A method of determining the buckling stress and the required cross-sectional area for centrallly loaded straight columns in elastic and inelastic range

A Method of Determining the Buckling Stress and the Required Cross-Sectional Area for Centrally Loaded Straight Columns in Elastic and Inelastic Range

Méthode pour déterminer la contrainte de flambage et la section d'une colonne rectiligne comprimée par une force centrée, dans les domaines élastiques et inélastiques

Ein Verfahren für die Bestimmung der Knickspannung und der erforderlichen Querschnittsfläche eines zentrisch belasteten geraden Stabes im elastischen und unelastischen Bereich

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1. Stress-Strain Function

In order to investigate the buckling strength of columns loaded in the inelastic range, it is advantageous to approximate the stress-strain diagram of the uniaxial state of stress occurring in the column with a suitable function and to deduce the corresponding buckling-stress formula. The function chosen should contain a sufficient number of free parameters. By choosing them appropriately, it is possible to make the values of the function coincide with the experimental values of the stress-strain diagram. In practice the stress at which the stress-strain diagram with increasing stress for the first time becomes parallel to the strain axis should be considered the upper limit of the buckling stress. This limit is either *yield point* or *compressive strength* depending on whether the material has a yield point or not. From this it appears that, in order to be suitable for this purpose, the so-called stress-strain function presenting the stress-strain diagram should be valid up to this stress limit for materials having a pronounced yield point and up to the compressive strength for other materials. For the sake of simplicity we speak in the following only of the yield point remembering that it will be replaced by compressive strength if no yield point exists.

For the approximation of the stress-strain diagram many different functions have been used[1][2]. In the following we present a new one which is meant particularly for elucidating problems in connection with the buckling phenomenon of columns[3]. Since not the stress-strain function itself but only its first derivative with respect to the strain is needed, the simplest way is to make a suitable assumption with respect to this derivative itself and to prove that the stress-strain function deduced from it by integration agrees with the test results. The expression of the derivative should be as simple as possible and only a function of stress.

We assume in the following that the expression of the derivative of the stress-strain function has the form

$$\frac{d\sigma}{d\epsilon} = \frac{a-\sigma}{b-c'\sigma}. \tag{1}$$

Here σ is the stress, ϵ the strain and a, b, c' are three free parameters, the values of which should be determined so that the stress-strain function deduced from (1) by integration suitably agrees with the stress-strain diagram. Below the yield point, the elasticity and the strength properties of the material are determined on the basis of its modulus of elasticity, its proportional limit and yield point stress. Therefore, we use these quantities for determining parameters a, b, c'.

For the yield-point stress $\sigma = \sigma_y$ we have $d\sigma/d\epsilon = 0$. By introducing these values into eq. (1) we obtain

$$a = \sigma_y. \tag{2}$$

When $\sigma = 0$, we have $d\sigma/d\epsilon = E$, hence, from eq. (1) it follows

$$E = \frac{a}{b} = \frac{\sigma_y}{b}. \tag{3}$$

By introducing the expressions of a and b and a new parameter $c = c'E$ into eq. (1) we obtain

$$\frac{d\sigma}{d\epsilon} = E\frac{\sigma_y - \sigma}{\sigma_y - c\sigma}. \tag{4}$$

In order to connect the third material constant, the proportional limit σ_p, with c, we need function $\sigma = \sigma(\epsilon)$ itself. Separating the variables the differential eq. (4) can be written as

$$d\epsilon = \frac{1}{E}\left[c + \frac{(1-c)\sigma_y}{\sigma_y - \sigma}\right] d\sigma.$$

The general solution of this is

$$\epsilon = \frac{1}{E}\left[c\sigma - (1-c)\sigma_y \ln(\sigma_y - \sigma) + C\right],$$

[1]) R. Mehmke, *Z. f. Math. u. Physik.* 1897, p. 327.

[2]) W. Osgood, *Journal of Aeron. Sciences.* 1946, p. 45.

[3]) See writer's paper in *Teknillinen Aikakauslehti*, Vol. 38 (1948), p. 9.

where C is the constant of integration. Since the stress-strain diagram goes through the origin of coordinates, the values $\epsilon=0$, $\sigma=0$ correspond to each other. From this condition follows for the constant of integration the expression

$$C = (1-c)\,\sigma_y \ln \sigma_y,$$

which, introduced into the general solution, gives as result the stress-strain function

$$\epsilon = \frac{1}{E}\left[c\,\sigma - (1-c)\,\sigma_y \ln\left(1-\frac{\sigma}{\sigma_y}\right)\right]. \tag{5}$$

With the value $c=1$ this is reduced to HOOKE's law.

The nature of the stress-strain curves according to eq. (5) may be seen from figure 1, where σ is plotted against ϵ. To determine the parameter c the difference δ between strain (5) and the strain $\epsilon_H=\sigma/E$ corresponding to HOOKE's law is formed and for that the expression

$$\delta = \epsilon - \epsilon_H = -\frac{1-c}{E}\left[\sigma + \sigma_y \ln\left(1-\frac{\sigma}{\sigma_y}\right)\right] \tag{6}$$

is obtained.

When $\sigma=\sigma_p$, the corresponding value of δ is denoted by δ_p. When these values are introduced into eq. (6), its both sides multiplied by the ratio E/σ_y and the equation is solved for c, we obtain

$$c = 1 + \frac{\dfrac{E\,\delta_p}{\sigma_y}}{\dfrac{\sigma_p}{\sigma_y} + \ln\left(1-\dfrac{\sigma_p}{\sigma_y}\right)}. \tag{7}$$

This formula shows how the parameter c depends on the proportional limit σ_p, the yield-point stress σ_y and the modulus of elasticity E. In addition, its value is affected by the quantity δ_p, which indicates the allowable deviation from HOOKE's law at the proportional limit. The usual definition is $\delta_p=0.00002$ to 0.0002. In table 1 are presented the values of the parameters c of some materials, calculated from formula (7).

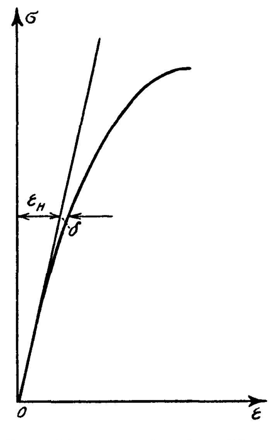

Fig. 1. Stress-strain diagram.

Table 1. E, σ_p, σ_y and c of some materials

Material	E kg/cm²	σ_p kg/cm²	σ_y kg/cm²	c
Finnish pine	125 000	160	450	0,875
Magnesium Alloy (Electron)	460 000	500	1000	0,857
Steel St 37	2 100 000	1920	2400	0,977
Steel St 52	2 100 000	2880	3600	0,977
Concrete	250 000	50	280	0

We see that the values c of all other materials are slightly less than unity except that of concrete, which is zero.

It should be observed that, for determining c, it is not necessary to use the proportional limit, but any point on the stress-strain diagram between the proportional limit and the yield point is applicable. The stress-strain function determined in this manner agrees in general better with the stress-strain diagram than that obtained with the aid of the proportional limit.

In order to get a general idea of the form of the stress-strain diagrams represented by function (5), we put it into a more suitable form for graphical representation by multiplying both sides by the ratio E/σ_y, which gives

$$\frac{E\epsilon}{\sigma_y} = c\frac{\sigma}{\sigma_y} - (1-c)\ln\left(1-\frac{\sigma}{\sigma_y}\right). \tag{8}$$

Compared with (5) this dimensionless form has the advantage that $E\epsilon/\sigma_y$ is a function only of the ratio σ/σ_y and parameter c. But on the right side of eq. (5) there are four variable quantities E, σ, σ_y and c. The stress-strain curves according to eq. (8) may be seen from figure 2, where σ/σ_y is plotted against $E\epsilon/\sigma_y$, c as parameter. We see that the greater the value of c is, the smaller is the deviation of the stress-strain curves from the broken line formed by

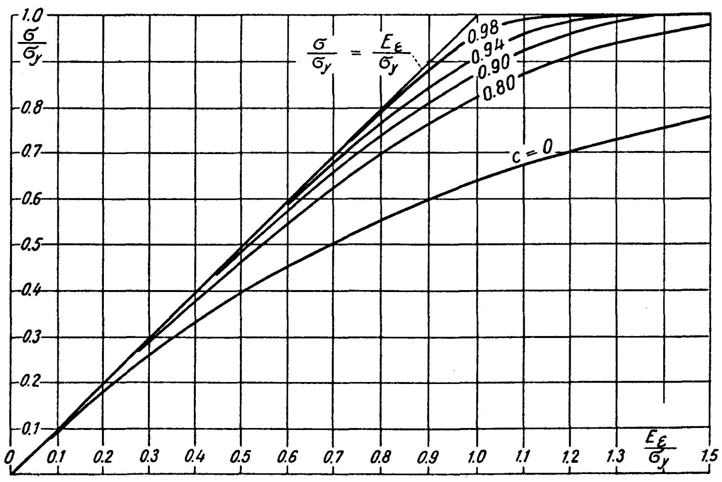

Fig. 2. Dimensionless stress-strain diagrams according to eq. (8).

HOOKE's line $\sigma/\sigma_y = E\,\epsilon/\sigma_y$ and the horizontal line $\sigma/\sigma_y = 1$ corresponding to the yield-point stress.

2. Tangent-Modulus Theory of Inelastic Buckling

The buckling force of a straight, centrally compressed, prismatical column in the elastic range is obtained from EULER's buckling formula[4])

$$F_c = \frac{\mu\,\pi^2 E\,J}{l^2}, \tag{9}$$

where E denotes the modulus of elasticity, J the smallest moment of inertia of the cross-section and l the length of the column. μ is the coefficient of restraint depending on the manner in which the ends of the column are fixed. The value of this coefficient varies within the ranges $^1/_4 \leqq \mu \leqq 4$. Figure 3 presents the value of μ in some modes of restraint. The case 2 of a column with hinged ends is very often encountered in practical applications and is called the *fundamental case* of buckling of a prismatical column.

Expressing in eq. (9) the moment of inertia J by the radius of gyration i and the area A of the cross section in the form $J = i^2 A$, the formula may be written in the form $F_c/A = \mu\,\pi^2 E/(l/i)^2$. Denoting further the average compressive stress by $F_c/A = \sigma_c$ and the slenderness ratio of the column by $l/i = \lambda$ we obtain for the buckling stress of the column the formula

$$\sigma_c = \frac{\mu\,\pi^2 E\,J}{\lambda^2}. \tag{10}$$

The buckling stress is independent of the shape of the cross section of the column.

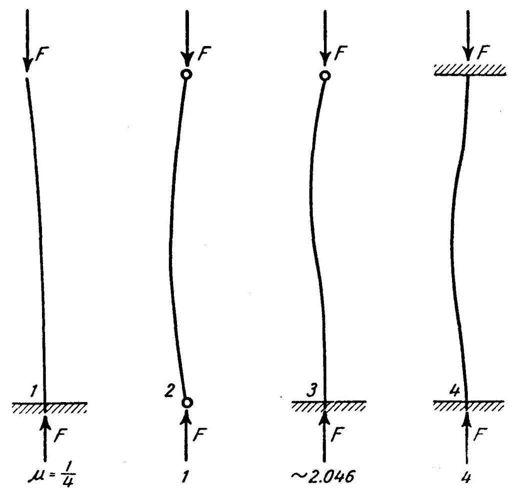

Fig. 3. Some fixing cases of the ends of the column.

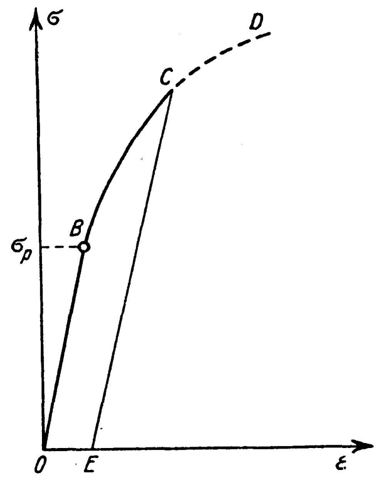

Fig. 4. Stress-strain diagram for increase and decrease of load.

[4]) L. EULER, *De curvis elasticis*. Lausanne and Geneva, 1744. — The EULER formula was derived in a later paper, *Sur la force de colonnes*, published 1759 in the *Mémoires de l'Académie de Berlin*.

EULER's formulas (9) and (10) are valid only as far as the compressive stress $\sigma_c < \sigma_p$. By introducing the expression of σ_c from (10) and solving the inequality for λ we obtain as the validity limit of formulas (9) and (10)

$$\lambda > \pi \sqrt{\frac{\mu E}{\sigma_y}}. \tag{11}$$

Let us now examine the determination of the buckling stress of a column buckling above the proportional limit. If the material is still perfectly elastic even beyond the proportional limit so that the diminishing stress in figure 4 follows the same stress-strain diagram CBO as obtained with increasing load, the buckling force and the buckling stress of the columns are, according to ENGESSER[5]), obtained from formulas (9) and (10) by replacing in them the modulus of elasticity E by the tangent modulus $d\sigma/d\epsilon$ of the stress-strain diagram at the point corresponding to the buckling stress. Denoting this tangent modulus by

$$\frac{d\sigma}{d\epsilon} = E_t \tag{12}$$

we obtain ENGESSER's formulas

$$F_t = \frac{\mu \pi^2 E_t J}{l^2} \tag{13}$$

and

$$\sigma_t = \frac{\mu \pi^2 E_t}{\lambda^2} \tag{14}$$

for determining the buckling force and the buckling stress beyond the proportional limit. Below the proportional limit E_t equals E and formulas (13) and (14) are changed into EULER's formulas (9) and (10). Consequently, ENGESSER's formulas (13) and (14) are valid both above and below the proportional limit. Both the buckling force and the buckling stress of a perfectly elastic column are independent of the shape of the cross section of the column.

By introducing the expression of the derivative $d\sigma/d\epsilon$ from eq. (4) into formula (12), the tangent modulus can be expressed in the form

$$E_t = E \frac{\sigma_y - \sigma}{\sigma_y - c\sigma}. \tag{15}$$

Introducing this into ENGESSER's formula (14) and taking into consideration that σ equals σ_t when buckling occurs, it follows for the buckling slenderness ratio of a centrally compressed straight column the formula

$$\lambda^2 = \frac{\mu \pi^2 E}{\sigma_t} \frac{\sigma_y - \sigma_t}{\sigma_y - c\sigma_t}. \tag{16}$$

This is valid when $0 \leqq \sigma_t \leqq \sigma_y$.

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[5]) F. ENGESSER, *Zeitschrift des Architekten- und Ingenieur-Vereins zu Hannover.* 1889, p. 455.

When eq. (16) is reduced, we obtain for buckling stress σ_t an equation of the second degree, the roots of which are

$$\sigma_t = \frac{\mu\,\pi^2 E + \sigma_y \lambda^2}{2\,c\,\lambda^2} (\pm) \sqrt{\frac{(\mu\,\pi^2 E + \sigma_y \lambda^2)^2 - 4\,\mu\,\pi^2 c\,E\,\sigma_y \lambda^2}{4\,c^2 \lambda^4}}. \tag{17}$$

For the square root a negative sign should be chosen, because the buckling stress must vanish when $\lambda \to \infty$.

In order to get an idea of the buckling stress according to eq. (16) we represent this equation graphically. For that purpose we first multiply both sides of the equation by $\sigma_y/\mu\,\pi^2 E$ thus giving it the dimensionless form

$$\frac{\sigma_y \lambda^2}{\mu\,\pi^2 E} = \frac{\sigma_y}{\sigma_t} \frac{1 - \frac{\sigma_t}{\sigma_y}}{1 - c\frac{\sigma_t}{\sigma_y}}. \tag{18}$$

Compared with eq. (16) this form has the advantage that $\sigma_y \lambda^2/\mu\,\pi^2 E$ is only a function of the parameter c and the ratio σ_t/σ_y. The variable quantities on the right side of eq. (16) are σ_t, σ_y, μ, E and c.

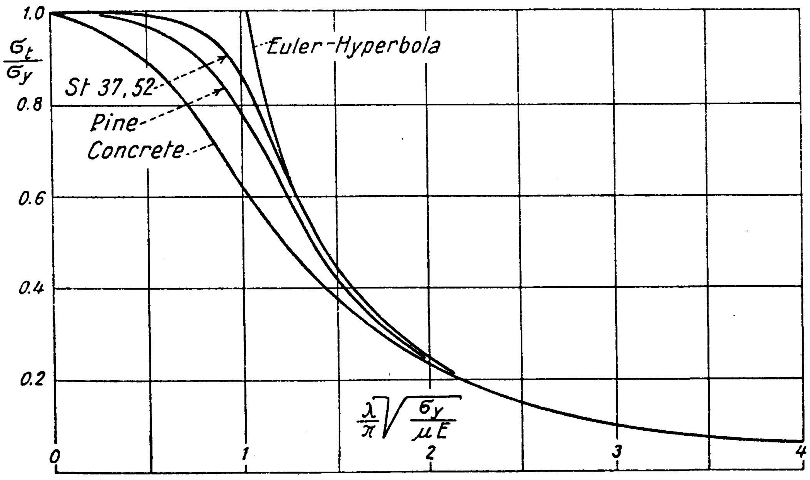

Fig. 5. Buckling stress diagrams in dimensionless form according to formula (17).

In figure 5 the ratio σ_t/σ_y is plotted against the dimensionless quantity $\lambda/\pi\sqrt{\sigma_y/\mu\,E}$. At great values of the slenderness ratio, when the buckling stress is low, the value of the tangent modulus is constant according to (15) and the buckling-stress diagrams corresponding to different values of c coincide with EULER's hyperbola. With decreasing slenderness ratio the buckling stress increases, the value of the tangent modulus decreases and the buckling stress diagrams deviate from EULER's hyperbola and approach the yield point stress of the material. How rapidly this will take place, depends on the value of parameter c. When $\lambda = 0$, σ_t becomes equal to σ_y for any value of c.

3. Double-Modulus Theory of Inelastic Buckling

ENGESSER's tangent-modulus theory was severely criticized, first by CONSIDÈRE[6]) and later by JASINSKY[7]), on the account that ENGESSER had assumed the material to be perfectly elastic even beyond the proportional limit and thus had not taken into consideration the effect of permanent deformations. In fact, when the compressive stress in figure 4 has increased up to point C above the proportional limit σ_p and the column bends, the decreasing stress on the convex side of the column does not under actual conditions follow the same curve CBO along which the stress has increased, but decreases from point C along the straight line CE, which has the same slope as part OB of the stress-strain diagram corresponding to HOOKE's law. Thus the phenomenon is irreversible by its nature. A certain strain ϵ corresponds to different values of stress depending on whether an increasing or a decreasing stress is considered. The stress is no more a single-valued function of the strain.

ENGESSER now completed his buckling theory by taking into consideration the effect of permanent deformations[8]). However, his theory did not receive the attention and acknowledgement it would have deserved, but was forgotten. The theory became once more known and its applicability stated in 1910 through the careful buckling tests made by v. KÁRMÁN[9]).

According to ENGESSER, the effect of permanent deformations in the inelastic buckling phenomenon can be taken into consideration by replacing the tangent modulus in formulas (13) and (14) by the so-called "double modulus", which is also called "reduced modulus". By employing the symbol E_r for it the formulas

$$F_r = \frac{\mu \pi^2 E_r J}{l^2} \tag{19}$$

and

$$\sigma_r = \frac{\mu \pi^2 E_r}{\lambda^2} \tag{20}$$

can be written for determining the buckling force and the buckling stress of a centrally loaded straight column. In the elastic range E_r equals E, while in the inelastic range E_r is variable and depends on $\sigma_r = F_r/A$ and on the shape of the cross section. In the inelastic range E_r is always greater than E_t and, consequently, the buckling stress will be slightly higher according to the

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[6]) M. CONSIDÈRE, *Congrès international des procédés de construction*, 1889. Comptes rendus, Annexe: *Résistance des pièces comprimées*, p. 381.

[7]) F. JASINSKY, *Annales des ponts et chaussées*, 1894, and *Schweizerische Bauzeitung*, Vol. 26 (1895), p. 172.

[8]) F. ENGESSER, *Schweizerische Bauzeitung*, Vol. 26 (1895), p. 24.

[9]) TH. v. KÁRMÁN, *Physikalische Zeitschrift*, Vol. 9 (1909), p. 136; and *Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens*, No. 81, Berlin 1910.

reduced modulus theory than according to the tangent modulus theory. For an I-section with infinitely thin web

$$E_r = \frac{2\,E\,E_t}{E + E_t}$$

according to v. KÁRMÁN.

By introducing into this the expression of E_t from equation (15) we obtain

$$E_r = E\frac{\sigma_y - \sigma}{\sigma_y - \frac{1+c}{2}\sigma}. \tag{21}$$

From this it appears that $E_r \to E$ when $\sigma \to 0$. E_r equals E at any value of σ, if $c = 1$. Comparison between the expression (21) of E_r and the expression (15) of E_t reveals in the case of the idealized I-section the interesting peculiarity that, when employing the stress-strain law (5), its tangent modulus and double modulus are similar in form except that parameter c of the tangent modulus has been replaced in expression (21) of the double modulus by $(1+c)/2$, which is greater than c if $c < 1$.

By replacing the tangent modulus E_t in eq. (14) by the double modulus E_r from eq. (21) and by denoting the corresponding buckling stress by σ_r, we may write for the buckling slenderness ratio of a column of idealized I-section the formula

$$\lambda^2 = \frac{\mu\,\pi^2\,E}{\sigma_r}\,\frac{\sigma_y - \sigma_r}{\sigma_y - \frac{1+c}{2}\sigma_r}. \tag{22}$$

This corresponds to formula (16) except that parameter c has been replaced by $(1+c)/2$ in formula (22).

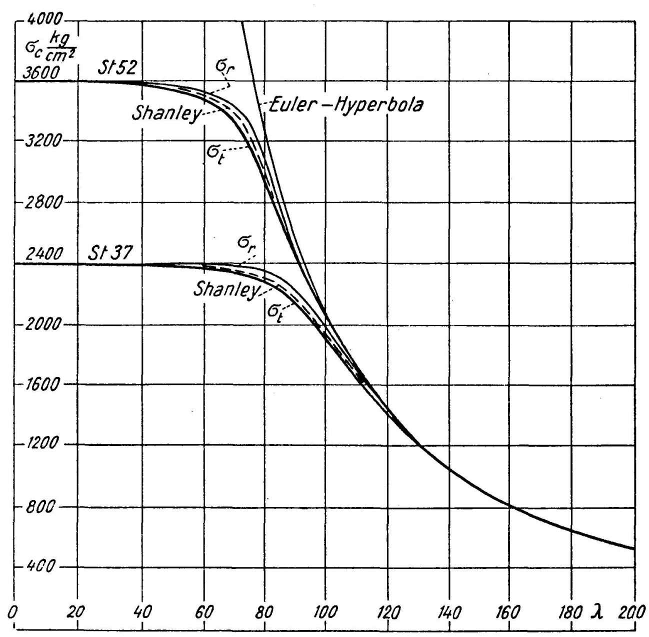

Fig. 6. Buckling stress diagrams for structural steels St 37 and St 52 according to tangent-modulus, double-modulus and SHANLEY's theories of inelastic buckling.

When eq. (22) is reduced, we obtain for the buckling stress σ_r an equation of the second degree, the roots of which are

$$\sigma_r = \frac{\mu \pi^2 E + \sigma_y \lambda^2}{(1+c)\lambda^2} (\pm) \sqrt{\frac{(\mu \pi^2 E + \sigma_y \lambda^2)^2 - 2\mu \pi^2 (1+c) E \sigma_y \lambda^2}{(1+c)^2 \lambda^4}}. \tag{23}$$

A negative sign should be chosen for the square root because the buckling stress must vanish when $\lambda \to \infty$.

The lower and upper full-line graphs in figure 6 represent the buckling stress of structural steels St 37 and St 52 according to formulas (16) and (22) or (17) and (23) by giving to the material constants the values of table 1. We see that the buckling stress is somewhat higher according to the double-modulus theory than according to the tangent-modulus theory. The relative deviation of the two curves is largest at $\lambda \approx 96$ for steel St 37 and at $\lambda \approx 80$ for steel St 52, both about 7,5%.

4. Shanley's Theory of Inelastic Buckling

After the double-modulus theory of Engesser had become generally known in 1910 through the investigations of v. Kármán, the scientists had been accustomed to consider it the final solution of the problem of inelastic buckling. Therefore, it was surprising when Shanley[10]) proved in 1946 that the double-modulus theory leads to discrepancies which could be removed only by renouncing some of the assumptions made in deriving this theory. In the theory it is assumed, as in the derivation of Euler's buckling formula for a perfectly elastic material, that the column remains straight until the buckling force F_r

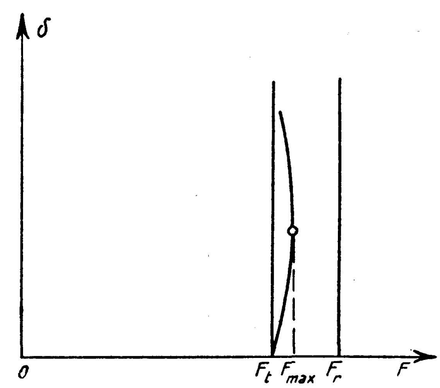

Fig. 7.

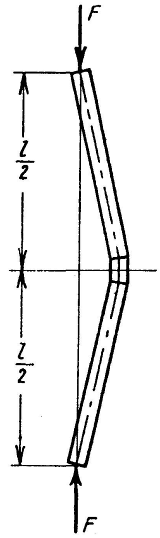

Fig. 8.

[10]) F. R. Shanley, *Journ. Aeronaut. Sci.*, Vol. 13 (1946), p. 678, and Vol. 14 (1947), p. 261.

is reached, and then suddenly bends. SHANLEY proved, however, that the column begins to bend already immediately after the buckling force F_t according to the tangent-modulus theory has been reached and the deflection increases gradually with increasing compressive force. The phenomenon is illustrated in figure 7, where the curves present schematically the deflection δ in the middle of the column as a function of the load F according to different theories. At any value of $F < F_t$ the column remains straight. When F equals F_t and when the linearized differential equation for the deflection curve is used, the deflection δ, according to ENGESSER's tangent-modulus theory, remains indefinite and, in addition to the straight form of equilibrium, there are also other equilibrium positions near that of the straight form under the same load. Thus the load F_t may be defined as the smallest load that can keep the column in slightly bent shape.

The column behaves in the same manner according to the double-modulus theory as according to the tangent-modulus theory. At any value of the compressive force $F < F_r$, the column remains straight. When $F = F_r$, the equilibrium of the column is indifferent and, in addition to the straight equilibrium position, slightly bent equilibrium forms are also possible under the same load. However, according to SHANLEY an exactly determined value of the deflection δ corresponds to each value of the load F when the load F_t is somewhat exceeded. The compressive force has a maximum value F_{max}, which is greater than the tangent-modulus load F_t but smaller than the double-modulus load F_r.

SHANLEY investigated the behaviour of the column in the inelastic range with the aid of a simplified type of column presented in figure 8. This column represents an infinitely stiff, straight column having in the middle an elastic-plastic hinge consisting of two small longitudinal elements. With this extremely simple model SHANLEY succeeded in elucidating qualitatively the principal properties of a column buckling in the inelastic range.

However, for explaining completely the inelastic buckling of a column, the model shown in figure 8 is not sufficient. In fact, the mathematical theory of the SHANLEY phenomenon is very complicated in the general case. When the column bends, the distribution of the stresses in the cross section will be nonlinear and is variable along the column. A so-called reversion range, where the contraction decreases with increasing compressive force, is formed in the middle of the column, on its convex side. When the column bends, the reversion range will gradually extend toward the ends and the inner parts of the column. Owing to the combined influence of all these facts, the effective flexural rigidity of the column depends in an unknown way on the load and the longitudinal coordinate of the column, thus making it difficult to clarify the phenomenon accurately.

On the basis of the stress-strain function (5), LARSSON[11]) has investigated

[11]) H. LARSSON, *Journ. Aeronaut. Sci.*, Vol. 23 (1956), pp. 867—873.

the buckling of a centrally loaded straight column in the inelastic range. The influence of the Shanley effect on the magnitude of the buckling force F_{max} of a column of idealized I-section appears from the diagrams in figure 9. They show how the quantity $(F_{max}-F_t)/(F_R-F_t)$ depends on the ratio F_t/F_0 where $F_0=\sigma_y A$ denotes the compressive force at which yielding of the column begins. By interpolating between the curves belonging to the parameter values $c=0.96$ and $c=0.99$, the dotted-line curve has been drawn whose $c=0.977$. This curve refers to the structural steels St 37 and St 52 given in table 1. The buckling stress diagrams in figure 6 represented by dotted lines have been constructed with the aid of this curve. We see that for an I-section, σ_{max} is in the inelastic range always nearer to the buckling stress σ_t than to the buckling stress σ_r. Larsson has also investigated the influence of the Shanley effect on the buckling stress of a rectangular cross section and found it to be very little higher than that of an I-section.

Summing up, we can say that the tangent-modulus load F_t does not accurately define the buckling load, but it can be regarded as a lower limit of the actual buckling force F_{max} of the column. *The tangent-modulus load F_t should therefore be considered the critical load F_c of a centrally loaded straight column.* For columns of such material that its stress-strain diagram can be presented by means of the stress-strain function (5), the critical buckling stress is obtained from formulas (16) or (17), where we now can write $\sigma_t=\sigma_c$ to indicate that a critical stress is in question.

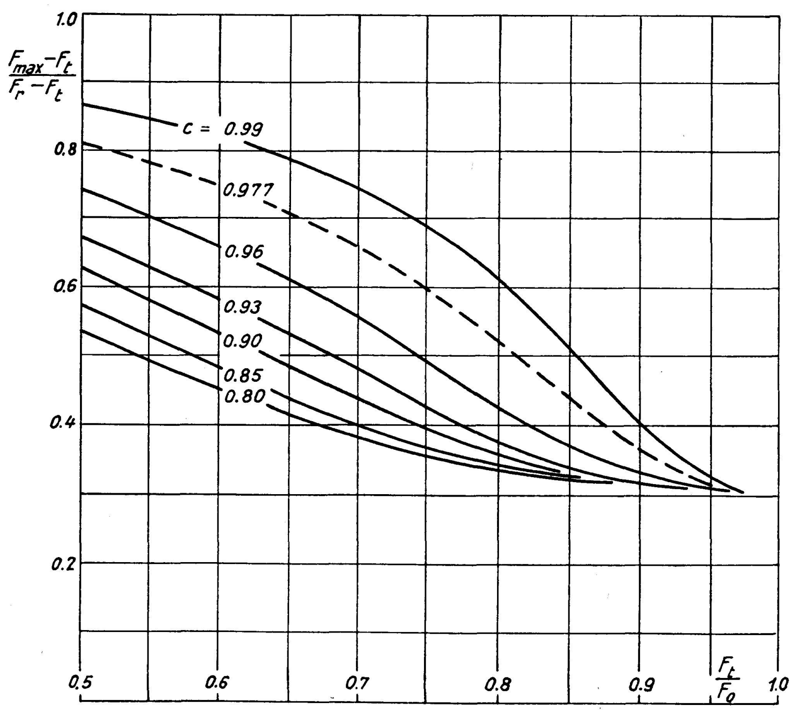

Fig. 9. Influence of the Shanley effect on the buckling force of a column of idealized I-section according to Larsson.

5. Navier-Rankine's Buckling Formula

The validity of EULER's buckling theory was subjected to doubt for a long time because for short columns it led to too high values of the buckling stress. Only when LAMARLE[12]) in 1845 had established the proportional limit as the limit of validity of EULER's formula, it was possible to understand why this formula could not give correct results in all cases. Therefore, before ENGESSER had presented his theory of inelastic buckling, attempts were made to find empirical buckling formulas suitable for the design of columns. One of the best known is NAVIER-RANKINE's buckling formula[13]).

$$\sigma_c = \frac{\sigma_y}{1 + \frac{\sigma_y}{\mu \pi^2 E} \lambda^2}. \tag{24}$$

In deducing this formula the column is usually assumed to be always somewhat eccentrically loaded, because it is not possible to produce a perfectly straight column and to have the force centrally applied with sufficient accuracy. An investigation of the strength of the column under these conditions results precisely in NAVIER-RANKINE's buckling formula (24).

The same formula is simply obtained as a special case of the buckling stress formula (16) by introducing $c=0$ or from formula (17) by a limiting process when $c \to 0$. From this it appears that NAVIER-RANKINE's formula can be applied to the computation of the buckling stress of only those materials whose tangent modulus decreases approximately linearly with increasing compressive stress and will be zero when $\sigma = \sigma_y$. According to table 1, concrete is such a material since its $c=0$.

6. The Required Cross-Sectional Area of the Column

When the column is to be dimensioned with respect to buckling, the following factors are usually known, the buckling force $F_c = \nu F$ where ν is the factor of safety and F the allowable load, the length of the column l, the constants E, σ_y and c characterizing the material used, and the coefficient of restraint μ of the column ends. The required cross-sectional area A of the column and the moment of inertia J are to be determined. Formulas (16) and (17), as all other buckling formulas, are inappropriate for this purpose,

[12]) E. LAMARLE, *Mémoires sur la flexion du bois.* Annales des travaux publics de Belgique, T. IV., p. 1. Brussels 1846.

[13]) The formula is known by several different names, such as NAVIER's, RANKINE's, SCHWARZ's or GORDON's formula. For its history cf. E. H. SALMON, *Columns,* Oxford Technical Publications. London 1921.

because the slenderness ratio of the column depends on the cross-sectional area and its moment of inertia, both of which are unknown.

In order to determine directly the required cross-sectional area of the column, we reduce the fraction in the tangent-modulus formula (15) to higher terms by the cross-sectional area A[14]). Taking into consideration that $\sigma = \sigma_c$ when buckling occurs, we obtain

$$E_t = E \frac{A\,\sigma_y - F_c}{A\,\sigma_y - c\,F_c}, \tag{25}$$

where $A\,\sigma_y$ means the compressive force at which yielding of the column begins and $F_c = A\,\sigma_c$ indicates the buckling force of the column. When the modulus of elasticity E in EULER's formula (9) is replaced, according to ENGESSER, by the tangent modulus (25) we obtain

$$F_c = \frac{\mu\,\pi^2\,E\,J}{l^2}\,\frac{A\,\sigma_y - F_c}{A\,\sigma_y - c\,F_c}. \tag{26}$$

In order to solve this equation for the area A, the moment of inertia J should be expressed as a function of A. Therefore, we introduce

$$A^2 = k\,J, \tag{27}$$

where k is the so-called *section number*[15]). It is a nondimensional quantity whose value depends only on the shape of the cross section. For geometrically isomorphic cross sections k is constant. By introducing the expression of J from (27) into eq. (26) we obtain

$$\frac{k\,F_c\,l^2}{\mu\,\pi^2\,E\,A^2} = \frac{A\,\sigma_y - F_c}{A\,\sigma_y - c\,F_c}.$$

When the fraction at the left-hand side is reduced to higher terms by the factor $F_c\,\sigma_y^2$, the numerator and the denominator of the fraction at the right-hand side are divided by F_c and the cross-sectional area corresponding to the pure compression is denoted by $F_c/\sigma_y = A_0$, it is possible, by using the abbreviations[16])

$$\frac{A}{A_0} = \omega, \qquad \frac{k\,\sigma_y^2\,l^2}{\mu\,\pi^2\,E\,F_c} = q \tag{28}$$

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[14]) The method of determining directly the required cross-sectional area of the column, to be presented in the following, is general and applicable in connection with any stress-strain function. Cf. author's investigation *Die Knickfestigkeit eines zentrisch gedrückten geraden Stabes im elastischen und unelastischen Bereich.* Doctor's thesis. Finland's Institute of Technology, Helsinki 1939, p. 93.

[15]) In newer German literature the section number has been denoted by Z. As Z in the English literature means the section modulus of the cross section, the older German symbol k has been used here in order to avoid confusion.

[16]) The quantity ω is identical with the buckling number of the German buckling specifications. Cf. DIN 4114.

to write the equation in the form $q/\omega^2 = (\omega - 1)/(\omega - c)$ or

$$q = \omega^2 \frac{\omega - 1}{\omega - c}. \tag{29}$$

This equation shows how the relative cross-sectional area $\omega = A/A_0$ depends on the quantity q that contains all the quantities given in connection with the problem of dimensioning. In order to get an idea of this relation, the function (29) has been represented graphically in figure 10 using c as parameter. When $q = 0$, ω equals 1 at any value of c. With increasing q all curves approach asymptotically the square parabola $\omega^2 = q$, which corresponds to the required area of the column according to EULER's buckling formula when the column is buckling in the elastic range. The influence of the parameter c on the area appears clearly from the slope of the curves. The greater c is, the smaller is the required area and the faster approaches the corresponding curve the parabola.

By expressing the quantities in inequality (11) by means of k, l and F, the inequality may be written in the form

$$q \geqq \left(\frac{\sigma_y}{\sigma_p}\right)^2. \tag{30}$$

If this inequality is satisfied, the column buckles elastically and its required moment of inertia is obtained from EULER's formula (9). If the condition (30) is not fulfilled, the column buckles in the inelastic range and its required cross-sectional area can be determined with the aid of the nomograms in figure 9.

If in figure 10 the quantity

$$\sqrt{q} = \frac{\sigma_y l}{\pi} \sqrt{\frac{k}{\mu E F_c}}$$

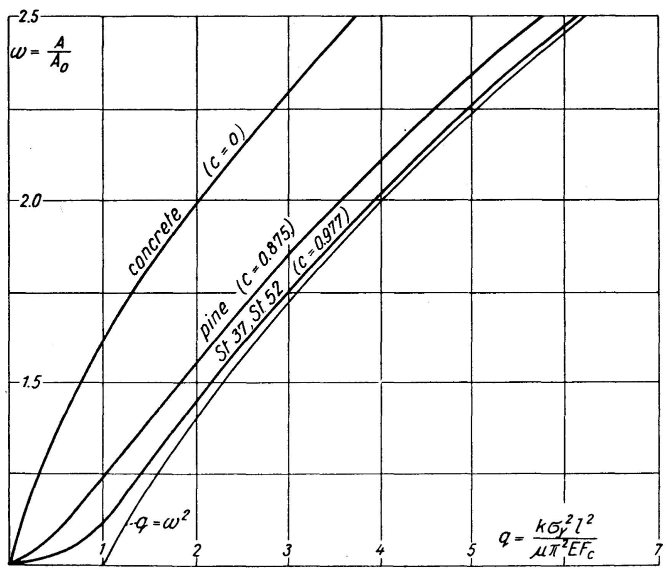

Fig. 10. Required relative cross-sectional area of the column according to formula (29).

had been used as abscissa, it would have been possible to represent EULER's formula (9) by a straight line through the origin. With increasing q the ω-curves would asymptotically approach this line.

By means of the curves in figure 10 it is easy to judge the choice of the cross section and the suitability of the material for the column.

The quantities on which q depends can be regarded as known in most cases concerning the design of columns. The value of the nondimensional quantity q can then be computed and the corresponding ratio A/A_0 determined. If this is large, the column will be heavy, its buckling stress small and the material of the column poorly utilized. In order to improve the column in this respect, the attempt should be made to decrease the value of q. This is best done by choosing a section form in which the material is put as far as possible from the neutral axis. The section number k and, at the same time, q will decrease. If the result desired is not obtained in this manner, the change of the material is still possible. The change of material affects the value A/A_0 in two ways since both q and c change, provided the latter does not happen to be the same for both materials. The conditions at the ends of the column and thereby the value of μ can possibly also be changed.

For the use of the nomograms in figure 10 the section number for different shapes of the cross section must be known. As has already been mentioned earlier, k is constant for geometrically isomorphic cross sections, such as circle and square. The use of the nomograms presented in figure 10 leads for such columns direct to the required cross-sectional area of the column[17]).

These cross-sectional shapes with constant section number occur, however, seldom in practice. In order to save weight and material and for constructional reasons it is attempted to use cross sections whose area is dispersed far from the neutral axis. The isomorphism of two such cross sections presupposes, in addition to the isomorphism of the external form, that also the corresponding ratios between the wall thicknesses and the external dimensions of both cross sections are the same. The sections used in practice belonging to the same category but different in size, do not fulfil, in general, this condition. Consequently, their section number is a variable quantity which is a function precisely of the ratio between the wall thickness and the external dimensions of the cross section. The use of the nomograms in figure 10 for such cross sections does not lead directly, in general, to the required cross-sectional area of the column. It should be checked by means of eq. (27) that the section number of the cross section obtained is actually the same as that used in connection with the diagrams in eq. (27). If that is not the case, the procedure should be repeated by using the new value k obtained from (27). This iteration method converges, however, very rapidly and often already the first determination of the cross section will be the final one.

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[17]) See author's paper in *Schweizerische Bauzeitung,* Vol. 119 (1942), p. 85.

Table 2. Section number k of various cross sections

Section	k	Section	k
1 $b:h=1$	6	8	4
2 $b:h=2:3$	7	9	6
3 $b:h=1:2$	11	10 $J_x = J_y$	1.2
4 $b=2h$	7.5	11	1.8
5 $b=h$	5	12	12
6	4.25	13 $h>b$	$12\frac{h}{b}$
7	7	14	4π

The above table 2 presents the section number k of some shapes of cross sections. In the cross sections 1 to 11, the influence of the wall thickness has been left out of consideration. Their k numbers are therefore rough approximate values only which correspond to average ratios, occurring in practice, between wall thicknesses and the external dimensions of the cross section.

When the required cross-sectional area of columns made of a certain material is to be determined, the previous method can still be considerably simplified in the following way. We introduce into the expression of the variable q in eq. (28) $F_c = \nu F$, where ν is the factor of safety and F the allowable compressive force and write as follows

$$q = \frac{\sigma_y{}^2}{\pi^2 \nu E} \frac{k l^2}{\mu F}.$$

The first factor indicates the quantities characterizing the properties of the material and the second the known quantities of design. By selecting for the latter quantity $k l^2/\mu F$ a sequence of suitable values and by multiplying them by the factor $\sigma_y{}^2/\pi^2 \nu E$, the magnitude of the variable q is obtained. The buckling numbers ω corresponding to these values of q can be taken from the nomogram of the material in question in figure 10. By choosing $\nu = 2$

as the factor of safety, the buckling numbers in table 3 are obtained in this way for structural steel St 37. For the sake of clearness it would also have been possible to add the slenderness ratios λ, but since the knowledge of them is not necessary in connection with this method of dimensioning, they have been left out.

Table 3. Buckling numbers of structural steel St 37 when $\nu = 2$ and allowable compressive stress $\sigma_a = \sigma_y/\nu = 1200$ kg/cm²

$\frac{kl^2}{\mu F}$ cm²/kg	ω	$\frac{kl^2}{\mu F}$ cm²/kg	ω
0	1	7	1,094
1	1,003	8	1,136
2	1,009	9	1,182
3	1,017	10	1,230
4	1,027	11	1,284
5	1,042	11,25	1,295
6	1,064		

When ω has been determined from table 3, the required cross-sectional area of the column is calculated from the formula

$$\frac{A}{\text{cm}^2} = \frac{\omega \cdot \frac{F}{\text{kg}}}{1200}. \tag{31}$$

After determining the cross-sectional area, the section number used in calculating the area must be checked with the aid of formula (27) $k = A^2/J$. If it is not the same as assumed at the beginning of the procedure, the determination of the cross section should be repeated by starting with this new section number obtained from formula (27). Because the value of k changes very slowly with the size of cross section, already the first approximation of the cross section is often sufficiently accurate.

Introducing the expression of q from eq. (28) and $F_c = \nu F$ into inequality (30) and solving this for the quantity $k l^2/\mu F$ we obtain

$$\frac{k l^2}{\mu F} \geqq \frac{\pi^2 \nu E}{\sigma_p{}^2}. \tag{32}$$

If this condition is fulfilled, the column buckles in the elastic range and its necessary moment of inertia is obtained from Euler's formula (9). If the condition (32) is not fulfilled, the column buckles in the inelastic range. Table 3 includes only those values of columns that fulfil the condition $k l^2/\mu F \leqq$ $\leqq 11.25$ cm²/kg for inelastic buckling.

As an example of the application of this method we determine the required cross-sectional area of a column made of steel St 37 when $F = 270\,000$ kg and $l = 325$ cm. The column ends are assumed to be hinged so that $\mu = 1$. As the cross-sectional shape of the column we choose an I-section of wide flange DIN IP, for which $k = 4.25$ according to table 2. On the basis of the given quantities we obtain

$$\frac{k\,l^2}{\mu\,F} = 1.66\frac{\text{cm}^2}{\text{kg}}.$$

Since this is smaller than 11.25 cm²/kg, the column buckles in the inelastic range. According to table 3, the buckling number $\omega = 1.007$ corresponds to the value $k\,l^2/\mu\,F = 1.66$ cm²/kg. From formula (31) we now obtain the required cross-sectional area of the column

$$A = \frac{\omega\,F}{1200}\,\text{cm}^2 = \frac{1.007 \cdot 270\,000}{1200}\,\text{cm}^2 = 226.8\ \text{cm}^2.$$

The cross section next in magnitude is IP 45, whose $A = 232$ cm³, $J = 12\,640$ cm⁴ and $k = A^2/J = 4.26$. Since this section number is almost the same as the assumed $k = 4.25$, the computed cross-sectional area is sufficiently accurate.

Summary

The author presents a new stress-strain function (5), which contains three parameters E, σ_y and c. The first two, the modulus of elasticity E and the yield point stress σ_y, have a quite determined physical meaning. The third parameter c depends primarily on the proportional limit σ_p of the material and some other quantities in the way shown by formula (7). The stress-strain functions corresponding to different values of c have been presented in a dimensionless form in figure 2. The first derivative of the stress-strain function with respect to the strain is very simple, it being a linear fraction of stress.

It follows from the stress-strain function (5) the simple expression (15) for ENGESSER's tangent modulus E_t. For an I-section with an infinitely thin web an equal expression (21) for the double modulus is obtained, where the parameter c is replaced by the value $(1+c)/2$. Formula (16) is valid for the buckling stress according to the tangent-modulus theory. The formula is simplified to NAVIER-RANKINE's buckling formula (24) when $c = 0$. Fig. 5 shows the buckling stress curves of different materials in the dimensionless form. According to the double-modulus theory, formula (22) is obtained for the buckling stress of a column of idealized I-section. This formula is exactly equal to (16) with the only exception that parameter c is replaced by $(1+c)/2$. The influence of the SHANLEY effect on the buckling stress of a column of idealized I-section has been presented according to LARSSON's investigations.

The buckling-stress diagrams of structural steels St 37 and St 52 according to different theories are given in figure 6. The buckling stress according to SHANLEY's theory lies slightly higher than that according to the tangent-modulus theory but is smaller than that according to the double-modulus theory and always nearer to the former. The buckling stress according to the tangent-modulus theory should be considered the critical stress of the column.

The investigation presents finally a method of determining the required cross-sectional area of the column without conventional trial and error. This new method which is quite general and applicable in connection with any stress-strain function is based on the fact that the buckling number ω of the column is given as a function of the quantity $k\,l^2/\mu\,F$. This quantity includes all the factors known in the given problem of design. The buckling numbers ω for structural steel St 37 appear in table 3. A numerical example illustrates the application of the method of dimensioning.

Résumé

L'auteur présente une nouvelle loi de déformation (5) sous la forme d'une équation qui comprend trois paramètres E, σ_y et c. Les deux premiers, le module d'élasticité E et la limite d'élasticité σ_y, ont une signification physique bien déterminée. Le troisième paramètre c dépend en premier lieu de la limite de proportionnalité σ_p et de certains autres facteurs, comme le montre la formule (7). Les fonctions de déformation correspondant aux différentes valeurs de c sont présentées sous une forme non dimensionnelle sur la figure 2. La première dérivée de la fonction de déformation par rapport à la tension est une fraction linéaire de cette dernière.

De la fonction de déformation, peut être déduite une expression simple pour le module de la tangente d'ENGESSER (15). Pour une section I dont l'âme est extrêmement mince, une expression semblable du module double (21) est obtenue, dans laquelle le paramètre c est remplacé par $(1+c)/2$. D'après la théorie du module de la tangente, la formule (16) est valable pour la tension de flambage. La formule est réduite à celle de NAVIER-RANKINE (24) quand on y fait $c=0$. La figure 5 représente les courbes de la tension de flambage des différentes matières, sous une forme non dimensionnelle. Suivant la théorie du module double, on peut obtenir la formule (22) pour la tension de flambage d'une section I. Cette formule est identique à (16), la seule exception étant que le paramètre c est alors remplacé par $(1+c)/2$. L'influence de l'effet de SHANLEY sur la tension de flambage d'une section I est présentée d'après les recherches de LARSSON. La figure 6 représente les courbes de la tension de flambage, d'après les différentes théories, pour les aciers spéciaux St 37 et St 52 utilisés en construction. La tension de flambage d'après la théorie de SHANLEY est un peu plus grande que d'après la théorie du module de la tangente, mais elle

est plus petite que d'après la théorie du module double et toujours plus proche de la précédente. La tension de flambage d'après la théorie du module de la tangente doit être considérée comme la tension critique de la barre.

L'étude expose enfin une méthode pour déterminer la section à prévoir pour la barre, sans recourir à une expérimentation conventionnelle. Cette nouvelle méthode, qui est tout à fait générale, est basée sur le fait que le taux de flambage (10) de la barre est donné sous forme de fonction de la quantité $kl^2/\mu F$. Cette quantité comprend tous les facteurs connus dans le présent problème de construction. Les taux de flambage (10) pour l'acier spécial St 37 sont indiqués dans le tableau 3. Un exemple numérique illustre l'application de la méthode.

Zusammenfassung

Der Verfasser stellt ein neues Formänderungsgesetz (5) mit drei freien Parametern E, σ_y und c dar. Die zwei erstgenannten, der Elastizitätsmodul E und die Stauchgrenze σ_y des Materials, haben eine genaue physikalische Bedeutung. Der Wert des Parameters c hängt in erster Linie von der Proportionalitätsgrenze σ_p des Materials und von einigen anderen Faktoren ab, wie genauer aus der Formel (7) hervorgeht. Die den verschiedenen Werten des Parameters c entsprechenden Druckstauchungsdiagramme sind in Fig. 2 in dimensionsloser Form dargestellt. Die erste Ableitung der Formänderungsfunktion ist eine lineare gebrochene Funktion.

Von der Formänderungsfunktion (5) folgt für den Engesserschen Tangentenmodul der Ausdruck (15). Für einen I-Querschnitt mit unendlich dünnem Steg erhält man einen ähnlichen Ausdruck (21) des Knickmoduls, der sich nur dadurch von Gl. (15) unterscheidet, daß an Stelle des Parameters c in Formel (21) der Wert $(1+c)/2$ steht. Für die Knickspannung nach der ersten Engesserschen Knicktheorie gilt die Formel (16). Wenn $c=0$, vereinfacht sich diese Formel zu der Knickformel (24) von Navier-Rankine. Die Knickspannungsdiagramme von verschiedenen Materialien sind in Fig. 5 in dimensionsloser Form dargestellt. Für die Knickspannung eines Stabes mit idealisiertem I-Querschnitt erhält man nach der Engesser-Kármánschen Knicktheorie die Formel (22), die dieselbe mathematische Form hat wie Gl. (16), nur mit dem Unterschied, daß in Gl. (22) der Parameter c durch den Wert $(1+c)/2$ ersetzt ist. Die Wirkung des Shanley-Effektes auf die Knickspannung des idealisierten I-Querschnittes ist nach den Untersuchungen von Larsson dargestellt. Die Knickspannungsdiagramme der Baustähle St 37 und St 52 sind in Fig. 6 nach den verschiedenen Knicktheorien wiedergegeben. Die Knickspannung nach der Knicktheorie von Shanley liegt etwas höher als die Engessersche Knickspannung, ist aber immer kleiner als die Engesser-Kármánsche Knickspannung. Als kritische Spannung soll die Engessersche Knickspannung betrachtet werden.

Am Ende des Aufsatzes wird ein Verfahren für die direkte Bestimmung des erforderlichen Querschnittes des Stabes dargestellt. Das Verfahren, das ganz allgemeingültig und in Zusammenhang mit jedem Formänderungsgesetz anwendbar ist, gründet sich darauf, daß die Knickzahl ω als Funktion der Größe $k\,l^2/\mu\,F$ dargestellt wird. Diese Größe enthält alle Faktoren, die bei den auf die Knickstäbe bezüglichen Konstruktionsaufgaben als bekannt angesehen werden können. Die Knickzahlen ω für den Baustahl St 37 sind in der Tabelle 3 gegeben. Als Beispiel für die Anwendung dieses Verfahrens wird die erforderliche Querschnittsfläche eines Knickstabes bestimmt.