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Am 17. März 1963 begeht Herr Professor Dr. J. Ackeret, Vorsteher des Instituts für Aerodynamik an der ETH in Zürich, seinen 65. Geburtstag. Einige Aufsätze aus dem Kreise seiner ehemaligen und gegenwärtigen Mitarbeiter mögen den Dank an den verehrten Lehrer und herzliche Wünsche an den Jubilaren überbringen. *Die Ehemaligen und die Mitarbeiter*

The Optimum Operation of Energy-restricted Rockets, Including Relativistic Effects

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Herrn Professor Dr. **J. Ackeret** zum 65. Geburtstag gewidmet

1. Relativistic analysis

If the total energy available for the propulsion of a rocket is restriced while the exhaust velocity may be varied, it has been shown by *Seifert* [1] that the rocket performance can be improved by certain «tayloring» of the exhaust speed. *Olds* [2] gave the solution as an exact result of the variational problem; it was found that the exhaust speed has to vary inversely with the mass of the rocket.

In view of the rather futuristic character of such devices it might be instructive to see how these results are modified in relativistic mechanics. The differential equation of rocket motion in the relativistic case was given by *Ackeret* [3]; excluding effects of resistance and gravity, it reads

$$(1) \qquad \frac{du}{1-u^2/c^2} = -w\,\frac{dm_0}{m_0}$$

where w is the exhaust velocity relative to the rocket, u is the rocket speed in the take-off frame of reference, and m_0 is the rest-mass of the rocket in the same frame.

Let us now assume that the rocket is provided in the take-off frame of reference with a certain amount of rest-energy E_0, which is to be used for propulsion. It can be postulated that the relation between the «consumption» of rest-energy E_0 and rest-mass m_0 will depend on w only (which is assumed to be the same for all particles expelled by the rocket). In other words, if this relation is to be independent of u, it can be determined either in flight or on a test stand in the take-off system. The energy balance is naturally simplest for the test stand case. Let dm_j be the rest-mass which is expelled and is given the energy dE_0; then

$$(2) \qquad dE_0 = dm_j\left[\frac{c^2}{\sqrt{1-w^2/c^2}} - c^2\right]$$

and the rest-mass of the rocket changes by

$$(3) \qquad -dm_0 = dm_j + \frac{dE}{c^2}$$

Elimination of dm_j gives

$$(4) \qquad dE_0 = -\,dm_0\,c^2\left(1-\sqrt{1-w^2/c^2}\right)$$

This solution holds exactly whether the rocket has a chemical fuel or an «inert» propellant which is energized by some source in the rocket; the only necessary assumption is that all mass particles leave the rocket with the same velocity w.

As a check, let the energy balance be made for the rocket in flight by an observer in the take-off frame, in terms of rest-mass and rest-energy in the take-off frame. The observer will notice a decrease of the rest-mass of the rocket, $-dm_0$, and an increase in the rest-mass of the exhaust products, dm_j. It can be easily deduced from Ackeret's fundamental equations that

$$(5) \qquad dm_j = -\,dm_0\sqrt{1-w^2/c^2}$$

holds true independently of u, and in accordance with Equations [3] and [4]. (In Ackerets' original notation, $dm_0 = dm_{01}$, and $dm_j = dm_{02}$). The observer will attribute the unbalance between these quantities to the decrease in rest-energy dE_0 and thus confirm Equation [4].

The variational problem of the energy-restricted flight can be formulated as follows: maximize u, from Equation [1], while keeping E_0 constant, with m_0 varying from m_{01} to m_{02}. By use of Equation [4], the problem is to extremize the following integral:

$$(6) \qquad \int_{m_{01}}^{m_{02}}\left\{\frac{w}{m_0} + \lambda\,c^2\left(1-\sqrt{1-w^2/c^2}\right)\right\}dm_0 = \text{extr.}$$

where λ is a Lagrangian multiplier. The solution is

$$(7) \qquad \frac{w}{\sqrt{1-w^2/c^2}} = \frac{k}{m_0}$$

(k = constant). The classical result of Olds is immediately confirmed. Solved for w, Equation (7) gives

$$(8) \qquad w = \frac{k}{\sqrt{m_0^2 + k^2/c^2}}$$

To discuss the results, it is necessary to express the constant k by the total energy E_0. The algebra becomes rather involved and will not be reproduced here. It turns out that a practical way to handle the complicated expressions is to introduce a velocity w^* by the equation

$$(9) \qquad E_0 = \left(m_{01} - m_{02}\right)c^2\left(1-\sqrt{1-w^{*2}/c^2}\right)$$

Thus, w^* is the «untaylored» constant exhaust speed which would have been obtained by uniform distribution of E_0 over the mass $m_{01} - m_{02}$. It can be shown that

$$(10) \qquad k = \frac{w^*}{\sqrt{1-w^{*2}/c^2}}\sqrt{m_{01}\,m_{02} + (m_{01}-m_{02})^2\,(w^{*2}/4c^2)}$$

The final velocity is found, after lengthy calculations, as follows: define

$$(11) \qquad u^* = w^*\,\frac{m_{01}-m_{02}}{\sqrt{m_{01}\,m_{02}}}$$

The final velocity is

$$(12) \qquad u = u^*\left(1+u^{*2}/4c^2\right)^{1/2}\left(1+u^{*2}/2c^2\right)^{-1}$$

which confirms that if $u^* \rightarrow \infty$ for $m_{02} \rightarrow 0$, then $u \rightarrow c$.

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Classically, $u \rightarrow u^*$, and Equation (11) represents indeed the same relation between u^* and w^* as would have been found classically, from Old's analysis.

It may be also of interest to find w_1 and w_2, respectively, at take-off and burnout, from Equations (8) and (10); the expressions are

$$(13)\quad \frac{w_1}{w^*} = \frac{\sqrt{m_{01}\, m_{02} + (m_{01} - m_{02})^2\, (w^{*2}/4c^2)}}{m_{01} - (m_{01} - m_{02})\, (w^{*2}/2c^2)}$$

$$(14)\quad \frac{w_2}{w^*} = \frac{\sqrt{m_{01}\, m_{02} + (m_{01} - m_{02})^2\, (w^{*2}/4c^2)}}{m_{02} + (m_{01} - m_{02})\, (w^{*2}/2c^2)}$$

The classical limits follow easily as well as the relativistic effect that for $m_{02} \rightarrow 0$, $w_2 \rightarrow c$.

2. Discussion

The formulas derived above promise extraordinary gains in final speed in comparison to the «untaylored» operation, if the mass ratio m_{01}/m_{02} is large; this conclusion is the same for the relativistic and for the classical results. If, however, a touch of realism is added, the discussion takes a different turn.

First, let us consider the case of chemical rocket propulsion. The energy limitation is serious, but the consumptions of energy and propellant mass are connected by a fundamental constant. Professor *E. L. Resler* of Cornell University devised a system which at least in principle can achieve the tayloring for chemical rockets (private communication); let the exhaust gases generate electric energy in the early stages of flight, with a magneto-hydrodynamic generator inserted; this energy can be stored and added in the late operations of the rocket, to increase w when it helps the most, namely, when the rocket mass is small.

In the case of nuclear propulsion, it is a fundamental fact that the limitation in total energy is irrelevant. Nuclear devices now in the planning stage are seriously power-limited; this gives rise to a different set of problems of optimal operation, which are amply discussed in the literature elsewhere.

The capability of tayloring w for a nuclear device implies that the ratio of the consumption of the nuclear fuel to the consumption of the energized inert mass can be varied at will. Once such a capability is achieved, designs will tend towards high energy content at take-off and comparatively low amounts of inert fuel mass. This reduces the ratio m_{01}/m_{02} and thereby the effectiveness of tayloring.

To illustrate this point, it suffices to consider the classical limit, when w^* becomes

$$(15)\quad w^*_{cl} = \sqrt{\frac{2E_0}{m_{01} - m_{02}}}$$

and the final speed for optimum exhaust velocity variation is

$$(16)\quad u_{cl} = \sqrt{2E_0}\sqrt{\frac{1}{m_{02}} - \frac{1}{m_{01}}}$$

Put

$$(17)\quad E_0 = \varepsilon\, m_f, \quad m_{02} \cong m_f + m_p$$

and consider the maximum of u_{cl} from Equation (16) with the take-off weight m_{01} an the payload m_p kept constant, while the amount m_f of nuclear fuel is varied. The optimum is found for

$$(18)\quad m_f = \sqrt{m_{01}\, m_p} - m_p$$

and the maximum of u_{cl} is

$$(19)\quad u_{max} = \sqrt{2\varepsilon}\left(1 - \sqrt{\frac{m_p}{m_{01}}}\right)$$

For *constant* exhaust velocity, equal to w^*, the final speed is

$$(20)\quad u_s = \sqrt{2\varepsilon}\sqrt{\frac{m_f}{m_{01} - m_p - m_f}}\log\frac{m_{01}}{m_p + m_f}$$

which gives a transcendental equation for the maximum with respect to m_f. The biggest gain of u_{max} from Equation (19) against the maximum of u_s from Equation (20) is found for $m_p \rightarrow 0$, where it turns out to be 20 %; for $m_p/m_{01} = 0{,}01$, the gain is only about 10 %. The reason is that m_f according to Equation (18) is large, and m_{01}/m_{02} is not too big.

For best operation, it will be advantageous to dump the used nuclear fuel, as has been pointed out by *Huth* [4]. The above analysis naturally does not apply for such systems. However, dumping might possibly be practical only in stages, in which case even parts of the power plant may be ejected. For each stage, the simple argument holds that tayloring is effective only for large m_{01}/m_{02}; i. e., staging makes tayloring even less useful.

In order to discuss Huth's case in more detail, let d_{m0} be eliminated from Equations (1) and (4) to yield

$$(21)\quad \frac{du}{1 - u^2/c^2} = \frac{w\, dE_0}{c^2\left(1 - \sqrt{1 - w^2/c^2}\right)}\frac{1}{m_0}$$

It can be concluded that this equation always holds if dE_0 is used to accelerate particles to the velocity w, whether additional rest-mass is dumped or not. Thus, du is maximized for a given dm_0 at any moment if

$$(22)\quad \frac{w\, dE_0/dm_0}{c^2\left(1 - \sqrt{1 - w^2/c^2}\right)} = \text{extr.}$$

Now, put

$$(23)\quad dm_0 = -\,dm_j - \frac{dE_0}{c^2}(1 + \varphi)$$

where $\varphi\, dE_0/c^2$ is the dumped rest-mass; this replaces Equation (3), and with Equation (2), which is naturally valid, Equation (4) will be changed to

$$(24)\quad dE_0 = -\,dm_0\, c^2\,\frac{1 - \sqrt{1 - w^2/c^2}}{1 + \varphi\left(1 - \sqrt{1 - w^2/c^2}\right)}$$

The quantity to be extremized in Equation (22) becomes

$$(25)\quad \frac{w}{1 + \varphi\left(1 - \sqrt{1 - w^2/c^2}\right)} = \text{extr.}$$

The solution is

$$(26)\quad 1 - \sqrt{1 - w^2/c^2} = \frac{1}{\varphi + 1} \equiv \varepsilon$$

or

$$(27)\quad \left(\frac{w}{c}\right)_{opt} = \sqrt{2\varepsilon - \varepsilon^2}$$

in accordance with Huth's result; as noted by Huth, the optimum occurs when the dumped mass and the energized inert mass are equal:

$$(28)\quad dm_j = \varphi\,\frac{dE_0}{c^2}$$

This truly significant optimum shows that exhaust velocity should not be maximized, but the expression (25), which is proportional to a properly defined «specific impulse». If staging is interpreted in the continuous limit (which is extreme) as a gradually ejected powerplant, φ is further increased, indicating even lower optimal exhaust velocities.

Huth's optimum is valid in any given moment, and unless an artificial total energy restriction makes its continuous application impossible, it is the best operation for the whole flight.

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