Deflection theory for arches

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DEFLECTION THEORY FOR ARCHES

THÉORIE DE LA DÉFORMATION DES ARCS

VERFORMUNGSTHEORIE FÜR BOGEN

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1. Introduction.

The development of the theoretical analysis for arches, especially for restrained concrete arches, does not entirely correspond with the progress in reinforced concrete arch construction. The customary method, not considering the new conformation of the arch-axis after loading has proved insufficient for the design of large-span arches with a low rise ratio, which are no more infrequent now. The ordinary analysis, neglecting the influence of arch-axis deformation on the values and the distribution of bending moments, has been found to lead, for these structures, to an inadmissible high rate of inaccuracy, as it will be shown by the numerical example, included in this paper. Therefore it is necessary to take into account the deformed configuration of the structure. — Different from suspension bridges, computation of arches by the exact method of analysis does not yield lower stresses and in consequence does not involve any saving of material. Reasons of safety and not reasons of economy require the refinement of the customary theory for the stress analysis.

The analysis presented in this paper is a Deflection Theory for flat arches, developed for a symmetrical, parabolic axis, applicable to both restrained and two-hinged types. By simply dropping the recognizable terms due to fixed ends, the formulas are reduced to those for the two-hinged arch. Moreover, permanent load is assumed to be uniformly distributed over the span. It is evident that the theory might have been developed for any other form of axis and distribution of permanent load without difficulties. Simplifications have been adopted only to make algebraic expressions easy to handle. — The law for variation of the moment of inertia I within the span has been assumed as $I_c = I \cdot \cos^2 \varphi$, for variation of the section as $F_c = F \cdot \cos \varphi$, I_c and F_c being the moment of inertia and the area for the crown-section, but there seems no difficulty to introduce different laws, e. g. the function $\frac{I_c}{I \cos^2 \varphi} = 1 - \varkappa \frac{x}{l}$ used in a similar form by Strassner for rational arch design by the customary method.

We are dealing with a rigid arch, as shown in fig. 1. The following nomenclature is common to the analysis presented.

E, I_c = Youngs modulus and the moment of inertia for the crown section,
F_c = the area of the crown section,
w = permanent load intensity at any point,
p = live load intensity at any point,

H_w = the horizontal thrust due to dead load, acting at the support,
H_p = the horizontal thrust due to live load, acting at the support,
H = the horizontal thrust due to permanent load, live load and secondary effects, i. e. temperature change and shrinkage effect,
H^0_w = the horizontal thrust of the arch, assumed as three-hinged, due to permanent load,
M^l, M^r = the bending moments at the left or right hand support,
M_0 = the bending moment at any point of the arch considered as a simple beam,
N = the normal component of the resultant thrust acting on a radial section of the arch,
η = the deflection of the arch.

2. Basic Equations.

The deflection curve of a circular bar, r being the radius of curvature, is given by the equation

$$\frac{d^2\eta}{ds^2} + \frac{\eta}{r^2} = \frac{M}{EI} - \frac{N}{rEF} \tag{1}$$

For flat arches the difference between circular and parabolic axis may be neglected.

The resulting moment M at any point of the span with deflection taken into account, is given by the expression

$$M = M_0 + H(y+\eta) + M^l - \frac{x}{l}(M^l - M^r) \tag{2}$$

Substituting Eq. (2) in Eq. (1) and introducing the relation $dx = ds \cdot \cos\varphi$, the equation of the deflection curve of a restrained arch may be written in the form

$$EI \cdot \cos^2\varphi \frac{d^2y}{dx^2} + EI\frac{\eta}{r^2} = M = M_0 + H(y+\eta) + M^l - \frac{x}{l}(M^l - M^r) - \frac{NI}{rF} \tag{1a}$$

For flat arches with parabolic axis we may write with sufficient accuracy

$$\frac{1}{\cos^2\varphi} = 1 + \mathrm{tg}^2\varphi \doteq 1 + \frac{16}{3}\left(\frac{f}{l}\right)^2; \quad \sin\varphi \cdot \cos\varphi = \frac{\mathrm{tg}\,\varphi}{1+\mathrm{tg}^2\varphi} \doteq \mathrm{tg}\,\varphi\left[1 - \frac{16}{3}\left(\frac{f}{l}\right)^2\right] \tag{3}$$

Introducing the abbreviation

$$c^2 = \left\{-\frac{H}{EJ_c} - \left[1 + \frac{16}{3}\left(\frac{f}{l}\right)^2\right]\frac{l}{r^2}\right\} \doteq -\frac{H}{EJ_c}, \text{ where } I_c = I \cdot \cos^2\varphi \text{ and } r = \frac{l^2}{8f} \tag{4}$$

and Eq. (3) in Eq. (1 a) we obtain the differential equation of the deflection curve

$$\frac{d^2\eta}{dx^2} + c^2\eta = \frac{M_0 + M^l}{EJ_c} - \frac{M^l - M^r}{EJ_c}\frac{x}{l} + \frac{H}{EI_c}y - \frac{H}{rEF_c} \tag{5}$$

The last term on the right hand, representing the longitudinal force correction has proved very small, compared with the other terms and may be omitted. The functions $M_0\left(\frac{x}{l}\right)$, depending upon loading condition and y, depending upon the form of the axis, practically may always be presented as polynoms of integer order of $\frac{x}{l}$. Hence, the disturbing term of Eq. (5) may be developed into a polynom for $\frac{x}{l}$, Eq. (5) taking the form

$$\frac{d^2\eta}{dx^2} + c^2\eta = \alpha + \beta\left(\frac{x}{l}\right) + \gamma\left(\frac{x}{l}\right)^2 + \ldots\ldots \nu\left(\frac{x}{l}\right)^n \tag{5a}$$

Introducing moreover the ratio $\frac{I_c}{I\cos^2\varphi}$ as variable and depending upon $\frac{x}{l}$, the coefficient of η becomes a function of $\left(\frac{x}{l}\right)$, $c_1 = c \cdot \sqrt{1 - \varkappa\left(\frac{x}{l}\right)}$. Eq. (5 a) is the basic equation of the Deflection Theory for rigid arches.

Assuming a parabolic axis

$$y = \frac{4f}{l^2}\, x\,(l - x)$$

and uniformely distributed total or partial load, the right hand side of Eq. (5a) reduces to a polynom of second order. The solution of the corresponding differential equation

$$\frac{d^2\eta}{dx^2} + c^2\eta = \alpha + \beta\left(\frac{x}{l}\right) + \gamma\left(\frac{x}{l}\right)^2 \tag{6}$$

is given by

$$\eta = C_1 \cos cx + C_2 \sin cx + \frac{1}{c^2}\left[\left(\alpha - \frac{2\gamma}{c^2 l^2}\right) + \beta\left(\frac{x}{l}\right) + \gamma\left(\frac{x}{l}\right)^2\right] \tag{7}$$

The constants C, α, β, γ, depend upon the loading and edge conditions. For partial load each span-segment, having a constant value of p, yields a differential equation (7) for the corresponding part of the deflection curve.

In order to evaluate the statically indeterminate forces and reactions, the arch has been made statically determinate, considering the horizontal thrust H and the bending moments M^l and M^r as redundant reactions. Their values are yielded by the restrained conditions of the ends of the arch:

1. Horizontal projection of the deformed arch-axis is equal to the span plus an eventual displacement of supports Δ.
2. 3. The ends of the arch are rigidly fixed.

The first condition may be written in the form:

$$\int_0^l (ds + \Delta\, ds)\cos(\varphi + \Delta\, d\varphi) = l + \Delta \tag{8}$$

Considering that with sufficient accuracy

$$\cos \Delta\, d\varphi \doteq 1\,; \qquad \sin \Delta\, d\varphi \doteq \Delta\, d\varphi$$

and neglecting the specific axial strain $\frac{\Delta\, ds}{ds}$ in comparison with the unity, we obtain with

$$\Delta\, d\varphi = \frac{d\eta \cdot \cos\varphi}{ds} \qquad \text{and} \qquad \int_0^l ds\cos\varphi = l$$

from Eq. (8)

$$\int_0^l \Delta\, ds \cdot \cos\varphi - \int_0^l d\eta \cdot \sin\varphi \cdot \cos\varphi = \Delta \tag{9}$$

Introducing Eq. (3) and the relations

$$\Delta ds = \frac{N}{EF} ds - \varepsilon_s \mp \varepsilon_t \cdot t^0; \quad \operatorname{tg} \varphi = \frac{4f}{l}\left(1 - \frac{2x}{l}\right); \quad N = H \cdot \cos\varphi - Q \cdot \sin\varphi$$

where ε_s is the shrinkage strain factor, ε_t the thermal coefficient, we obtain from Eq. (9) the basic equation for the horizontal thrust in the form:

$$\frac{H \cdot l}{EF_c} - \frac{4f}{l}\int_0^l \frac{Q}{EF_c}\left(1-\frac{2x}{l}\right)dx - \frac{4f}{l}\int_0^l \left(1-\frac{2x}{l}\right)d\eta = \left[1+\frac{16}{3}\left(\frac{f}{l}\right)^2\right]\cdot(\varepsilon_s \cdot l \pm \varepsilon_t \cdot t^0 \cdot l + \Delta) \quad (10)$$

Conditions 2. and 3. yield the equation

$$\left(\frac{d\eta}{dx}\right)_{x=0} = 0; \qquad \left(\frac{d\eta}{dx}\right)_{x=l} = 0 \quad (11)$$

resp.

$$C_2 \cdot c + \frac{1}{c^2}\frac{\beta}{l} = 0; \quad C_1 \cdot c \sin cl - C_2 \cdot c \cos cl - \frac{1}{c^2}\frac{1}{l}[\beta + 2\gamma] = 0 \quad (12)$$

By Eqs. (7), (10) and (12) the distribution of stress and strain for a restrained symmetrical flat arch with parabolic axis is completely determined. We shall now consider the different loading conditions.

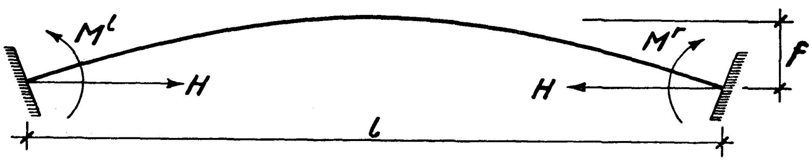

Fig. 1.

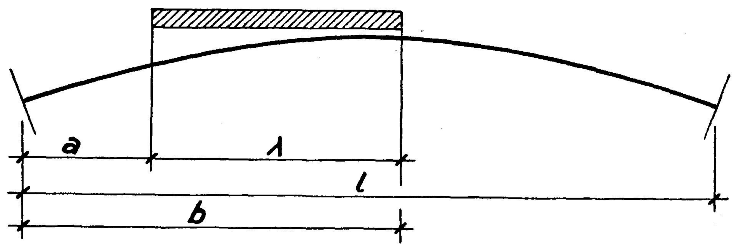

Fig. 2.

3. Application to the Case of Full Loading.

The bending moment M and the shear V are given by the familiar expressions

$$M_0 = \frac{wx}{2}(l-x); \quad V = w\left(\frac{l}{2} - x\right) \quad (13)$$

so that

$$\alpha = \frac{M^l}{EJ_c} \quad \text{and} \quad \beta = -\gamma = \frac{4f}{EJ_c}[H_w - H_w^0] \quad (14)$$

The constants C_1 and C_2 yield from the edge conditions: $\eta = 0$ for $x = 0$ and $x = l$:

$$C_1 = -\frac{1}{c^2}\left[\alpha - \frac{2\gamma}{(cl)^2}\right]; \quad C_2 = C_1 \cdot \operatorname{tg}\frac{cl}{2} = -\frac{1}{c^2}\left[\alpha - \frac{2\gamma}{(cl)^2}\right] \cdot \operatorname{tg}\frac{cl}{2} \quad (15)$$

Introducing these values in Eq. (12) we obtain

$$C_2 = -\frac{1}{c^2}\frac{\beta}{cl}; \quad C_1 = -\frac{1}{c^2}\frac{\beta}{cl}\operatorname{ctg}\frac{cl}{2} \quad (16)$$

The equation of the deflection curve takes the form

$$\eta = \frac{4f}{cl}\left(\frac{H_w^0}{H_w} - 1\right)\left\{\operatorname{ctg}\frac{cl}{2} - \operatorname{ctg}\frac{cl}{2}\cos cx - \sin cx + cl\left[\left(\frac{x}{l}\right) - \left(\frac{x}{l}\right)^2\right]\right\} \tag{17}$$

Its first differential

$$d\eta = -\frac{4f}{cl}\left(\frac{H_w^0}{H_w} - 1\right)\left[\cos cx \cdot d(cx) - \operatorname{ctg}\frac{cl}{2}\sin cx\, d(cx) - d(cx) + \frac{2x}{l}\, d(cx)\right] \tag{18}$$

Substituting $x = \frac{l}{2}$ in Eq. (17) we obtain the deflection of the crown:

$$\eta_0 = \frac{4f}{cl}\left(\frac{H_w^0}{H_w} - 1\right)\left[\frac{cl}{4} - \frac{1}{\sin\frac{cl}{2}}\left(1 - \cos\frac{cl}{2}\right)\right] \tag{19}$$

The second term on the left of Eq. (1a) is small, compared with the first and has been omitted [see Eq. (4)]. This approximation yields the simplified formula for the bending moments at any point of the span

$$M = EI_c\frac{d^2\eta}{dx^2} = \frac{4f}{cl}(H_w - H_w^0)\left[\operatorname{ctg}\frac{cl}{2}\cdot\cos cx + \sin cx - \frac{2}{cl}\right] \tag{20}$$

Substituting $x = 0$, resp. $x = \frac{l}{2}$, we obtain the moment at the support

$$M_{(x=0)} = \frac{4f}{(cl)^2}(H_w - H_w^0)\left[cl\cdot\operatorname{ctg}\frac{cl}{2} - 2\right] \tag{20a}$$

and the moment at the mid-point of the span:

$$M_{\left(x=\frac{l}{2}\right)} = \frac{4f}{(cl)^2}(H_w - H_w^0)\left[\frac{cl}{\sin\frac{cl}{2}} - 2\right] \tag{20b}$$

In order to obtain the working formula for H we have to introduce Eq. (18) in Eq. (10) and to perform the integration with regard to the relations (14). Eq. (10) takes the form:

$$\left(H_w - \frac{2}{3}w\cdot f\right)\frac{I_c}{F_c} + \frac{32f^2}{(cl)^3}(H_w - H_w^0)\left[\frac{2}{cl} - \frac{cl}{6} - \operatorname{ctg}\frac{cl}{2}\right] = \left[1 + \frac{16}{3}\left(\frac{f}{l}\right)^2\right]\left(\varepsilon_s \pm \varepsilon_t\cdot t^0 + \frac{\Delta}{l}\right)EI_c \tag{21}$$

The horizontal thrust due to permanent load, without secondary effects, may be obtained by writing zero for the right hand terms of Eq. (21).

The solution of this transcendent equation has to be determined graphically, considering the two left hand terms as functions $\varphi(H)$ and $-\psi(H)$, evaluating them for different values of H and plotting these functions in a coordinate system, the axis of H being the axis of abscissae. The real value of H yields as the abscissa of the point of intersection of the functions $\varphi(H)$ and $-\psi(H)$. The horizontal thrust due to permanent load in conjunction with secondary effects is obtained as the abscissa corresponding to a difference

$$\left[1 + \frac{16}{3}\left(\frac{f}{l}\right)^2\right](\varepsilon_s \pm \varepsilon_t\, t^0)\, EI_c$$

between the ordinates of $\varphi(H)$ and $-\psi(H)$.

4. Application to the Case of Partial Live Load (Fig. 2).

In the Deflection Theory the bending moment is not simply proportional to the load p that produces it, being affected by the permanent load stresses. Consequently, influence lines can not be used. Stresses producible by combinations of loadings can not be found by adding algebraically the respective stresses producible by the component loadings, but are to be determined directly for the combined-loaded structure. In the subsequent treatment, partial loading of a length λ will be considered. The expressions for M_0, V, α, β, γ, are different for the differently loaded span-segments of the structure. We may write

$$\begin{aligned} 0 \gtreqless x \leqq a \qquad & M_0 = Ax \\ a \gtreqless x \leqq b \qquad & M_0 = Ax - \frac{1}{2} p (x-a)^2 \\ b \gtreqless x \leqq l \qquad & M_0 = (p\lambda - A)(l-x) \end{aligned} \tag{22}$$

where

$$A = p\lambda \left[1 - \frac{1}{l}\left(a + \frac{1}{2}\lambda\right)\right]$$

The influence of shears due to live load upon the value of the horizontal thrust may be neglected with sufficient accuracy. With regard to the Eqs. (5) and (22) we obtain for:

$0 \gtreqless x \leqq a$:

$$\alpha_1 = \frac{1}{EJ_c} M^l; \qquad \beta_1 = \frac{4f}{EJ_c}\left[H_p + \frac{Al - (M^l - M^r)}{4f}\right]; \quad \gamma_1 = -\frac{4f}{EJ_c} H_p$$

$a \gtreqless x \leqq b$:

$$\alpha_2 = \frac{1}{EJ_c}(M^l - \tfrac{1}{2} p a^2); \; \beta_2 = \frac{4f}{EJ_c}\left[H_p + \frac{(A + pa)l - (M^l - M^r)}{4f}\right]; \; \gamma_2 = -\frac{4f}{EJ_c}\left(H_p + \frac{pl^2}{8f}\right) \tag{24}$$

$b \gtreqless x \leqq l$:

$$\alpha_3 = \frac{1}{EJ_c}[M^l + (p\lambda - A)l]; \; \beta_3 = \frac{4f}{EJ_c}\left[H_p + \frac{(A - p\lambda)l - (M^l - M^r)}{4f}\right]; \quad \gamma_3 = -\frac{4f}{EJ_c} H_p$$

Considering the combination of partial live load and full permanent load, we have to add to the expressions (24) the respective values according to (14).

The deflection curve consists of three segments. Edge conditions and conditions of continuity yield the equations for the constants $C_1, \ldots C_6$:

$x = 0$, $\eta = 0$: $\quad C_1 + \frac{1}{c^2}\left(\alpha_1 - 2\frac{\gamma_2}{(cl)^2}\right) = 0$

$x = a$, $\eta_1 = \eta_2$:

$$(C_1 - C_3)\cos ca + (C_2 - C_4)\sin ca - \frac{1}{c^2}\left[(\alpha_2 - \alpha_1) + \frac{a}{l}(\beta_2 - \beta_1) + \frac{(ca)^2 - 2}{(cl)^2}(\gamma_2 - \gamma_1)\right] = 0$$

$x = a$; $\frac{dy_1}{dx} = \frac{dy_2}{dx}$:

$$(C_1 - C_3)\sin ca + (C_2 - C_4)\cos ca + \frac{1}{c^2}\frac{1}{cl}\left[(\beta_2 - \beta_1) + \frac{2a}{l}(\gamma_2 - \gamma_1)\right] = 0$$

$x = b$, $\eta_2 = \eta_3$: $\qquad$ (25)

$$(C_3 - C_5)\cos cb + (C_4 - C_6)\sin cb - \frac{1}{c^2}\left[(\alpha_3 + \alpha_2) + \frac{b}{l}(\beta_3 - \beta_2) + \frac{(cb)^2 - 2}{(cl)^2}(\gamma_3 - \gamma_2)\right] = 0$$

$x=b$; $\frac{dy_2}{dx}=\frac{dy_3}{dx}$:

$$(C_3-C_5)\sin cb+(C_4-C_6)\cos cb+\frac{1}{c^2}\frac{1}{cl}\left[(\beta_3-\beta_2)+\frac{2b}{l}(\gamma_3-\gamma_2)\right]=0$$

$$x=l;\ \eta_3=0:\quad C_5\cos cl+C_6\sin cl+\frac{1}{c^2}\left[\alpha_3+\beta_3+\frac{(cl)^2-2}{(cl)^2}\gamma_3\right]=0$$

resp., introducing Eqs. (24):

$$\begin{aligned} C_1 &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[M^l+\frac{8f}{(cl)_2}H_p\right] \\ (C_1-C_3)\cos ca+(C_2-C_4)\sin ca &= \frac{1}{EJ_c}\frac{pl^4}{(cl)^4} \\ (C_1-C_3)\sin ca-(C_2-C_4)\cos ca &= 0 \\ (C_3-C_5)\cos cb+(C_4-C_6)\sin cb &= -\frac{1}{EJ_c}\frac{pl^4}{(cl)^4} \\ (C_3-C_5)\sin cb-(C_4-C_6)\cos cb &= 0 \\ C_5\cos cl+C_6\cdot\sin cl &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[M^r+\frac{8f}{(cl)^2}H_p\right] \end{aligned} \tag{25a}$$

For partial live load in conjunction with dead load, the first and the last equation (25 a) change into:

$$\begin{aligned} C_1 &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[M^l+\frac{8f}{(cl)^2}(H-H_w^0)\right] \\ C_5\cos cl+C_6\cdot\sin cl &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[M^r+\frac{8f}{(cl)^2}(H-H_w^0)\right] \end{aligned} \tag{25b}$$

The solution of Eqs. (25 a) yields:

$$\begin{aligned} C_1 &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[M^l+\frac{8f}{(cl)^2}H_p\right] \\ C_2 &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[(M^r-M^l\cos cl)\frac{1}{\sin cl}+\frac{8f}{(cl)^2}H_p\operatorname{tg}\frac{cl}{2}+\frac{pl^2}{(cl)^2}\operatorname{ctg}cl(\cos cb-\cos ca)+\right. \\ &\qquad\left.+\frac{pl^2}{(cl)^2}\sin cb-\sin ca)\right] \\ C_3 &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[M^l+\frac{8f}{(cl)^2}H_p+\frac{pl^2}{(cl)^2}\cos ca\right] \\ C_4 &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[(M^r-M^l\cos cl)\frac{1}{\sin cl}+\frac{8f}{(cl)^2}H_p\operatorname{tg}\frac{cl}{2}+\frac{pl^2}{(cl)^2}\operatorname{ctg}cl(\cos cb-\cos ca)+\right. \\ &\qquad\left.+\frac{pl^2}{(cl)^2}\sin cb\right] \\ C_5 &= -\frac{1}{c^2}\cdot\frac{1}{EJ_c}\left[M^l+\frac{8f}{(cl)^2}H_p-\frac{pl^2}{(cl)^2}(\cos cb-\cos ca)\right] \\ C_6 &= -\frac{1}{c^2}\cdot\frac{1}{EJ_c}\left[(M^r-M^l\cos cl)\frac{1}{\sin cl}+\frac{8f}{(cl)^2}H_p\operatorname{tg}\frac{cl}{2}+\frac{pl^2}{(cl)^2}\operatorname{ctg}cl(\cos cb-\cos ca)\right] \end{aligned} \tag{26}$$

For live load and dead load in conjunction we have to substitute in the solutions (26) the value $(H - H_w^0)$ instead of H_p.

For a loading condition as shown in Fig. 3 we may introduce $a = 0$ and $b = \lambda$. The deflection curve consists of only two segments, continuous in $x = \lambda$. To evaluate the 4 constants we have the equations:

$$\begin{aligned} C_1 &= -\frac{1}{c^2} \cdot \frac{1}{EJ_c}\left[M^l + \frac{8f}{(cl)^2} H_p + \frac{pl^2}{(cl)^2}\right] \\ (C_1 - C_3)\cos c\lambda + (C_2 - C_4)\sin c\lambda &= -\frac{1}{EJ_c}\frac{pl^4}{(cl)^4} \\ (C_1 - C_3)\sin c\lambda - (C_2 - C_4)\cos c\lambda &= \emptyset \\ C_3 \cos cl + C_4 \sin cl &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[M^r + \frac{8f}{(cl)^2} H_p\right] \end{aligned} \qquad (27)$$

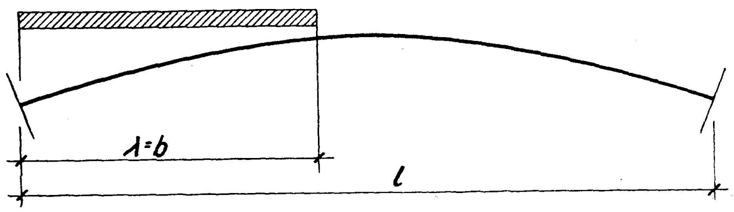

Fig. 3.

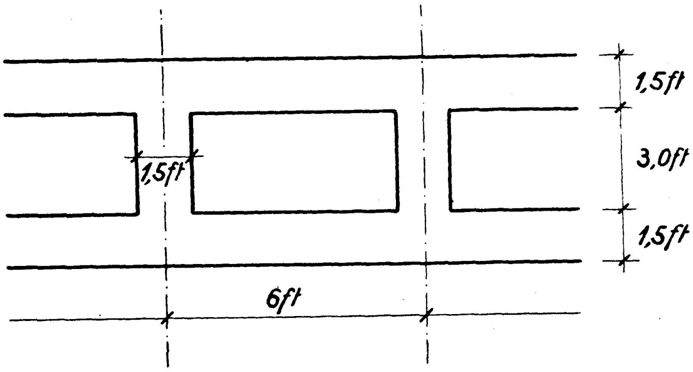

Fig. 4.

Their solution yields:

$$\begin{aligned} C_1 &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[M^l + \frac{8f}{(cl)^2} H_p + \frac{pl^2}{(cl)^2}\right] \\ C_2 &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[(M^r - M^l\cos cl)\frac{1}{\sin cl} + \frac{8f}{(cl)^2} H_p \operatorname{tg}\frac{cl}{2} + \frac{pl^2}{(cl)^2}\operatorname{ctg} cl(\cos c\lambda - 1) + \frac{pl^2}{(cl)^2}\sin c\lambda\right] \\ C_3 &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[M^l + \frac{8f}{(cl)^2} H_p - \frac{pl^2}{(cl)^2}(\cos c\lambda - 1)\right] \\ C_4 &= -\frac{1}{c^2}\frac{1}{EJ_c}\left[(M^r - M^l\cos cl)\frac{1}{\sin cl} + \frac{8f}{(cl)^2} H_p \operatorname{tg}\frac{cl}{2} + \frac{pl^2}{(cl)^2}\operatorname{ctg} cl(\cos c\lambda - 1)\right] \end{aligned} \qquad (28)$$

The bending moments at the abutements may now be determined by Eqs. (12). For partial load as shown in Fig. 2 these equations take the form:

$$\begin{aligned} C_2 &= -\frac{1}{c^2}\frac{1}{EJ_c} H_p + \frac{1}{cl}(Al - M^l + M^r)\Big] \\ C_5 \sin cl - C_6 \cdot \cos cl &= -\frac{1}{c^2}\frac{1}{EJ_c}\left\{\frac{4f}{cl} H_p - \frac{1}{cl}\left[(A - p\lambda)\, l - M^l + M^r\right]\right\} \end{aligned} \qquad (29)$$

With regard to Eq. (23) and (24) we obtain from (29):

$$M^l(cl\cos cl-\sin cl)+M^r(\sin cl-cl)=$$
$$=\frac{pl^2}{cl}[\cos c(l-b)-\cos c(l-a)]-\frac{1}{2}pl\lambda\left[2\left(1-\frac{a}{l}\right)-\frac{\lambda}{l}\right]\sin cl-\frac{4f}{cl}H_p(cl\sin cl+2\cos cl-2)$$
$$M^l(\sin cl-cl)+M^r(cl\cos cl-\sin cl)= \qquad (30)$$
$$=\frac{pl^2}{cl}[\cos ca-\cos cb]-\frac{1}{2}pl\lambda\left[2\frac{a}{l}+\frac{\lambda}{l}\right]\sin cl-\frac{4f}{cl}H_p(cl\sin cl+2\cos cl-2)$$

With $a=0$ and $b=\lambda$ we obtain the respective equations for loading-conditions as shown in Fig. 2:

$$M^l(cl\cos cl-\sin cl)+M^r(\sin cl-cl)=$$
$$=\frac{pl^2}{cl}[\cos c(l-\lambda)-\cos cl]-\frac{1}{2}pl\lambda\left[2-\frac{\lambda}{l}\right]\sin cl-\frac{4f}{cl}H_p(cl\sin cl+2\cos cl-2) \qquad (30\text{a})$$
$$M^l(\sin cl-cl)+M^r(cl\cos cl-\sin cl)=$$
$$=\frac{pl^2}{cl}(1-\cos c\lambda)-\frac{1}{2}p\lambda^2\sin cl-\frac{4f}{cl}H_p(cl\sin cl+2\cos cl-2)$$

In order to evaluate the horizontal thrust, we have to introduce the relations

$V_P=A$ for the segment $0\gtreqless x\leqq a$, $V_P=A-p\cdot(x-a)$ for the segment $a\gtreqless x\leqq b$ and $V_P=A-p\cdot\lambda$ for the segment $b\gtreqless x\leqq l$

into Eq. (10). The basic equation for H takes the form:

$$\frac{H_p\cdot l}{EF_c}-\frac{4f}{EF_c}p\frac{\lambda}{l}\left[b\left(1-\frac{a}{l}\right)-\frac{1}{2}\lambda\left(1+\frac{2}{3}\frac{\lambda}{l}\right)\right]-\frac{4f}{l}\left\{-C_1\int_0^a\sin cx\,d(cx)+C_2\int_0^a\cos cx\,d(cx)\right.$$
$$+\frac{\beta_1}{c^2l}\int_0^a dx+\frac{2\gamma_1}{(cl)^2}\int_0^a x\,dx+\frac{2}{cl}C_1\int_0^a cx\sin cx\,d(cx)-\frac{2}{cl}C_2\int_0^a cx\cdot\cos cx\,d(cx)-\frac{2\beta_1}{(cl)^2}\int_0^a x\,dx$$
$$-\frac{4\gamma_2}{c^2l^3}\int_0^a x^2\,dx-C_3\int_a^b\sin cx\,d(cx)+C_4\int_a^b\cos cx\,d(cx)+\frac{\beta_2}{c^2l}\int_a^b dx+\frac{2\gamma_2}{(cl)^2}\int_a^b x\,dx \qquad (32)$$
$$+\frac{2}{cl}C_3\int_a^b cx\sin cx\,d(cx)-\frac{2}{cl}C_4\int_a^b cx\cos cx\,d(cx)-\frac{2\beta_2}{(cl)^2}\int_a^b x\,dx-\frac{4\gamma_2}{c^2l^3}\int_a^b x^2\,dx$$
$$-C_5\int_b^l\sin cx\,d(cx)+C_6\int_b^l\cos cx\,d(cx)+\frac{\beta_3}{c^2l}\int_b^l dx+\frac{2\gamma_3}{(cl)^2}\int_b^l x\,dx+\frac{2}{cl}C_5\int_b^l cx\sin cx\,d(cx)$$
$$\left.-\frac{2}{cl}C_6\int_b^l cx\cos cx\,d(cx)-\frac{2\beta_3}{(cl)^2}\int_b^l x\,dx-\frac{4\gamma_3}{c^2l^3}\int_b^l x^2\,dx\right\}=0$$

By performing the integrations the basic equation reduces finally to:

$$\left\{H_p-\frac{4pf}{l}\frac{\lambda}{l}\left[b\left(1-\frac{a}{l}\right)-\frac{1}{2}\lambda\left(1+\frac{2}{3}\frac{\lambda}{l}\right)\right]\right\}\frac{J_c}{F_c} \qquad (33)$$
$$-\frac{4f}{(cl)^2}\left\{\frac{pl^2}{6}\left[\left(\frac{b}{l}\right)^2\left(3-2\frac{b}{l}\right)-\left(\frac{a}{l}\right)^2\left(3-2\frac{a}{l}\right)\right]+M^l+M^r+\frac{4}{3}H_p\cdot f\right\}=0$$

For the combination of partial live load, dead load, shrinkage effect, temperature change and displacement of supports the H-equation may be written in the form

$$\left\{\left(H-\frac{2}{3}wf\right)-\frac{4pf}{l}\frac{\lambda}{l}\left[b\left(1-\frac{a}{l}\right)-\frac{1}{2}\lambda\left(1+\frac{2}{3}\frac{\lambda}{l}\right)\right]\right\}\frac{J_c}{F_c}$$
$$-\frac{4f}{(cl)^2}\left\{\frac{pl^2}{6}\left[\left(\frac{b}{l}\right)^2\left(3-2\frac{b}{l}\right)-\left(\frac{a}{l}\right)^2\left(3-2\frac{a}{l}\right)\right]+M^l+M^r+\frac{4}{3}(H-H_w^0)f\right\} \quad (33a)$$
$$=EJ_c\left[1+\frac{16}{3}\left(\frac{f}{l}\right)^2\right]\left(\varepsilon_s\pm\varepsilon_t\cdot t^0+\frac{\Delta}{l}\right)$$

For live load according to Fig. 3, this equation reduces, with $a = 0$ and $b = \lambda$ to

$$\left[\left(H-\frac{2}{3}wf\right)-\frac{4f}{l}p\lambda\frac{\lambda}{l}\left(\frac{1}{2}-\frac{1}{3}\frac{\lambda}{l}\right)\right]\frac{J_c}{F_c} \quad (33b)$$
$$-\frac{4f}{(cl)^2}\left[\frac{p\lambda^2}{6}\left(3-2\frac{\lambda}{l}\right)+M^l+M^r+\frac{4}{3}f(H-H_w^0)\right]=EJ_c\left[1+\frac{16}{3}\left(\frac{f}{l}\right)^2\right]\left(\varepsilon_s\pm\varepsilon_t\cdot t^0+\frac{\Delta}{l}\right)$$

Omitting the influence of axial forces and substituting zero for the moments at the supports, we obtain from (33) the familiar expression for the hori-

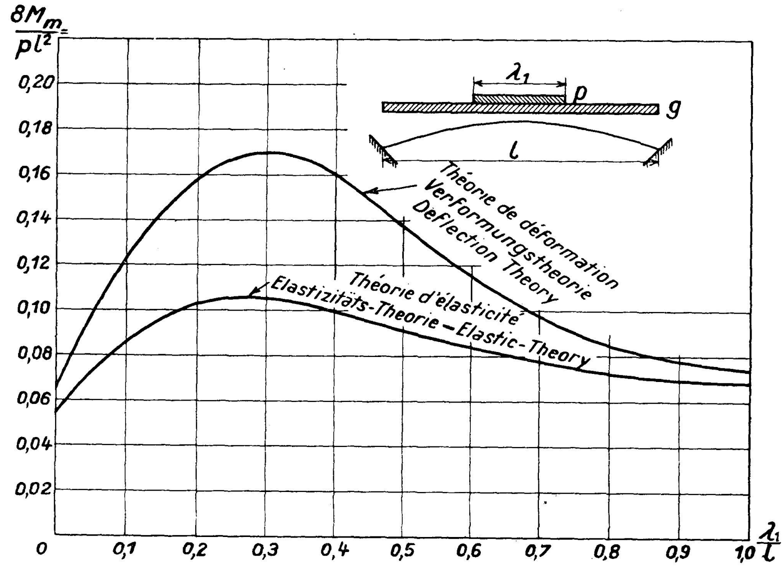

Fig. 5.

Moments M_m à la clef dûs à la charge permanente et divers cas de charge roulante.
Momente M_m im Scheitel für ständige Last und für verschiedene Fälle der Verkehrslast.
Moments M_m at the crown for dead load and various amounts of live load.

zontal thrust of a two-hinged arch, computed by methods of customary statical analysis.

Equations (30) and (33) are sufficient for the complete analysis of the stress-distribution of an arch by the Deflection Theory. The solution of this three equations however can not be found directly and has to be performed graphically, the moments at the abutments to be determined first for series of values H and the diagrams $M^l(H)$ and $M^r(H)$ plotted, whereupon functions $\varphi(H)$ and $-\psi(H)$ may be computed in order to find their point of intersection, as it was stated above.

Resultant bending moments at any point of the span are given by the relation

$$M = EJ_c \frac{d^2 \eta}{dx^2} = H\left(C_n \cos cx + C_{n+1} \cdot \sin cx + \frac{2 EJ_c}{(cl)^2} \frac{\gamma_n}{H}\right) \tag{34}$$

the index n being the number of the considered span-segment.

5. Practical Application to a Concrete Arch of Large Span.

In order to consider the practical applicability of the Deflection Theory and to establish data for comparison, the formulas developed in this paper have been applied to the analysis of a concrete arch of large span. Stresses were first computed by the ordinary Elastic Theory. This preliminary analysis yields the approximate loading conditions to be used as a guide for assuming load length in the more exact analysis. The stresses were then evaluated by the Deflection Theory, using the formulas and procedure developed in this paper.

General Data. The following are the dimensional constants:

$$1 = 450 \text{ ft}, \ f = 50 \text{ ft}, \ r = \frac{l^2}{8f} = 506{,}3 \text{ ft}, \ l/f = 9, \ 1 + \frac{16}{3}\left(\frac{f}{l}\right)^2 = 1{,}07.$$

Dimensions of crown-section see Fig. 4 (width of the considered section B = 6 ft).

$J_c = 100 \text{ ft}^4$, $F_c = 22{,}5 \text{ ft}^2$, $E = 5 \cdot 10^8 \text{ lb/ft}^2$, $E \cdot J_c = 50 \cdot 10^9 \text{ lb. ft}^2$, $E \cdot F_c = 11{,}3 \cdot 10 \text{ lb}$

$$\nu = \frac{1}{1 + \frac{45}{4} \frac{J_c}{F_c \cdot f}} = 0{,}9804.$$

The following are the loading constants (values per 6ft width):

Permanent load $w = 4950$ lb/ft, Live load $p = 600$ lb/ft.

$$H_w^0 = -\frac{w l^2}{8f} = -2{,}506 \cdot 10^6 \text{ lbs}.$$

Design I (Elastic Theory). For the influence of a single load, acting at the distance x from the left support, computed by methods of customary statical analysis, we obtain:

$$\text{reaction at the left support} \quad A = \frac{P}{l^3}(l - x)^2 (l + 2x)$$

$$\text{horizontal thrust} \quad H = -\frac{15}{4} \frac{P}{f l^3} \nu (l - x)^2 x^2 \tag{35}$$

$$\text{moment at the left support} \quad M^l = -\frac{P}{l^3} x (b - x)^2 \left(l - \frac{5}{2} \nu x\right)$$

The horizontal thrust due to uniformly distributed dead load w is

$$H_w = -\frac{1}{8} \frac{w l^2}{f} \nu = -2{,}456 \cdot 10^6 \text{ lbs},$$

the corresponding moment at the support:

$$M^l = -\frac{1}{12} w l^2 (1 - \nu) = -1{,}63 \cdot 10^6 \text{ lb. ft},$$

the moment at the crown:

$$M_m = +\frac{1}{24} w l^2 (1-\nu) = +0{,}815 \cdot 10^6 \text{ lb.ft},$$

the horizontal thrust due to temperature variation of $\pm 40^\circ$ F, considered to act through the neutral point of the structure

$$H_t = \mp \frac{EJ_c}{f^2} \frac{45}{4} \varepsilon_t t^0 \cdot \nu = \mp 0{,}053 \cdot 10^6 \text{ lbs},$$

the thermal coefficient of concrete ε_t having generally been assumed as $6 \cdot 10^{-6}$ per degree F. The effect of shrinkage may be considered as equi-

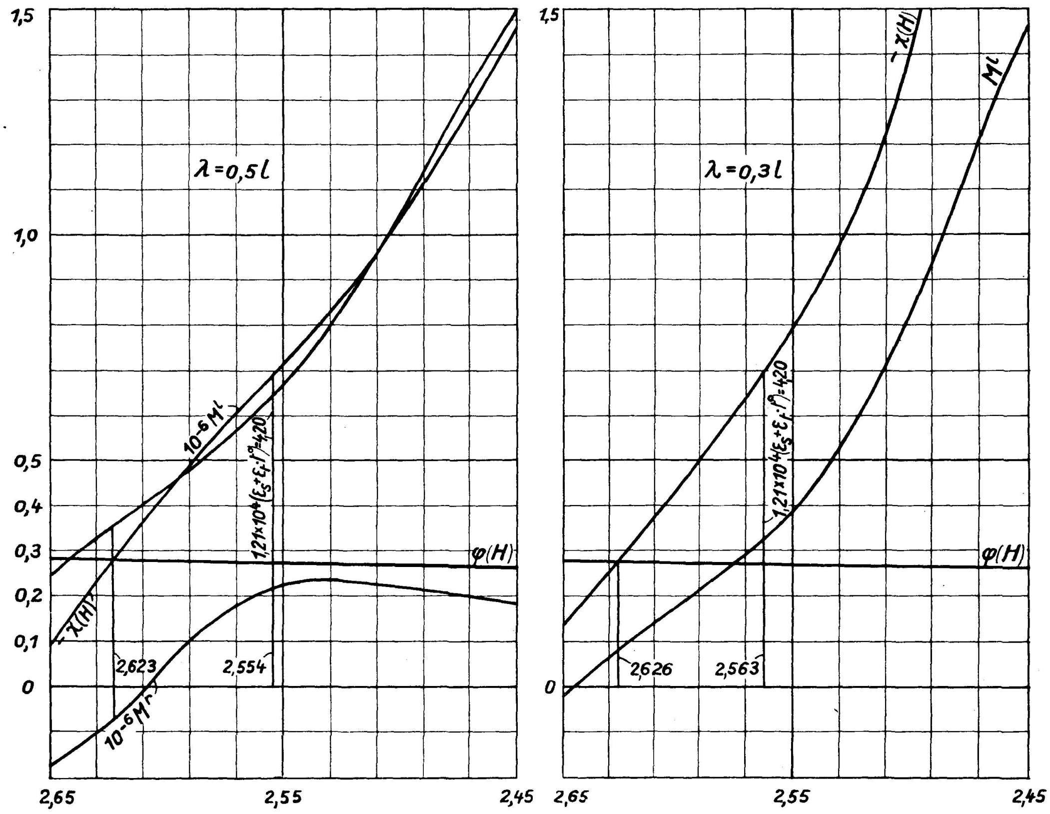

Fig. 6a. Fig. 6b.
Solution graphique des équations 30a et 33b.
Graphische Auflösung der Gleichungen 30a und 33b.
Graphic solution of Eq. 30a and 33b.

valent to a drop in temperature of 18° F. The horizontal thrust due to shrinkage yields:

$$H_s = +\frac{45}{4} \frac{EJ_c}{f^2} \varepsilon_s \cdot \nu = +0{,}024 \cdot 10^6 \text{ lbs}.$$

Introducing $P = p \cdot dx$ in Eq. (35) and performing the integration from $x = 0$ to $x = \lambda$, we obtain the values H, A, M^l, for partial, uniformly distributed load of the length λ (see Fig. 3):

$$A = p\lambda \left[1 - \left(\frac{\lambda}{l}\right)^2 + \frac{1}{2}\left(\frac{\lambda}{l}\right)^3\right]$$

$$H = -\frac{pl^2}{8f} \nu \left(\frac{\lambda}{l}\right)^3 \left[10 - 15\left(\frac{\lambda}{l}\right) + 6\left(\frac{\lambda}{l}\right)^2\right] \qquad (36)$$

$$M^l = -\frac{pl^2}{2}\left(\frac{\lambda}{l}\right)^2 \left[1 - 2{,}97\left(\frac{\lambda}{l}\right) + 2{,}95\left(\frac{\lambda}{l}\right)^2 - 0{,}98\left(\frac{\lambda}{l}\right)^3\right]$$

For maximum negative moment at the left support a length of span extending from the support to the right must be covered with live load. The partial loading always stops at the zero-section of the influence line. This yields, with regard to (34)

$$\lambda = \frac{5\,l}{2\,\nu} = 0{,}45\,l$$

Introducing this load-length into (36), we obtain the maximum negative bending moment $M^l = -0{,}01737\ p \cdot l^2 = -\frac{1}{57{,}6}\,p \cdot l^2 = -2{,}11 \,.\, 10^6$ lb. ft, and the corresponding horizontal thrust $H = -0{,}398\,\frac{p\,l^2}{8\,f} = -0{,}30 \cdot 10^6$ lbs.

Maximum positive moment at the centre of the span is produced when the central section of the span is symmetrically loaded with uniform live load for a length λ_1. The value of this moment yields from $M = M_0 + H \cdot f + M^l$, by introducing the equations (36) evaluated for the load p of a length $\frac{l+\lambda}{2}$ and the load $-p$ of a length $\frac{l-\lambda}{2}$:

$$M_m = \frac{1}{8}\,pl^2\left[0{,}4\,\frac{\lambda_1}{l} - \left(\frac{\lambda_1}{l}\right)^2 + 0{,}738\left(\frac{\lambda_1}{l}\right)^3 - 0{,}138\left(\frac{\lambda_1}{l}\right)^5\right] \qquad (37)$$

The corresponding horizontal thrust

$$H = -\frac{pl^2}{8f}\,\nu\left(\frac{\lambda_1}{l}\right)\left[\frac{15}{8} - \frac{10}{8}\left(\frac{\lambda_1}{l}\right)^2 + \frac{3}{8}\left(\frac{\lambda_1}{l}\right)^4\right] \qquad (38)$$

The function (37) has a maximum for $\lambda_1 = 0{,}28\ l$. Substituting this value in (37) we obtain the maximum positive moment at the centre of span $M_{max} = \frac{1}{8}\,pl^2 \,.\, 0{,}05 = 0{,}00625\ p \cdot l^2 = \frac{1}{160}\,p \cdot l^2 = 0{,}76 \,.\, 10^6$ lb.ft. The ratio $\varkappa = \frac{8\,M_m}{pl^2}$ as a function of $\left(\frac{\lambda_1}{l}\right)$ has been plotted in Fig. 5.

By combination of the most unfavourable loading-conditions we obtain the maximum values of the bending moments at the supports and at the crown and the corresponding values of the horizontal thrust and reactions at the supports. Both, at the support and at the crown, maximum moments are produced by permanent and live load at lowest temperature and with consideration of shrinkage effect. We find:

at the support:

$$M^l_{min} = -1{,}67 \cdot 10^6 - 0{,}67 \cdot 50 \cdot 5{,}3 \cdot 10^4 - 0{,}67 \cdot 50 \cdot 2{,}4 \cdot 10^4 - 2{,}11 \cdot 10^6 = -6{,}35 \cdot 10^6\,\text{lbft.}$$
$$H = -2{,}45 \cdot 10^6 - 5{,}3 \cdot 10^4 - 2{,}4 \cdot 10^4 - 0{,}12 \cdot 10^6 = -2{,}49 \cdot 10^6\ \text{lbs.}$$
$$A = 4950 \cdot 225 + 600 \cdot 0{,}45 \cdot 450 \cdot 0{,}776 = 1{,}20 \cdot 10^6\ \text{lbs.}$$

at the crown:

$$M_{max} = 0{,}83 \cdot 10^6 + 0{,}33 \cdot 50 \cdot 5{,}3 \cdot 10^4 + 0{,}33 \cdot 50 \cdot 2{,}4 \cdot 10^4 + 0{,}76 \cdot 10^6 = 2{,}88 \cdot 10^6\ \text{lbft.}$$
$$H = -2{,}45 \cdot 10^6 + 5{,}3 \cdot 10^4 + 2{,}4 \cdot 10^4 - 0{,}13 \cdot 10^6 = -2{,}50 \cdot 10^6\ \text{lbs.}$$

D e s i g n II. By referring to Eq. (21) we obtain the basic equation for the horizontal thrust due to dead load and secondary effects. For different values of H from — 2300 kips to —2800 kips quantities cl and their trigonometric functions have been evaluated and are tabulated below.

H lbs	$(cl)^2$	cl	$\sin cl$	$\cos cl$	$\sin \frac{cl}{2}$	$\cos \frac{cl}{2}$	$\mathrm{tg} \frac{cl}{2}$	$\mathrm{ctg} \frac{cl}{2}$	$H - H_w^0$ lbs
$-2{,}80 \cdot 10^6$	11,34	3,37	−0,2264	−0,9740	0,9930	−0,1140	− 8,72	−0,1147	$-0{,}294 \cdot 10^6$
−2,75	11,14	3,34	−0,1971	−0,9804	0,9951	−0,0990	− 10,05	−0,0995	−0,244
−2,70	10,93	3,31	−0,1676	−0,9858	0,9965	−0,0841	− 11,85	−0,0844	−0,194
−2,65	10,74	3,28	−0,1380	−0,9904	0,9976	−0,0691	− 14,43	−0,0693	−0,144
−2,60	10,53	3,25	−0,1082	−0,9947	0,9985	−0,0542	− 18,43	−0,0543	−0,094
−2,55	10,33	3,21	−0,0684	−0,9977	0,9994	−0,0342	− 29,22	−0,0342	−0,044
−2,50	10,13	3,18	−0,0384	−0,9993	0,9998	−0,0192	− 52,09	−0,0192	+0,006
−2,45	9,93	3,15	−0,0084	−1,0000	1,0000	−0,0042	−238,10	−0,0042	+0,056
−2,40	9,73	3,12	+0,0218	−0,9998	0,9999	+0,0108	+ 92,62	+0,0108	+0,106
−2,35	9,53	3,09	+0,0519	−0,9987	0,9997	+0,0258	+ 38,76	+0,0258	+0,156
−2,30	9,33	3,05	+0,0917	−0,9958	0,9990	+0,0458	+ 21,81	+0,0458	+0,206

The graphical solution of the H-equation, performed by aid of the tabulated values, yields:

for dead load: $H_w = -2{,}465 \cdot 10^6$ lbs, M^l (by Eq. 20a) $= -1{,}67 \cdot 10^6$ lb. ft, and
M_m (by Eq. 20b) $= +0{,}97 \cdot 10^6$ lb. ft.

for dead load, drop of temperature (40° F) and shrinkage effect (18° F):
$H = -2{,}402 \cdot 10^6$ lbs, M^l (by Eq. 20a) $= -4{,}21 \cdot 10^6$ lb. ft and
M_m (by Eq. 20b) $= +2{,}39 \cdot 10^6$ lb. ft.

To obtain the maximum negative moment at the left support it is necessary to load fully the span with dead load and a segment of it with live load at lowest temperature. The load length for which the moment at the support is maximum, must be determined by trial for certain length λ of a continous advancing live load. Moments $M^{l,r}$ are calculated for different values of H and $\frac{\lambda}{l}$ by Eq. (30 a). The graphical solution of the H-equation

$$\left[\frac{H}{10^6} - 0{,}165 - 0{,}04\left(\frac{\lambda}{l}\right)^2\left(1{,}5 - \frac{\lambda}{l}\right)\right] - \frac{45}{(cl)^2}\left[40{,}4\left(\frac{\lambda}{l}\right)^2\left(1{,}5 - \frac{\lambda}{l}\right) + \frac{200}{3}\frac{H - H_w^0}{10^6} + \frac{1}{10^6}(M^l + M^r)\right]$$

$$= \varphi(H) - \psi(H) = 1{,}21 \cdot 10^4\left[\varepsilon_s \pm \varepsilon_t \cdot t^0 + \frac{\Delta}{l}\right]$$

performed by aid of the computed values $M^{l,r}$ yields the real horizontal thrust and therewith the real moments at the supports. The maximum negative moment at the left support occurs when $\lambda = 0{,}5\, l$. The horizontal thrust produced by dead and live load is found to be $H = -2{,}623 \cdot 10^6$ lbs., the corresponding moment being $M^l = -3{,}63 \cdot 10^6$ lb.ft., these values due to dead and live load in conjunction with drop of temperature and shrinkage effect are $H = -2{,}554 \cdot 10^6$ lbs. and $M^l = -6{,}43 \cdot 10^6$ lb.ft. The graphical solution of Eq. (30 a) and (33 b) for $\frac{\lambda}{l} = 0{,}5$ has been plotted in Fig. 6 a. — Comparison of maximum negative moments at the supports computed by the Deflection Theory with the corresponding values obtained by the customary methods does not show any essential difference, whereas these differences increase towards the centre of the span to a considerable rate (see Fig. 7).

Maximum positive moment at the crown is produced by symmetrically loading the central section of the span with live load for a length λ_1 in con-

Comparaison des moments calculés d'après la théorie de l'élasticité et d'après la théorie de la déformation (Moments en 10^6 lb. ft.).

Vergleich der Momente, berechnet nach der Elastizitäts- und nach der Verformungstheorie (Momente in 10^6 lb. ft.).

Comparison of moments by elastic and deflection theories (Moments in 10^6 lb. ft.).

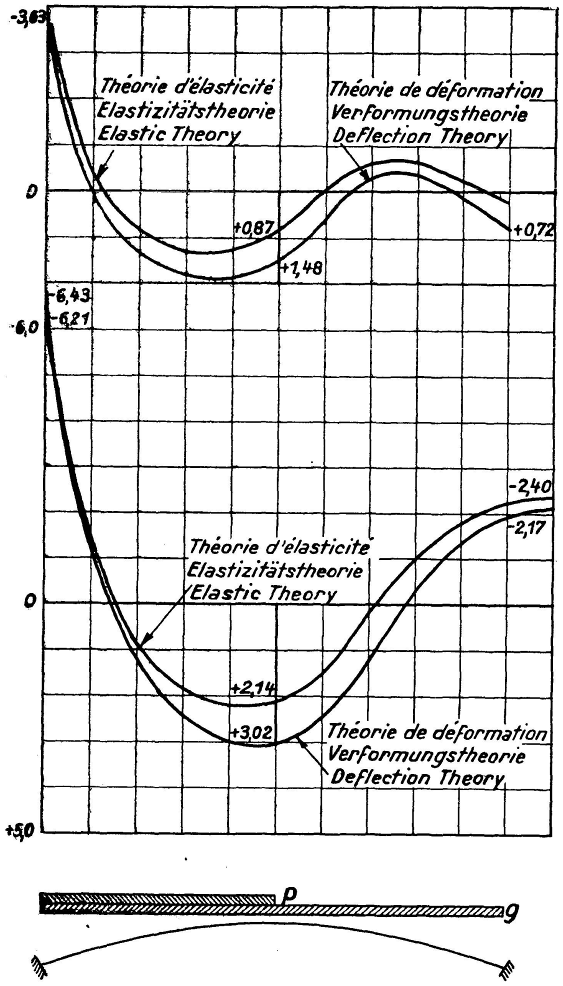

Fig. 7a und 7b.

Charge permanente, charge roulante $\lambda = 0,5$ l.

Ständige Last, Verkehrslast $\lambda = 0,5$ l.

Dead load, live load $\lambda = 0,5$ l.

Fig. 7a. Sans effets secondaires.
Ohne Nebenwirkungen.
Without secondary effects.

Fig. 7b. A basse température et avec retrait.
Für tiefste Temperatur und Schwinden.
At lowest temperature and with shrinkage.

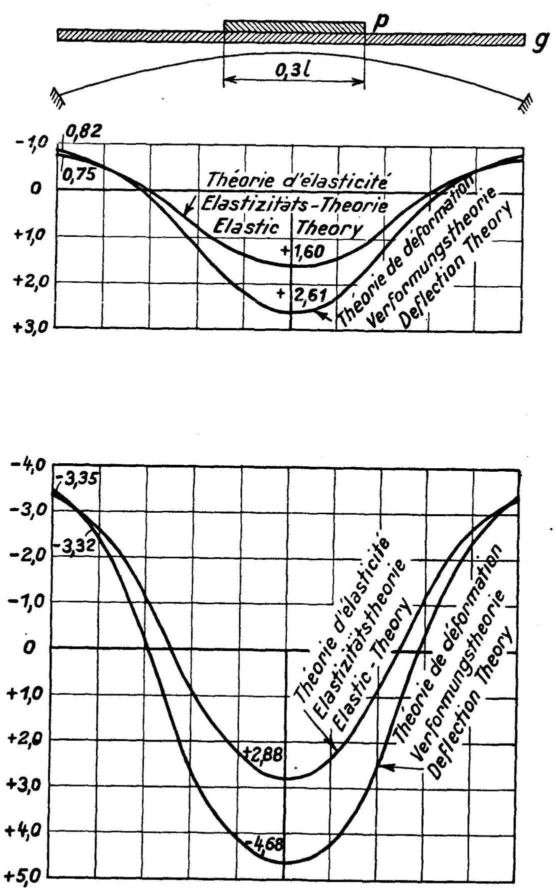

Fig. 8a und 8b.

Fig. 8a. Sans effets secondaires.
Ohne Nebenwirkungen.
Without secondary effects.

Fig. 8b. A basse température, avec retrait, charge permanente et charge roulante $\lambda_1 = 0,3$ l.

Für tiefste Temperatur und Schwinden, ständige Last, Verkehrslast $\lambda_1 = 0,3$ l.

At lowest temperature and with shrinkage, dead load, live load $\lambda_1 = 0,3$ l.

junction with dead load, drop of temperature and shrinkage effect. In order to find the load length producing maximum positive moment it is necessary to assume different trial values λ_1 and to calculate the corresponding values M^l by use of the formulas (30):

$$M^l\,cl(\cos cl - 1) = \frac{pl^2}{cl}\left[\cos\frac{c(l-\lambda_1)}{2} - \cos\frac{c(l+\lambda_1)}{2}\right] - \frac{1}{2}pl\lambda_1 \sin cl - \frac{4f}{cl}(H - H_w^0)(cl\sin cl + 2\cos cl - 2),$$

(33 a):

$$\left\{\frac{H}{10^6} - 0{,}165 - 0{,}03\left(\frac{\lambda_1}{l}\right)\left[\left(1-\frac{\lambda_1}{l}\right)^2 - 2\left(\frac{\lambda_1}{l}\right)\left(1+\frac{2}{3}\frac{\lambda_1}{l}\right)\right]\right\} - \frac{45}{(cl)^2}\left\{\frac{pl^2}{24}\left(1+\frac{\lambda_1}{l}\right)^2\left[\left(2-\frac{\lambda_1}{l}\right) - \left(\frac{1-\frac{\lambda_1}{l}}{1+\frac{\lambda_1}{l}}\right)^2\left(2+\frac{\lambda_1}{l}\right)\right] + \frac{200}{3}\frac{H-H_w^0}{10^6} + \frac{2}{10^6}M^l\right\} = 1{,}21\cdot 10^4\left[\varepsilon_s \pm \varepsilon_t\cdot t^0 + \frac{\Delta}{l}\right],$$

(26) and (34). The function $M_m\left(\frac{\lambda_1}{l}\right)$ has been calculated by the Deflection Theory for a few trial load-length, plotting the ratio $\frac{8\,M_m}{p\,l^2}$ in Fig. 5. The maximum occurs for $\lambda_1 = 0{,}3\,l$. The corresponding values H and M_m, produced by dead load and live load are: $H = -2{,}626\cdot 10^6$ lbs., $M_m = +2{,}61\cdot 10^6$ lb.ft., in conjunction with lowest temperature and shrinkage effect $H = -2{,}563\cdot 10^6$ and $M_m = +4{,}68\cdot 10^6$ lb.ft. Fig. 8 shows the moment-diagrams for these two loading conditions, for comparison computed both by the customary and by the Deflection Theory. Considerable differences between approximative and exact values prove the vital importance of exact analysis and the great influence of deflections on the value and distribution of bending moments for flat arches. Differences in crown deflection reach 70 %, differences in moments about 60 %.

For flat, large-span arches the customary stress analysis has proved to be a not very close approximation and it is evident, that deductions relying on this method are not to be treated as axiomatical. This is to be noticed for the estimation of the different proceedings for „amelioration" of thrust-lines and shape of arches, worked out especially in Germany and France. For arches of large span and a low rise ratio the character of the line of thrust and its connection with rational arch design has to be explored anew with regard to the results of the Deflection Theory.

Summary.

The method presented here for calculating the stresses and change of shape in arch bridges is derived from the differential equation of the circular bar. This, in conjunction with geometrical conditions concerning the ends of the arch, yields the differential equation for the deflection curve of a restrained arch. Its solution and the conditions of equilibrum allow the stresses to be determined. In applying the result to the analysis of a concrete arch, the considerable influence of deflection on the stresses is shown. The Deflection Theory is found to yield an average increase of 50 % and

more in the bending moments near the crown, as previously found by the customary method of statics.

Résumé.

La méthode ici présentée pour le calcul des efforts et des déformations dans les ponts en arc est déduite de l'équation différentielle de la barre circulaire. La méthode employée, concurremment avec la considération des conditions géométriques relatives aux extrémités de l'arc, permet d'obtenir l'équation différentielle de la courbe de déformation d'un arc encastré. La solution et les conditions d'équilibre permettent de déterminer les efforts. En appliquant les résultats obtenus à l'étude analytique d'un arc en béton armé, l'auteur met en évidence l'influence considérable de la déformation sur les efforts. Il constate que la théorie de la déformation donne, par rapport aux méthodes ordinaires de la statique une augmentation moyenne de 50 % et plus pour les moments fléchissants aux environs de la clef.

Zusammenfassung.

Die vorliegende Methode zur Berechnung der Spannungen und der Formänderung von Bogenbrücken ist von der Differentialgleichung des kreisförmigen Stabes abgeleitet. Dies in Verbindung mit geometrischen Bedingungen betreffend die Bogenenden, ergibt die Differentialgleichung für die Verformungskurve eines eingespannten Bogens. Ihre Lösung und die Gleichgewichtsbedingungen erlauben die Bestimmung der Spannungen. Die Anwendung des Resultates auf die Berechnung eines Betonbogens zeigt den bedeutenden Einfluß der Verformung auf die Spannungen. Auf Grund der Verformungstheorie ergibt sich gegenüber den gewöhnlichen Methoden der Statik eine durchschnittliche Vergrößerung der Biegungsmomente im Scheitel von 50 % und mehr.