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Bending of rectangular orthotropic plates under concentrated load with two opposite edges simply supported

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Summary

The problem is solved by making use of the singular solution for the bending of an infinitely long simply supported orthotropic strip under concentrated load. The second and higher derivatives of this singular part of the solution are summed and this enables accurate values to be readily obtained for the normal forces which act at the edges of the plate. Knowledge of these forces is required especially for the design of the support structure and the longitudinal and transverse reinforcing members in bridge construction.

Numerical illustrative examples are provided which indicate that the maximum values of the normal forces can be calculated very accurately from the first few terms of the series. Rather more terms are required to calculate the maximum value of the deflection to within the same accuracy.

1. Introduction

The problem of the bending of a uniformly orthotropic plate of constant thickness has attracted attention in recent years because of its general technological importance, e. g. integrally stiffened plates in aeronautical engineering [1], bridge structures in civil engineering [2, 3], grillages in general structural engineering [4, 5], etc. In the present paper, attention is given to the case where a rectangular plate is subjected to a concentrated normal load and where two opposite edges are simply supported. This case has special significance in civil engineering practice and in particular to bridges where it is necessary to design for an abnormal loading condition.

Although formal solutions are available, attention is continually being drawn [2, 3, 6, 7] to the practical difficulties of calculating sufficiently accurate values for the physical quantities and especially for the normal forces at the edges of the plate. These normal forces depend upon the third derivative of the deflection w and it is of interest to note that the convergence of the single series solution for the deflection of an isotropic plate, as given by Timoshenko and Woinowsky-Krieger [5, p. 142], is controlled by the factor $1/n^3$, where n is the number of terms in the series. Of these normal forces, a knowledge of the shearing force is required during the design of the longitudinal and transverse members of a bridge structure, while the Kirchhoff edge reaction is required for the design of the support structure. (It must be noted, however, that in bridge construction it is more common to encounter some degree of skewness and this presents a considerably more difficult problem and requires special investigation. In this connection, it is very appropriate to draw attention to the distinguished pioneer work of Professor Henry Favre [8, 9, 10]).

It has already been noted [5] that the singular solution for the bending of an infinitely long simply supported strip can be used to advantage in the solution of the corresponding finite plate, although actual activity seems to have been restricted to the method of images with its inherent limitations. In the present paper, this combination of solutions is effected in a much more general way for orthotropic plates. Full advantage is then taken of the fact that the second and higher derivatives of this singular part of the solution can be summed explicitly [11] and this enables accurate values to be readily obtained for the normal forces. Indeed, it is now a peculiar feature that it is considerably easier to calculate accurate values of the normal forces than for the deflections.

While two opposite edges are always assumed to be simply supported, the remaining edges of the plate may be any combination of simple support, free, clamped or elastically restrained. Furthermore, for the orthotropic plate there are essentially three different algebraic forms of solution depending upon whether the roots of the characteristic equation are real and unequal, real and equal or are complex. Although the present ideas are applicable to all combinations of these factors, the presentation is simplified by confining attention to those cases where the remaining edges are either both simply supported or both free and where the roots of the characteristic equation are complex (as in bridge construction).

Numerical illustrative examples are provided for a square plate with a single concentrated load. An idea of the accuracy which is achieved is given by the case of the centrally loaded plate with all edges simply supported where, for the particular orthotropy considered, the maximum value of the Kirchhoff edge reaction is determined correct to four significant figures by the first term, whereas the maximum value of the deflection is correct only to the first significant figure. This is because of the slow convergence of the series for the singular part of the solution and, in this respect, it is worth noting that this series can be summed by numerical means quite independently of the remainder of the analysis. All the computations were carried out by Mr. B. C. Merrifield who provided valuable assistance also by checking many of the expressions.

The method can be readily developed to deal with simply supported right bridges with intermediate column supports.

2. Notation

a, b	dimensions of the plate in the x and y directions respectively
A_n, B_n, A'_n, B'_n	constants which are associated with the nth term of the infinite series
D	a constant relating to the flexural rigidity
D_x, D_y	flexural rigidities in the x and y directions respectively
D_{xy}	plate shear rigidity
D_1	coupling rigidity arising from the Poisson ratio effect
H	$= D_1 + 2D_{xy}$
M_x, M_y, M_{xy}	bending and twisting moments of the xOy co-ordinate system
n	integer
P	magnitude of applied concentrated load
Q_x, Q_y	shearing forces of the xOy co-ordinate system
V_x, V_y	Kirchhoff edge reactions
w	normal deflection $= w_i + w_f$
w_i, w_f	normal deflections associated with infinite strip and finite plate respectively
x, y	rectangular Cartesian co-ordinates
$\alpha, \beta, \alpha', \beta'$	constants defined by equations (4.5)
γ_n	$= n\pi/b$
λ, μ	constants defined by equation (4.4)
η, ξ	co-ordinates which define the position of the applied concentrated load

3. Fundamental Equations for Orthotropic Plates

The fundamental equations for orthotropic plates are given in the book by Timoshenko and Woinowsky-Krieger [5]. They are briefly summarised below.

It is assumed that the directions of orthotropy coincide with the rectangular co-ordinate system xOy. The expressions for the bending and twisting moments are

$$(3.1a) \qquad M_x = -\left(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right),$$

$$(3.1b) \qquad M_y = -\left(D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2}\right),$$

$$(3.1c) \qquad M_{xy} = 2\, D_{xy} \frac{\partial^2 w}{\partial x\, \partial y},$$

where D_x, D_y are the flexural rigidities in the x and y directions respectively, D_{xy} is the plate shear rigidity and D_1 is the coupling rigidity arising from the Poisson ratio effect. If these equations are substituted into the usual differential equation of element equilibrium we obtain, in the absence of normal loading, the following governing equation for orthotropic plates

$$(3.2) \qquad D_x \frac{\partial^4 w}{\partial x^4} + 2\, H \frac{\partial^4 w}{\partial x^2\, \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = 0,$$

where the notation

$$(3.3) \qquad H = D_1 + 2\, D_{xy}$$

is introduced.

The shearing forces are given by

$$Q_x = -\frac{\partial}{\partial x}\left(D_x \frac{\partial^2 w}{\partial x^2} + H \frac{\partial^2 w}{\partial y^2}\right), \tag{3.4a}$$

$$Q_y = -\frac{\partial}{\partial y}\left(D_y \frac{\partial^2 w}{\partial y^2} + H \frac{\partial^2 w}{\partial x^2}\right), \tag{3.4b}$$

and the Kirchhoff edge reactions by

$$V_x = Q_x - \frac{\partial M_{xy}}{\partial y} = -\frac{\partial}{\partial x}\left\{D_x \frac{\partial^2 w}{\partial x^2} + (H + 2\, D_{xy}) \frac{\partial^2 w}{\partial y^2}\right\}, \tag{3.5a}$$

$$V_y = Q_y - \frac{\partial M_{xy}}{\partial x} = -\frac{\partial}{\partial y}\left\{D_y \frac{\partial^2 w}{\partial y^2} + (H + 2\, D_{xy}) \frac{\partial^2 w}{\partial x^2}\right\}. \tag{3.5b}$$

4. Infinite Strip Under Concentrated Load

Woinowsky-Krieger [11] presents a solution for the infinitely long, simply supported, orthotropic strip under a concentrated load. This singular solution is in the form of a simple series which is summed for the second derivatives of the deflection. The main details are recorded below where the suffix i denotes relevance to the infinitely long strip.

The co-ordinate axes xOy are taken as shown in Fig. 1, the concentrated load P is situated a distance η along the Oy axis and the width of the strip is denoted by b.

Fig. 1. Infinitely long strip under concentrated load. ss = simply supported

Consider a solution of the form

$$w_i = \sum_{n=1}^{\infty} X_n(x) \sin \gamma_n y \tag{4.1}$$

where

$$\gamma_n = \frac{n\pi}{b}. \tag{4.2}$$

Equation (4.1) satisfies the boundary conditions of simple support

$$w_i = \frac{\partial^2 w_i}{\partial y^2} = 0 \tag{4.3}$$

along the edges $y = 0, b$ and when it is substituted into equation (3.2) it yields an ordinary differential equation which determines the function $X_n(x)$. In considering the roots of the resulting characteristic equation it is convenient to introduce the notation

$$\lambda = (D_x/D_y)^{\frac{1}{4}}, \quad \mu = H/(D_x D_y)^{\frac{1}{2}}, \tag{4.4}$$

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = \frac{b\lambda}{\pi}\left\{\mu \mp (\mu^2 - 1)^{\frac{1}{2}}\right\}^{\frac{1}{2}}, \quad \text{when } \mu > 1 \tag{4.5a}$$

$$\left.\begin{matrix}\alpha'\\ \beta'\end{matrix}\right\} = \frac{b\lambda}{\pi}\left\{2/(1 \pm \mu)\right\}^{\frac{1}{2}}, \quad \text{when } \mu < 1. \tag{4.5b}$$

There are now three forms of solution according to whether $\mu > 1$, $\mu = 1$ or $\mu > 1$, but to reduce the algebraic detail attention is confined to the case which commonly occurs in bridge construction where $\mu < 1$. The roots of the characteristic equation are then complex and the solution, equation (4.1), for the infinitely long strip may be written as

$$w_i = \frac{Pb}{2\pi^2 \sqrt{D_x D_y}} \times \sum_{n=1}^{\infty} \frac{1}{n^3}\left(\alpha' \cos\frac{nx}{\beta'} + \beta' \sin\frac{nx}{\beta'}\right) e^{-\frac{nx}{\alpha'}} \sin\gamma_n \eta \sin\gamma_n y \tag{4.6}$$

which is valid only for $x \geqslant 0$. For negative x,

$$w_i(-x) = w_i(x). \tag{4.7}$$

The first three derivatives of w_i are required in the evaluation of the other physical quantities. The derivatives with respect to x are listed below for ready reference,

$$\frac{\partial w_i}{\partial x} = -\frac{Pb}{2\pi^2\sqrt{D_x D_y}}\left(\frac{\alpha'}{\beta'} + \frac{\beta'}{\alpha'}\right) \times \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{nx}{\beta'} e^{-\frac{nx}{\alpha'}} \sin\gamma_n \eta \sin\gamma_n y, \tag{4.8a}$$

$$\frac{\partial^2 w_i}{\partial x^2} = -\frac{Pb}{2\pi^2\sqrt{D_x D_y}}\left(\frac{1}{\beta'^2} + \frac{1}{\alpha'^2}\right) \times \sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha' \cos\frac{nx}{\beta'} - \beta' \sin\frac{nx}{\beta'}\right) e^{-\frac{nx}{\alpha'}} \sin\gamma_n \eta \sin\gamma_n y. \tag{4.8b}$$

$$\frac{\partial^3 w_i}{\partial x^3} = -\frac{Pb}{2\pi^2\sqrt{D_x D_y}}\left(\frac{1}{\beta'^2} + \frac{1}{\alpha'^2}\right) \times \sum_{n=1}^{\infty}\left\{\left(\frac{\beta'}{\alpha'} - \frac{\alpha'}{\beta'}\right)\sin\frac{nx}{\beta'} - 2\cos\frac{nx}{\beta'}\right\} e^{-\frac{nx}{\alpha'}} \sin\gamma_n \eta \sin\gamma_n y. \tag{4.8c}$$

Equations (4.8) are again valid only for $x \geqslant 0$. For negative x

$$\left.\frac{\partial w_i}{\partial x}\right|_{-x} = -\left.\frac{\partial w_i}{\partial x}\right|_{x}, \quad \left.\frac{\partial^2 w_i}{\partial x^2}\right|_{-x} = \left.\frac{\partial^2 w_i}{\partial x^2}\right|_{x}, \quad \left.\frac{\partial^3 w_i}{\partial x^3}\right|_{-x} = -\left.\frac{\partial^3 w_i}{\partial x^3}\right|_{x}. \tag{4.9}$$

The derivatives with respect to y follow very simply and are not recorded.

The second derivatives may be summed and expressed in the following explicit form

$$\frac{\partial^2 w_i}{\partial x^2} = -\frac{P}{8\, b D_x}\left\{\alpha' \ln\frac{\varrho_1}{\varrho_2} + \beta'(\varphi_1 - \varphi_2)\right\}, \tag{4.10a}$$

$$\frac{\partial^2 w_i}{\partial y^2} = -\frac{P}{8\, b\sqrt{D_x D_y}}\left\{\alpha' \ln\frac{\varrho_1}{\varrho_2} - \beta'(\varphi_1 - \varphi_2)\right\}, \tag{4.10b}$$

$$\frac{\partial^2 w_i}{\partial x \partial y} = -\frac{P}{8\pi\sqrt{D_x D_y}}\left(\frac{\alpha'}{\beta'} + \frac{\beta'}{\alpha'}\right)\ln\frac{\varrho_3}{\varrho_4}, \tag{4.10c}$$

where,

$$\left.\begin{matrix}\varrho_1^2\\ \varrho_2^2\end{matrix}\right\} = \left\{\cosh\frac{x}{\alpha'}\cos\frac{x}{\beta'} - \cos\frac{\pi(y \pm \eta)}{b}\right\}^2 + \sinh^2\frac{x}{\alpha'}\sin^2\frac{x}{\beta'}, \tag{4.11a}$$

$$\left.\begin{matrix}\varrho_3^2\\ \varrho_4^2\end{matrix}\right\} = \left\{\cosh\frac{x}{\alpha'}\cos\left(\frac{x}{\beta'} \pm \frac{\pi\eta}{b}\right) - \cos\frac{\pi y}{b}\right\}^2 + \sinh^2\frac{x}{\alpha'}\sin^2\left(\frac{x}{\beta'} \pm \frac{\pi\eta}{b}\right), \tag{4.11b}$$

$$\left.\begin{matrix}\varphi_1\\ \varphi_2\end{matrix}\right\} = \tan^{-1}\frac{\sinh\frac{x}{\alpha'}\sin\frac{x}{\beta'}}{\cosh\frac{x}{\alpha'}\cos\frac{x}{\beta'} - \cos\frac{\pi(y \pm \eta)}{b}}. \tag{4.11c}$$

It is useful to note the following derivatives of the above quantities

$$\frac{\partial}{\partial x}\log \varrho_1 = \frac{1}{2\,\varrho_1^2}\left[-2\cos\frac{\pi(y+\eta)}{b}\,\times\right.$$

(4.12 a)
$$\times\left\{\frac{1}{\alpha'}\sinh\frac{x}{\alpha'}\cos\frac{x}{\beta'}-\frac{1}{\beta'}\cosh\frac{x}{\alpha'}\sin\frac{x}{\beta'}\right\}+$$

$$\left.+\frac{1}{\alpha'}\sinh\frac{2x}{\alpha'}-\frac{1}{\beta'}\sin\frac{2x}{\beta'}\right],$$

$$\frac{\partial}{\partial y}\log \varrho_1 =$$

(4.12b)
$$=\frac{1}{\varrho_1^2}\,\frac{\pi}{b}\sin\frac{\pi(y+\eta)}{b}\left\{\cosh\frac{x}{\alpha'}\cos\frac{x}{\beta'}-\cos\frac{\pi(y+\eta)}{b}\right\},$$

$$\frac{\partial \varphi_1}{\partial x} = \frac{1}{2\,\varrho_1^2}\left[-2\cos\frac{\pi(y+\eta)}{b}\,\times\right.$$

(4.12 c)
$$\times\left\{\frac{1}{\alpha'}\cosh\frac{x}{\alpha'}\sin\frac{x}{\beta'}+\frac{1}{\beta'}\sinh\frac{x}{\alpha'}\cos\frac{x}{\beta'}\right\}+$$

$$\left.+\frac{1}{\alpha'}\sin\frac{2x}{\beta'}+\frac{1}{\beta'}\sinh\frac{2x}{\alpha'}\right],$$

(4.12 d)
$$\frac{\partial \varphi_1}{\partial y} = -\frac{1}{\varrho_1^2}\,\frac{\pi}{b}\sin\frac{\pi(y+\eta)}{b}\sinh\frac{x}{\alpha'}\sin\frac{x}{\beta'}\,.$$

5. Rectangular Plate under Concentrated Load

The Levy type single series solution for the rectangular plate is written in the form

(5.1)
$$w = w_i + w_f$$

where the suffix f denotes relevance to the finite plate. The function w_i is given by equation (4.6) while w_f is given by

$$w_f = \frac{Pb}{2\,\pi^2\sqrt{D_x D_y}}\,\times$$

(5.2)
$$\times\sum_{n=1}^{\infty}\frac{1}{n^3}\left\{\left(\alpha' A_n\cos\frac{nx}{\beta'}+\beta' B_n\sin\frac{nx}{\beta'}\right)e^{-\frac{nx}{\alpha'}}+\right.$$

$$\left.+\left(\alpha' A'_n\cos\frac{nx}{\beta'}+\beta' B'_n\sin\frac{nx}{\beta'}\right)e^{\frac{nx}{\alpha'}}\right\}\sin\gamma_n\eta\sin\gamma_n y\,.$$

Equation (5.2) satisfies the boundary conditions of simple support along the edges $y = 0, b$, c. f. equation (4.3), and the constants A_n, A'_n, B_n and B'_n are to be determined so that the boundary conditions for $w = w_i + w_f$ are satisfied along the remaining edges $x = \xi$ and $x = \xi - a$, see Fig. 2. It is useful to note the following derivatives with respect to x,

$$\frac{\partial w_f}{\partial x} = -\frac{Pb}{2\,\pi^2\sqrt{D_x D_y}}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\left\{A_n\left(\frac{\alpha'}{\beta'}\sin\frac{nx}{\beta'}+\cos\frac{nx}{\beta'}\right)+\right.\right.$$

(5.3 a)
$$+B_n\left(\frac{\beta'}{\alpha'}\sin\frac{nx}{\beta'}-\cos\frac{nx}{\beta'}\right)\Bigg\}e^{-\frac{nx}{\alpha'}}+\left\{A'_n\left(\frac{\alpha'}{\beta'}\sin\frac{nx}{\beta'}-\cos\frac{nx}{\beta'}\right)-\right.$$

$$\left.\left.-B'_n\left(\frac{\beta'}{\alpha'}\sin\frac{nx}{\beta'}+\cos\frac{nx}{\beta'}\right)\right\}e^{\frac{nx}{\alpha'}}\right]\sin\gamma_n\eta\sin\gamma_n y\,,$$

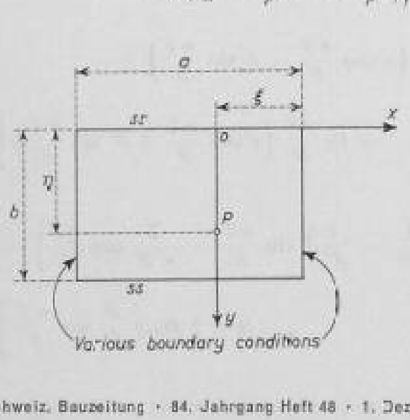

Fig. 2. Rectangular plate under concentrated load.
ss = simply supported

$$\frac{\partial^2 w_f}{\partial x^2} = -\frac{Pb}{2\,\pi^2\sqrt{D_x D_y}}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left\{\frac{A_n}{\beta'}\left(\frac{\alpha'}{\beta'}-\frac{\beta'}{\alpha'}\right)\cos\frac{nx}{\beta'}-\right.\right.$$

$$\left.-\frac{2A_n}{\beta'}\sin\frac{nx}{\beta'}+\frac{B_n}{\alpha'}\left(\frac{\alpha'}{\beta'}-\frac{\beta'}{\alpha'}\right)\sin\frac{nx}{\beta'}+\frac{2B_n}{\alpha'}\cos\frac{nx}{\beta'}\right\}e^{-\frac{nx}{\alpha'}}+$$

(5.3b)
$$+\left\{\frac{A'_n}{\beta'}\left(\frac{\alpha'}{\beta'}-\frac{\beta'}{\alpha'}\right)\cos\frac{nx}{\beta'}+\frac{2A'_n}{\beta'}\sin\frac{nx}{\beta'}+\frac{B'_n}{\alpha'}\times\right.$$

$$\left.\left.\times\left(\frac{\alpha'}{\beta'}-\frac{\beta'}{\alpha'}\right)\sin\frac{nx}{\beta'}-\frac{2B'_n}{\alpha'}\cos\frac{nx}{\beta'}\right\}e^{\frac{nx}{\alpha'}}\right]\sin\gamma_n\eta\sin\gamma_n y\,,$$

$$\frac{\partial^3 w_f}{\partial x^3} = -\frac{Pb}{2\,\pi^2\sqrt{D_x D_y}}\sum_{n=1}^{\infty}\left[\left\{-\frac{A_n}{\beta'^2}\left(\frac{\alpha'}{\beta'}-\frac{3\beta'}{\alpha'}\right)\sin\frac{nx}{\beta'}-\right.\right.$$

$$-\frac{A_n}{\alpha'\beta'}\left(\frac{3\alpha'}{\beta'}-\frac{\beta'}{\alpha'}\right)\cos\frac{nx}{\beta'}+\frac{B_n}{\alpha'\beta'}\left(\frac{\alpha'}{\beta'}-\frac{3\beta'}{\alpha'}\right)\cos\frac{nx}{\beta'}-$$

(5.3 c)
$$\left.-\frac{B_n}{\alpha'^2}\left(\frac{3\alpha'}{\beta'}-\frac{\beta'}{\alpha'}\right)\sin\frac{nx}{\beta'}\right\}e^{-\frac{nx}{\alpha'}}+\left\{-\frac{A'_n}{\beta'^2}\left(\frac{\alpha'}{\beta'}-\frac{3\beta'}{\alpha'}\right)\times\right.$$

$$\times\sin\frac{nx}{\beta'}+\frac{A'_n}{\alpha'\beta'}\left(\frac{3\alpha'}{\beta'}-\frac{\beta'}{\alpha'}\right)\cos\frac{nx}{\beta'}+\frac{B'_n}{\alpha'\beta'}\left(\frac{\alpha'}{\beta'}-\frac{3\beta'}{\alpha'}\right)\times$$

$$\left.\left.\times\cos\frac{nx}{\beta'}+\frac{B'_n}{\alpha'^2}\left(\frac{3\alpha'}{\beta'}-\frac{\beta'}{\alpha'}\right)\sin\frac{nx}{\beta'}\right\}e^{\frac{nx}{\alpha'}}\right]\sin\gamma_n\eta\sin\gamma_n y\,.$$

Once again, the derivatives with respect to y follow very easily and are not recorded.

While the boundary conditions along $x = \xi$ and $x = \xi - a$ may be any combination of simple support, free, clamped or elastically restrained, attention is confined here to the two very important cases which occur in the constructional industry where both these edges are either simply supported or are free.

5.1 *Determination of constants when all edges are simply supported*

The constants A_n, A'_n, B_n and B'_n are first determined for the case where all the edges are simply supported so that

(5.4)
$$w_f = -w_i\,,\quad \frac{\partial^2 w_f}{\partial x^2} = -\frac{\partial^2 w_i}{\partial x^2}$$

along the edges $x = \xi$ and $x = \xi - a$. When equations (4.6), (4.8b), (5.2) and (5.3b) are substituted into equations (5.4) there results independent sets of four simultaneous equations which determine the value of the A_n, A'_n, B_n and B'_n for the desired n. A typical set of equations is

(5.5)
$$\begin{bmatrix} c_n(\xi) & c_n(-\xi) & s_n(\xi) & -s_n(-\xi) \\ c_n(\xi-a) & c_n(-(\xi-a)) & s_n(\xi-a) & -s_n(-(\xi-a)) \\ c'_n(\xi) & c'_n(-\xi) & s'_n(\xi) & -s'_n(-\xi) \\ c'_n(\xi-a) & c'_n(-(\xi-a)) & s'_n(\xi-a) & -s'_n(-(\xi-a)) \end{bmatrix}\times$$

$$\times\begin{bmatrix} A_n \\ A'_n \\ B_n \\ B'_n \end{bmatrix} = \begin{bmatrix} a_n(\xi) \\ a_n(-(\xi-a)) \\ a'_n(\xi) \\ a'_n(-(\xi-a)) \end{bmatrix}$$

where

$$
(5.6)\quad \begin{cases}
c_n(\xi) = \alpha' \cos \frac{n\xi}{\beta'}\, e^{-\frac{n\xi}{\alpha'}}, \\
s_n(\xi) = \beta' \sin \frac{n\xi}{\beta'}\, e^{-\frac{n\xi}{\alpha'}}, \\
c'_n(\xi) = \left\{\frac{1}{\beta'}\left(\frac{\alpha'}{\beta'} - \frac{\beta'}{\alpha'}\right)\cos\frac{n\xi}{\beta'} - \frac{2}{\beta'}\sin\frac{n\xi}{\beta'}\right\} e^{-\frac{n\xi}{\alpha'}}, \\
s'_n(\xi) = \left\{\frac{1}{\alpha'}\left(\frac{\alpha'}{\beta'} - \frac{\beta'}{\alpha'}\right)\sin\frac{n\xi}{\beta'} + \frac{2}{\alpha'}\cos\frac{n\xi}{\beta'}\right\} e^{-\frac{n\xi}{\alpha'}}, \\
a_n(\xi) = -\left(\alpha'\cos\frac{n\xi}{\beta'} + \beta'\sin\frac{n\xi}{\beta'}\right) e^{-\frac{n\xi}{\alpha'}}, \\
a'_n(\xi) = -\left(\frac{1}{\beta'^2} + \frac{1}{\alpha'^2}\right)\left(\alpha'\cos\frac{n\xi}{\beta'} - \beta'\sin\frac{n\xi}{\beta'}\right) e^{-\frac{n\xi}{\alpha'}}.
\end{cases}
$$

It is worth noting that equations (5.5) and (5.6) are independent of the value of η. Consequently, the values of the A_n, A'_n, B_n and B'_n are independent of the position of the load P along the y axis.

5.2 *Determination of constants when two opposite edges are simply supported and remaining edges are free*

The constants A_n, A'_n, B_n and B'_n are similarly determined for the case where the remaining edges are free. Along these edges $x = \xi$ and $x = \xi - a$ we have $M_x = V_x = 0$ so that, on substitution from

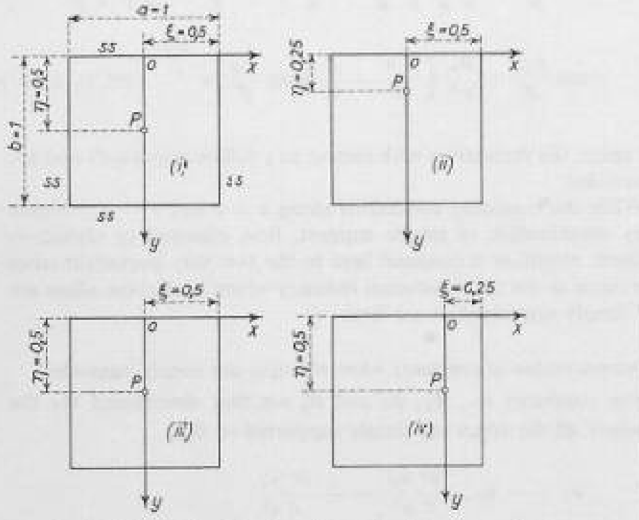

Fig. 3. Position of load and orthotropic plate dimensions for numerical examples where all edges are simply supported. Note: $D_x = 2D$, $D_y = D$ in cases *(iii)* and *(iv)*. *ss* = simply supported

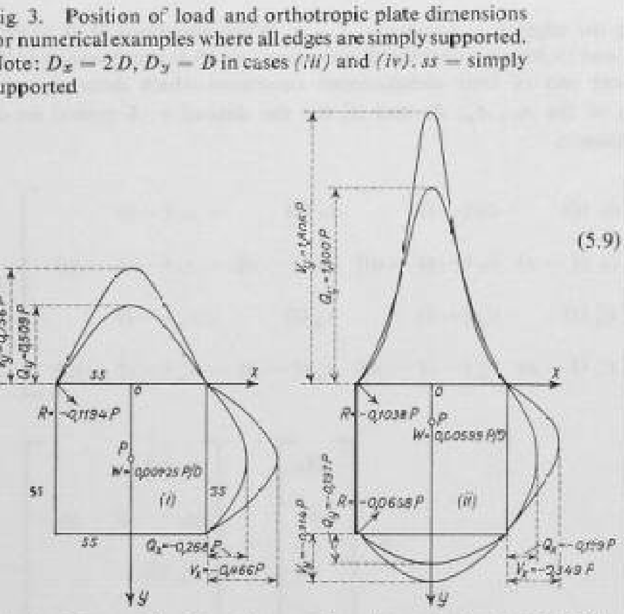

Fig. 4. Numerical values for examples where all edges are simply supported. The corner reactions R are given by $\pm\, 2\, M_{xy}$. R positive indicates opposite sense to P. $a = b = 1$, $D_x = D$, $D_y = 2D$, $D_1 = 0{,}15\,D$, $D_{xy} = 0{,}425\,D$

equations (3.1a) and (3.5a), it is necessary for

$$
(5.7a)\qquad D_x \frac{\partial^2 w_f}{\partial x^2} + D_1 \frac{\partial^2 w_f}{\partial y^2} = -\left(D_x \frac{\partial^2 w_i}{\partial x^2} + D_1 \frac{\partial^2 w_i}{\partial y^2}\right),
$$

$$
(5.7b)\qquad \frac{\partial}{\partial x}\left\{D_x \frac{\partial^2 w_f}{\partial x^2} + (H + 2\,D_{xy}) \frac{\partial^2 w_f}{\partial y^2}\right\} = - \frac{\partial}{\partial x}\left\{D_x \frac{\partial^2 w_i}{\partial x^2} + (H + 2\,D_{xy}) \frac{\partial^2 w_i}{\partial y^2}\right\}.
$$

When the appropriate equations from paras. 4 and 5 are substituted into equations (5.7) there again results independent sets of four simultaneous equations

$$
(5.8)\quad \begin{bmatrix}
c_n(\xi) & c_n(-\xi) & s_n(\xi) & -s_n(-\xi) \\
c_n(\xi - a) & c_n(-(\xi - a)) & s_n(\xi - a) & -s_n(-(\xi - a)) \\
c'_n(\xi) & -c'_n(-\xi) & s'_n(\xi) & s'_n(-\xi) \\
c'_n(\xi - a) & -c'_n(-(\xi - a)) & s'_n(\xi - a) & s'_n(-(\xi - a))
\end{bmatrix} \times
\begin{bmatrix} A_n \\ A'_n \\ B_n \\ B'_n \end{bmatrix} =
\begin{bmatrix} a_n(\xi) \\ a_n(-(\xi - a)) \\ a'_n(\xi) \\ -a'_n(-(\xi - a)) \end{bmatrix}
$$

where

$$
(5.9)\quad \begin{cases}
c_n(\xi) = \left[\frac{D_x}{\beta'}\left\{\left(\frac{\alpha'}{\beta'} - \frac{\beta'}{\alpha'}\right)\cos\frac{n\xi}{\beta'} - 2\sin\frac{n\xi}{\beta'}\right\} + \frac{D_1 \pi^2}{b^2}\alpha'\cos\frac{n\xi}{\beta'}\right] e^{-} \\
s_n(\xi) = \left[\frac{D_x}{\alpha'}\left\{\left(\frac{\alpha'}{\beta'} - \frac{\beta'}{\alpha'}\right)\sin\frac{n\xi}{\beta'} + 2\cos\frac{n\xi}{\beta'}\right\} + \frac{D_1 \pi^2}{b^2}\beta'\sin\frac{n\xi}{\beta'}\right] e^{-} \\
c'_n(\xi) = \Big[\frac{D_x}{\beta'}\left\{\frac{1}{\beta'}\left(\frac{\alpha'}{\beta'} - \frac{3\beta'}{\alpha'}\right)\sin\frac{n\xi}{\beta'} + \frac{1}{\alpha'}\left(\frac{3\alpha'}{\beta'} - \frac{\beta'}{\alpha'}\right)\cos\frac{n\xi}{\beta'}\right\} \\
\qquad + (H + 2\,D_{xy})\frac{\pi^2}{b^2}\left(\frac{\alpha'}{\beta'}\sin\frac{n\xi}{\beta'} + \cos\frac{n\xi}{\beta'}\right)\Big] e^{-} \\
s'_n(\xi) = \Big[-\frac{D_x}{\alpha'}\left\{\frac{1}{\beta'}\left(\frac{\alpha'}{\beta'} - \frac{3\beta'}{\alpha'}\right)\cos\frac{n\xi}{\beta'} - \frac{1}{\alpha'}\left(\frac{3\alpha'}{\beta'} - \frac{\beta'}{\alpha'}\right)\sin\frac{n\xi}{\beta'}\right\} \\
\qquad + (H + 2\,D_{xy})\frac{\pi^2}{b^2}\left(\frac{\beta'}{\alpha'}\sin\frac{n\xi}{\beta'} - \cos\frac{n\xi}{\beta'}\right)\Big] e^{-} \\
a_n(\xi) = -\Big[D_x\left(\frac{1}{\beta'^2} + \frac{1}{\alpha'^2}\right)\left(\alpha'\cos\frac{n\xi}{\beta'} - \beta'\sin\frac{n\xi}{\beta'}\right) + \\
\qquad + D_1\frac{\pi^2}{b^2}\left(\alpha'\cos\frac{n\xi}{\beta'} + \beta'\sin\frac{n\xi}{\beta'}\right)\Big] e^{-} \\
a'_n(\xi) = \left(\frac{\alpha'}{\beta'} + \frac{\beta'}{\alpha'}\right)\Big[D_x\left\{\left(\frac{1}{\alpha'^2} - \frac{1}{\beta'^2}\right)\sin\frac{n\xi}{\beta'} - \frac{2}{\alpha'\beta'}\cos\frac{n\xi}{\beta'}\right\} - \\
\qquad - (H + 2\,D_{xy})\frac{\pi^2}{b^2}\sin\frac{n\xi}{\beta'}\Big] e^{-}
\end{cases}
$$

Table 1: All edges simply supported

Series truncated after n =	Case (i) $w(0;\,0{,}5)\,D/P$	Maximum value V_x/P	Maximum value V_y/P	At corner $2M_{xy}/P$	Case (ii) $w(0;\,0{,}25)\,D/P$	Max. value V_y/P	At corner $2M_{xy}/P$
1	0,008709 (0,009247)	0,4656	0,7564	0,1202	0,004354 (0,005988)	1,8028	0,0947
3	0,009094 (0,009247)	0,4666	0,7564	0,1194	0,005845 (0,005989)	1,8060	0,1038
5	0,009177 (0,009247)	0,4664	0,7564	0,1194	0,005886 (0,005989)	1,8060	0,1038
7	0,009207 (0,009247)	0,4664	0,7564	0,1194	0,005950 (0,005989)	1,8060	0,1038
9	0,009221 (0,009247)	0,4664	0,7564	0,1194	0,005957 (0,005989)	1,8060	0,1038
11	0,009229 (0,009247)	0,4664	0,7564	0,1194	0,005971 (0,005989)	1,8060	0,1038
15	0,009237 (0,009247)	0,4664	0,7564	0,1194	0,005979 (0,005989)	1,8060	0,1038
21	0,009241 (0,009247)	0,4664	0,7564	0,1194	0,005983 (0,005989)	1,8060	0,1038

The values in parenthesis are obtained by summing independently the series for w_t up to $n = 160$.

Table 2: All edges simply supported but $D_x = 2D$ and $D_y = D$

Series truncated after n =	Case (ii) $w(0;\,0{,}5)\,D/P$	Maximum value V_x/P	Maximum value V_y/P	At corner $2M_{xy}/P$	Case (iv) $w(0;\,0{,}5)\,D/P$	Max. value V_x/P	At corner $2M_{xy}/P$
1	0,008486 (0,009246)	0,7342	0,4678	0,1239	0,005263 (0,005023)	1,5357	0,1197
3	0,009030 (0,009247)	0,7574	0,4664	0,1192	0,005772 (0,005989)	1,7458	0,0999
5	0,009148 (0,009247)	0,7567	0,4664	0,1194	0,005889 (0,005989)	1,7985	0,1047
7	0,009190 (0,009247)	0,7564	0,4664	0,1194	0,005932 (0,005989)	1,8073	0,1037
9	0,009210 (0,009247)	0,7564	0,4664	0,1194	0,005952 (0,005989)	1,8073	0,1039
11	0,009221 (0,009247)	0,7564	0,4664	0,1194	0,005963 (0,005989)	1,8066	0,1038
15	0,009232 (0,009247)	0,7564	0,4664	0,1194	0,005974 (0,005989)	1,8060	0,1038
21	0,009239 (0,009247)	0,7564	0,4664	0,1194	0,005981 (0,005989)	1,8060	0,1038

The values in parenthesis are obtained by summing independently the series for w_t up to $n = 145$.

Table 3: Two opposite edges simply supported and remaining edges free

Series truncated after n =	Case (i) $w(0;\,0{,}5)\,D/P$	Max. value V_y/P	At corner $2M_{xy}/P$	Case (ii) $w(0;\,0{,}25)\,D/P$	Max. value V_y/P	At corner $2M_{xy}/P$
1	0,01230 (0,01283)	0,7774	0,0403	0,006148 (0,007781)	1,8177	0,0382
3	0,01268 (0,01283)	0,7774	0,0395	0,007646 (0,007790)	1,8179	0,0412
5	0,01276 (0,01283)	0,7774	0,0395	0,007688 (0,007790)	1,8179	0,0412
7	0,01279 (0,01283)	0,7774	0,0395	0,007751 (0,007790)	1,8179	0,0412
9	0,01281 (0,01283)	0,7774	0,0395	0,007758 (0,007790)	1,8179	0,0412
11	0,01282 (0,01283)	0,7774	0,0395	0,007772 (0,007790)	1,8179	0,0412
15	0,01282 (0,01283)	0,7774	0,0395	0,007780 (0,007790)	1,8179	0,0412
21	0,01282 (0,01283)	0,7774	0,0395	0,007784 (0,007790)	1,8179	0,0412

The values in parenthesis are obtained by summing independently the series for w_t up to $n = 76$.

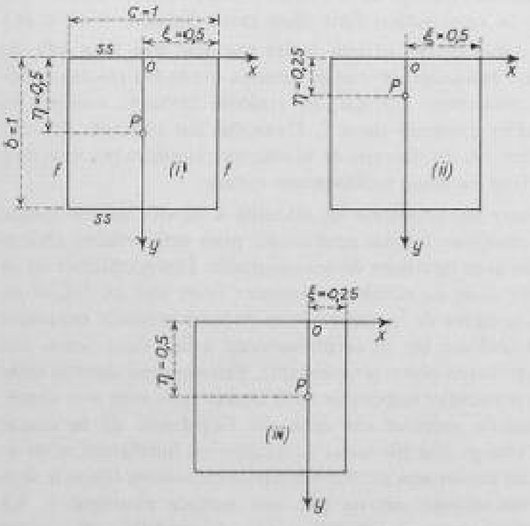

Fig. 5. Position of load and orthotropic plate dimension for numerical examples when two opposite edges are simply supported and remaining edges are free. ss = simply supported, f = free

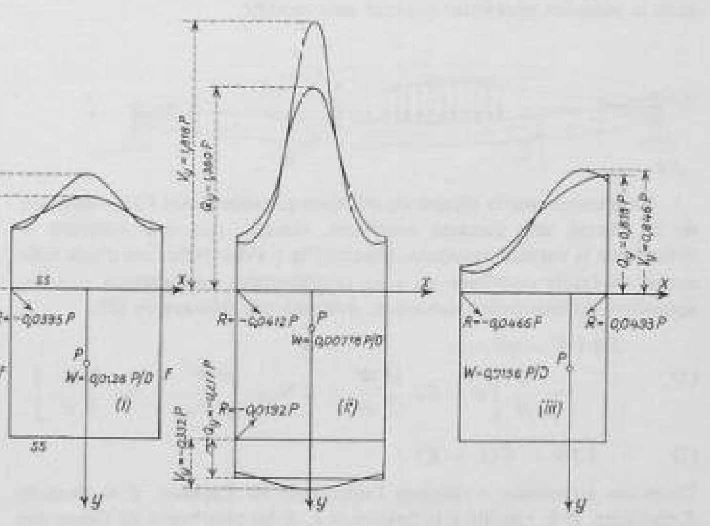

Fig. 6. Numerical values for examples when two opposite edges are simply supported and remaining edges are free. The corner reactions R are given by $\pm 2\,M_{xy}$. R positive indicates opposite sense to P. $a = b = 1$, $D_x = D$, $D_y = 2\,D$, $D_1 = 0{,}15\,D$, $D_{xy} = 0{,}425\,D$

Once again, the values of the A_n, A'_n, B_n and B'_n are independent of the position of the load P along the y axis.

6. Numerical Examples

Illustrative numerical examples are now given for both the above cases for various positions of the applied concentrated load P.

All the examples refer to a square plate where

$$(6.1a) \qquad a = b = 1\,, \quad D_1 = 0{,}15\ D\,, \quad D_{xy} = 0{,}425\ D$$

and, unless otherwise stated,

$$(6.1b) \qquad D_x = D\,, \qquad D_y = 2\ D\,.$$

In all cases

$$(6.1c) \qquad H = D \quad \text{and} \quad \mu = \sqrt{2}\,/\,2$$

see equations (3.3) and (4.4).

6.1 *All edges simply supported*

Four positions of the applied concentrated load are considered as shown in Fig. 3, where in cases (iii) and (iv) it is noted that $D_x = 2D$ and $D_y = D$. Cases (i) and (iii) provide identical situations as does case (ii) with case (iv), the values of the physical quantities are, however, derived by quite different numerical processes and so this provides a useful check on the calculations. The results of the calculations are sketched in Fig. 4.

With conventional methods it is a matter of some difficulty to calculate accurate values for Q_x, Q_y and V_x, V_y. It is a feature of the present method, however, that it is considerably easier to calculate these values than to calculate the deflection w (0; η) underneath the applied load to within the same degree of accuracy. This is illustrated in Tables 1 and 2 where values of the physical quantities are given for various truncations of the series. It is noted, incidentally, that it is necessary to consider even values of n only for case (ii) and also that an expeditious choice of the co-ordinate axes leads even more quickly to accurate results.

6.2 *Two opposite edges simply supported and remaining edges free*

Three positions of the applied concentrated load are considered as shown in Fig. 5. The results of the calculations are sketched in Fig. 6, Table 3 illustrates the convergence of the physical quantities for various truncations of the series.

References

[1] *W. H. Hoppmann, N. J. Huffington, L. S. Magness:* A study of orthogonally stiffened plates. «Jour. Appl. Mechs.», *23* (1956), 343.

[2] *A. Coull:* The stress analysis of orthotropic bridge slabs. «Quart. Jour. Mech. Appl. Maths.», *17* (1964), 437.

[3] *R. E. Rowe:* The distribution of shear forces and bearing reactions in simply supported bridges. Cement and Concrete Research Assoc. Tech. Rep. 353 (1961).

[4] *E. Lightfoot:* A grid framework analogy for laterally loaded plates. «Internat. Jour. Mech. Sciences», *6* (1964), 201.

[5] *S. Timoshenko, S. Woinowsky-Krieger:* Theory of plates and shells. McGraw-Hill 2nd Ed. (1959).

[6] *R. F. S. Hearmon:* Applied anisotropic elasticity. Oxford Univ. Press (1961), 123.

[7] *J. R. Vinson, M. A. Brull:* New techniques of solution for problems in the theory of orthotropic plates. Proc. Fourth US Nat. Cong., Berkeley. Pergamon (1962), 817.

[8] *H. Favre:* Contribution à l'étude des plaques obliques. «Schweiz. Bauztg.», *120* (1942), 35–36, 51–54 and 60.

[9] *H. Favre:* Le calcul des plaques obliques par la méthode des équations aux différences. Publ. Int. Assoc. Bridge and Structural Engrg., Zurich (1943), 91.

[10] *H. Favre:* Sur l'introduction des co-ordonnées cartésiennes obliques dans la théorie de l'élasticité. «Bull. Tech. Suisse Romande» (1946).

[11] *S. Woinowsky-Krieger:* Über die Biegung des orthotropen Plattenstreifens durch Einzellasten. «Ing. Archiv.», *25* (1957), 97.

Sur l'aplanissement élastique d'une coque de faible courbure

DK 624.074.4

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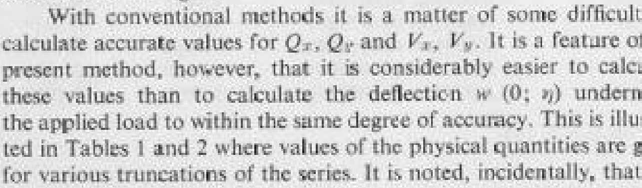
Introduction et équations générales

Dans la théorie bidimensionnelle sur la flexion des plaques minces sollicitées par une charge p on suppose l'épaisseur H de celles-ci faible par rapport aux deux autres dimensions. Lorsque le déplacement latéral W des points du plan moyen est de l'ordre de grandeur de H, mais néanmoins $|\operatorname{grad} W|^2 \ll 1$, on utilise comme point de départ les équations différentielles *non-linéaires* de *v. Kármán* [1], dans lesquelles intervient à côté de W la fonction d'Airy Φ de l'effet membrane. Comme dans la pratique on se borne à des déplacements faibles d'une part et que d'autre part on cherche à eviter les systèmes non-linéaires dont l'intégration n'est pas toujours aisée, nous avons été conduit à étudier dans la suite un problème quelque peu modifié.



Supposons que la plaque en question possède avant l'appliquation de la charge une certaine courbure, donnée par une fonction w définissant la surface moyenne initiale (Fig.). Pour la flexion d'une telle coque de faible courbure on a en coordonnées cartésiennes x, y les équations différentielles suivantes, données par *Marguerre* [2]:

$$(1) \qquad \Delta\Delta\,(W - w) = = \frac{1}{D}\left[p + N_x \frac{\partial^2 W}{\partial x^2} + 2\,N_{xy}\frac{\partial^2 W}{\partial x\,\partial y} + N_y \frac{\partial^2 W}{\partial y^2}\right],$$

$$(2) \qquad \Delta\Delta\Phi = E\,(k - K)\,.$$

Dans ces équations Δ désigne l'opérateur de *Laplace*, E le module d'élasticité, D la rigidité à la flexion et k, K les courbures de *Gauss* des surfaces w, W. Les fonctions $N_x = F\partial^2\Phi/\partial y^2$, $N_{xy} = -F\partial^2\Phi/\partial x\,\partial y$, $N_y = F\partial^2\Phi/\partial x^2$ sont les forces normales et tangentielles dûes à l'effet de membrane avec F comme aire de section effective par unité de longueur (dans le cas homogène on aurait $F = H$).

Supposons maintenant que la coque, sous l'influence de la charge p, ait été aplanie, c'est-à-dire que $W = 0$. Pour la surface initiale w et la fonction d'*Airy* Φ on obtient alors le système

$$(3) \qquad \Delta\Delta\, w = -\frac{p}{D}\,,$$

$$(4) \qquad \Delta\Delta\,\Phi = Ek = E\left[\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right].$$

La première de ces équations est au signe près identique à l'équation de base de la théorie linéaire de la flexion. Rappelons toutefois que contrairement à la supposition faite dans cette dernière ($|w| \ll H$) la grandeur du déplacement w dans notre cas peut très bien être du même ordre que l'épaisseur H. Les équations (3) et (4) résolues successivement se présentent comme un système linéaire, malgré les termes nonlinéaires contenus dans k. Dans des cas concrets de conditions aux limites, où les moyens de la théorie classique peuvent être utilisés l'intégration est donc relativement simple.

Vient s'ajouter un problème de stabilité à savoir qu'une coque mince de faible courbure initiale peut sauter pour une certaine charge dans une position avec courbure de sens opposée. Des problèmes de ce genre sont traités dans de nombreux travaux (voir par ex. [3]) et en particulier dans le cadre de la théorie non-linéaire générale présentée par *Mushtari* et *Galimov* [4]. Si ce phénomène a lieu dans notre cas, l'équilibre de la position plane sera instable. Envisageons dans la suite seulement le cas particulier important de la charge uniforme p = const. La courbure initiale critique est celle où l'équilibre de la coque aplanie par une charge dite elle aussi «critique» est indifférent, c'est-à-dire où il existe au moins une position d'équilibre voisine (dans le sens du calcul des variations), décrite par une surface élastique ζ. La fonction ζ satisfait donc à l'équation (1), à condition d'y poser $W = \zeta$, $\Delta\Delta w = -p/D$. Les forces normales sont à tirer de la fonction d'*Airy* Φ obtenue par l'intégration de (3) et (4). Ces forces étant proportionnelles à p^2EF/D^2, il est indiqué d'introduire des «forces» n_x, n_{xy}, n_y et un paramètre λ sans dimensions afin d'éviter la