bar{r} e^{i\bar{\theta}}, \quad z = r e^{i\theta},$$

so that

$$\bar{r} = r^{\pi/\alpha}, \quad \bar{\theta} = \frac{\pi\theta}{\alpha}. \tag{36}$$

Proceeding with the mapping the expression for M becomes:

$$\begin{aligned} M(r, \theta; \rho, \varphi) &= \frac{(1+\nu)}{4\pi} \log \frac{r^{\frac{2\pi}{\alpha}} - 2 r^{\pi/\alpha} \rho^{\pi/\alpha} \cos\frac{\pi}{\alpha}(\theta+\varphi) + \rho^{\frac{2\pi}{\alpha}}}{r^{\frac{2\pi}{\alpha}} - 2 r^{\pi/\alpha} \rho^{\pi/\alpha} \cos\frac{\pi}{\alpha}(\theta-\varphi) + \rho^{\frac{2\pi}{\alpha}}}, \\ &= \frac{(1+\nu)}{4\pi} \log \frac{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) - \cos\frac{\pi}{\alpha}(\theta+\varphi)}{\cosh\left(\frac{\pi}{\alpha}\log\frac{r}{\rho}\right) - \cos\frac{\pi}{\alpha}(\theta-\varphi)}. \end{aligned} \tag{37}$$

The bending moments M_r and M_θ are readily obtained from the moment sum M by the following differential operations:

$$\begin{aligned} M_r &= \frac{1}{2} M + \frac{(1-\nu)}{4(1+\nu)} \left(\frac{r}{\rho} - \frac{\rho}{r}\right) \frac{\partial M}{\partial \left(\frac{r}{\rho}\right)}, \\ M_\theta &= \frac{1}{2} M - \frac{(1-\nu)}{4(1+\nu)} \left(\frac{r}{\rho} - \frac{\rho}{r}\right) \frac{\partial M}{\partial \left(\frac{r}{\rho}\right)}. \end{aligned} \tag{38}$$

The corresponding results are identical with eqs. (26).

VI. Acknowledgements

The results presented in this paper were obtained in the course of research on Moment Influence Surfaces sponsored by the National Science Foundation (Grant NSF-G 2949). The author would like to express his sincere appreciation to Dr. Bruno Thürlimann for his assistance and suggestions. The checking of the theoretical derivations and the numerical computations were done by Mr. Akira Nitta, research assistant at Fritz Engineering Laboratory, Lehigh University.

VII. References

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Summary

Green's function for the deflection of a sectorial plate with simply supported radial edges and a clamped circumferential edge is obtained in two different forms, that is, in double and single series form. Using the second solution and taking its limit as the radius of the sector increases to infinity, Green's functions for bending and twisting moments of a wedge-shaped plate are derived in closed form.

With this solution, moment influence surfaces as well as moment surfaces can be developed with a detailed discussion of the stress singularities at the corner.

Résumé

La fonction de flexion de Green pour une dalle en forme de secteur avec bords radiaux simplement posés et bord circulaire encastré peut être obtenue de deux manières, par sommation double et sommation simple.

En employant la deuxième forme et en tenant compte de la valeur limite pour un rayon de grandeur infinie, il est possible d'établir sous forme explicite les fonctions de Green concernant les moments de flexion et de torsion sur une dalle en forme de coin.

A partir de cette solution, il est possible d'indiquer les aires d'influence pour les moments, ainsi que les aires des moments, avec discussion détaillée des singularités à la pointe de la dalle.

Zusammenfassung

Die Greensche Durchbiegungsfunktion für eine Sektorplatte mit einfach gelagerten Radialrändern und eingespanntem Kreisrand kann auf zwei Arten erhalten werden: nämlich in doppelter und einfacher Summendarstellung.

Durch Verwendung der zweiten Form und unter Berücksichtigung des Grenzwertes für unendlich großen Radius können in geschlossener Form die Greenschen Funktionen für Biege- und Verdrehungsmomente an einer keilförmigen Platte abgeleitet werden.

Mit dieser Lösung können Einflußflächen für die Momente wie auch Momentenflächen angegeben werden mit einer detaillierten Diskussion der Singularitäten an der Spitze der Platte.