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Autor: Hartinger, Jürgen / Kortschak, Dominik

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B. Wissenschaftliche Mitteilungen

JÜRGEN HARTINGER[a], DOMINIK KORTSCHAK[b]

Quasi-Monte Carlo Techniques and Rare Event Sampling*

1 Introduction

In today's simulation literature a lot of attention is attracted to the estimation of small probabilities. This in particular plays a role in managing rare event risk in insurance, but also has applications in areas like queueing theory. Explicit or easily computable solutions are then typically not available such that simulation is required even for simple problems. In this article we are concerned with a standard problem within rare event sampling. Let $Y_1, Y_2, \ldots$ be independent, identically distributed random variables (with generic random variable Y) with cumulative distribution function F and tail $\bar{F} = 1 - F$, and N an integervalued random variable (independent of the Y_i's) on a suitable probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The goal is the evaluation of

$$z(u) = \mathbb{P}(S_N > u) = \mathbb{P}(Y_1 + \ldots + Y_N > u). \tag{1}$$

Observe that $\lim_{u \to \infty} z(u) = 0$, such that for large u we have the problem of simulating a rare event. Even this simple problem has practical relevance, e. g. for the estimation of the probability of ruin in classical insurance risk models, the steady state waiting times for queues, see e. g. (AS03), or for the valuation of catastrophe risk bonds within a collective risk model, e. g. (AHT03; AHT04). The typical magnitude for $z(u)$ in applications ranges from 10^{-2} to 10^{-10}.
Let us first recall the behavior of a crude Monte Carlo estimator, namely $Z_1(u) = I_{\{S_N > u\}}$. Z_1 is Bernoulli distributed with parameter $z(u)$ and variance $z(u)(1 - z(u))$. Since the goal is to have a competitive relative error, our quantity of interest is the (squared) coefficient of variation $COV^2(Z_1(u)) = \frac{Var(Z_1(u))}{z^2(u)} = \frac{1-z(u)}{z(u)}$. For $u \to \infty$, we have $COV^2(Z_1(u)) \sim z(u)^{-1}$. Hence, asymptotically the number of paths needed to guarantee a fixed relative Monte Carlo error grows to infinity and, technically, we face a (nontrivial) variance reduction problem. In the literature, the following efficiency classes for suitable estimators are distinguished with respect to the behavior of their squared COV: We say $Z(u)$

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- is *logarithmically efficient* if $\forall\epsilon > 0 : \overline{\lim}_{u\to\infty} \frac{Var(Z(u))}{z^{2-\epsilon}(u)} < \infty$,
- has *bounded relative error* if $\frac{Var(Z(u))}{z^2(u)} < \infty$,
- has *vanishing relative error* if $\lim_{u\to\infty} \frac{Var(Z(u))}{z^2(u)} = 0$.

The design of good estimators heavily depends on the existence of exponential moments of the random variable Y. In the light tail case, (i. e. $\exists\, t > 0 : E[\exp(tY_1)] < \infty$), estimators with bounded relative error may be found by an exponential change of measure, as determined by a saddlepoint method, see e. g. (SIG76; AS03). For heavy tails, methods for the subclass of subexponential distributions were intensively studied in the last decade. A distribution function F is said to be subexponential ($F \in \mathcal{S}$) if for fixed $n \geq 2$:

$$\lim_{u\to\infty} \frac{\mathbb{P}(Y_1 + \cdots + Y_n > u)}{\mathbb{P}(\max\{Y_1, \ldots, Y_n\} > u)} = n.$$

Hence, the tail of the sum of iid random variables behaves asymptotically like the tail of maximum of the summands. Important elements of $\mathcal{S}$ are

- the Lognormal distribution: $Y = e^X$, where X is Gaussian,
- Weibull-type distributions: $\bar{F} \sim cu^{1+\gamma-\beta}e^{-x^\beta}, 0 < \beta < 1$,
- Regularly varying distributions with index $\alpha > 0 : \bar{F}(u) = u^{-\alpha}L(u)$, where $L(u)$ is slowly varying, (i. e. for all $t > 0 : \lim_{u\to\infty} L(tu)/L(u) = 1$).

For a thorough introduction to subexponential distributions and modeling with heavy tails consult the monograph (EKM97).
Good Monte Carlo estimators in this setting are either obtained by importance sampling through twisting the hazard rate, see e. g. (JS02; HS03; AKR05), or by conditional Monte Carlo methods, see (AB97; ABH00; AK06).
In the sequel, we are mainly interested in recent work by Asmussen and Kroese (AK06), who propose the following two estimators for $z(u)$ given in (1):

$$Z_2(u) = N\ \bar{F}(\max\{u - S_{N-1}, M_{N-1}\}),$$

and its control variate counterpart in terms of N,

$$Z_3(u) = N\ \bar{F}(\max\{u - S_{N-1}, M_{N-1}\}) + (E[N] - N)\bar{F}(u),$$

where $M_n = \max\{Y_1, \ldots, Y_n\}$. They show that $Z_2(u)$ has bounded relative error for regularly varying Y (under mild conditions on N) and is logarithmically efficient for Weibull Y, given that $2^{1+\beta} < 3$ and N bounded. In Hartinger and Kortschak (HK09) it is shown that $Z_2(u)$ has bounded relative error for the lognormal (again under mild conditions on N) and the Weibull case (given

$\beta < \log(3/2)/\log(3)$ and N bounded). Furthermore, it is shown that under the same conditions, $Z_3(u)$ has vanishing relative error.
Section 2 reviews the quasi-Monte Carlo methodology and shows that asymptotically the effective dimension of Z_2 is 1. The speed of convergence to this limit is analyzed for Pareto Y. In Section 3 numerical results comparing Monte Carlo and randomized quasi-Monte Carlo (QMC) methods (in the sense of Wang and Fang (WF03)) are given.

2 Rare event sampling and effective dimension

In contrast to Monte Carlo methods, the quasi-Monte Carlo integration error can be bounded deterministically due to the famous Koksma-Hlawka Theorem (HL61) by the product of the discrepancy of the used sequences and the integrand's variation in the sense of Hardy and Krause, $V(f)$. For a thorough introduction consult the monograph (NIE92). Let $\{x_m\}_{1\le m\le M}$ be a point sequence in $[0,1)^s$, $D^*_M(x_m)$ denote the star discrepancy of $(x_1,\ldots,x_M)$ and $V(f)<\infty$. Then,

$$\left|\int_{[0,1]^s} f(x)dx - \frac{1}{M}\sum_{i=1}^{M} f(x_i)\right| \le V(f)D^*_M(x_m). \tag{2}$$

Since the best known sequences (so-called low discrepancy sequences) have a discrepancy of order $\mathcal{O}(\log^s(M)/M)$, QMC techniques are at least asymptotically superior to Monte Carlo simulation, the probabilistic error of which is known to be of order $\mathcal{O}(1/\sqrt{M})$. It was frequently shown empirically that there are extremely high dimensional problems ($s = 360$ and more, e. g. (PT95)) occurring in mathematical finance, where QMC methods outperform Monte Carlo algorithms by far for reasonable M. One way to classify types of integrands that are particularly well suited for QMC integration is the notion of effective dimension based on the ANOVA decomposition, e. g. (CMO97; WF03).
Let $f(\mathrm{x})$ be a function in $\mathcal{L}^2(U^s)$ and $\nu \subseteq \{1,\ldots,s\} = S, |\nu|$ its cardinality, x_ν the $|\nu|$-dimensional vector having the coordinates of x with the indices of ν and U^ν denoting the corresponding unit cube. Denote the integral value $I(f)=\int_{U^s} f(\mathrm{x})d\mathrm{x}$ by $f_\emptyset$ and let $f_\nu(\mathrm{x})= \int_{U^{S\setminus\nu}} f(\mathrm{x})d\mathrm{x}_{S\setminus\nu} - \sum_{\gamma\subset\nu} f_\gamma(\mathrm{x})$. Then the ANOVA decomposition is defined by $f(\mathrm{x}) = \sum_{\emptyset\neq\nu\subset S} f_\nu(\mathrm{x})$. Let $Var(f)$ be the variance $Var(f(U))$, where U denotes a uniformly distributed random variable on the corresponding unit cube. It is well known that with these definitions $\int_0^1 f_\nu(\mathrm{x})dx_j = 0$ for all $j\in\nu$, $\int_{U^s} f_\nu(x)f_\gamma(x)d\mathrm{x} = 0$ for $\nu\neq\gamma$ and that $Var(f) = \sum_{\nu\subseteq S} Var(f_\nu)$.

This gives the following natural notions of the importance of the coordinates:

- For $0 < p < 1$ (typically p close to one), the effective truncation dimension of the function f is defined by the smallest integer d_t, such that there exists a set ν with cardinality d_t and $\sum_{\gamma \subseteq \nu} Var(f_\gamma) > p\, Var(f)$.
- The smallest integer d_s such that $\sum_{0 \leq |\nu| \leq d_s} Var(f_\nu) > p\, Var(f)$ holds, is called effective superposition dimension.

2.1 *The effective dimension of Z_2*

Asmussen and Kroese (AK06) remark that the asymptotic behavior of the subexponential case in (1), indicates that a substantial part of the variability of Z_2, may be due to the variability in N, which is their motivation to propose the estimator Z_3. This remark motivates to calculate the effective dimension of Z_2 explicitly:

Lemma 2.1 Let $F \in S$ and $E[z^N] < \infty$ for some $z > 1$. The ANOVA term of $Z_2(u)$ corresponding to N is given by

$$g_0(n) := \mathbb{P}(S_n > u) - \mathbb{P}(S_N > u) = \bar{F}^{n*}(u) - z(u).$$

Remark 2.1 Note that $g_0(n)$ is a deterministic function of the variable n.

Proof. We have

$$\begin{aligned} g_0(n) &= E[N\, \bar{F}(M_{N-1} \vee (u - S_{N-1})) - E[Z_2] | N = n] \\ &= nE[\mathbb{P}(S_n > u, M_n = X_n | X_1, \ldots, X_{n-1})] - E[Z_2] \\ &= n\mathbb{P}(S_n > u, M_n = X_n) - \mathbb{P}(S_N > u) = \mathbb{P}(S_n > u) - \mathbb{P}(S_N > u). \end{aligned}$$

□

Lemma 2.2 Let $F \in S$ and $E[z^N] < \infty$ for some $z > 1$. The asymptotic variance of the random variable $g_0(N)$ is given by

$$\lim_{u \to \infty} \frac{Var[g_0(N)]}{\bar{F}(u)^2} = Var[N].$$

Proof. Define $\mathbb{P}(N = n) = p_n$.

$$
\begin{aligned}
\lim_{u\to\infty} \frac{Var[g_0(N)]}{\bar{F}(u)^2} &= \lim_{u\to\infty} \left(\sum_{n=0}^{\infty} p_n \frac{\bar{F}^{n*}(u)^2}{\bar{F}(u)^2} - \frac{\mathbb{P}(S_N > u)^2}{\bar{F}(u)^2} \right) \\
&= \lim_{u\to\infty} \sum_{n=0}^{\infty} p_n \frac{\bar{F}^{n*}(u)^2}{\bar{F}(u)^2} - \lim_{u\to\infty} \frac{\mathbb{P}(S_N > u)^2}{\bar{F}(u)^2} \\
&= \sum_{n=0}^{\infty} p_n \left(\lim_{u\to\infty} \frac{\bar{F}^{n*}(u)}{\bar{F}(u)} \right)^2 - E[N]^2 \qquad (3) \\
&= \sum_{n=0}^{\infty} n^2 p_n - E[N]^2 \qquad (4) \\
&= E[N^2] - E[N^2] = Var[N].
\end{aligned}
$$

By assumption $E[z^N] < \infty$, interchanging limit and summation in (3) is justified by dominated convergence as $(\bar{F}^{n*}(u)/\bar{F}(u))^2 \le Kz^n$. Equation (3) follows from $\lim_{u\to\infty} \mathbb{P}(S_N > u)/\bar{F}(u) = E[N]$ and Equation (4) by $\lim_{u\to\infty} \mathbb{P}(S_n > u)/\bar{F}(u) = n$. □

The asymptotic variance of $Z_2(u)$ is obtained by the following theorem:

Theorem 2.3 (HK09). If either

- Y is regularly varying and $E[z^N] < \infty$,
- Y is Lognormal and $E[z^N] < \infty$,
- Y is Weibull and N bounded,

then

$$\lim_{u\to\infty} \frac{Var[Z_2(u)]}{P(S_N > u)^2} = \frac{Var[N]}{E[N]^2}.$$

Now, it is easy to show that in the limit $u \to \infty$ the effective dimension of $Z_2(u)$ in both senses converges to 1 for any choice of p.

Corollary 2.4 Let N be nondegenerate. Under the conditions of Theorem 2.3, we have

$$\lim_{u\to\infty} \frac{Var[g_0(N)]}{Var[Z_2(u)]} = 1.$$

Proof.

$$\lim_{u\to\infty}\frac{Var[g_0(N)]}{Var[Z_2(u)]}=\lim_{u\to\infty}\frac{Var[g_0(N)]/\bar{F}(u)^2}{Var[Z_2(u)]/\bar{F}(u)^2}=\frac{Var[N]}{Var[N]}=1.$$

□

In the last part of this section, let us have a look at the convergence speed of the effective dimension to 1 for the case that the Y_i are regularly varying, fulfilling a certain regularity condition.

Theorem 2.5 Let $\bar{F}(x)$ be regularly varying with index $\alpha>0$ and regularly varying density $f(x)$. Further let $E[N^{3(\alpha+\delta+1)}]<\infty$, for a $\delta>0$. Then there exists a u_0 and a constant k, such that for all $u>u_0$:

$$0\leq 1-\frac{Var[g_0(N)]}{Var[Z_2(u)]}\leq\begin{cases}ku^{-1} & if\, E[Y_1]<\infty\\ k\bar{F}(u) & if\, \alpha<1\end{cases}.$$

Further if $\alpha=1$ and $E[Y_1]=\infty$ then there exists a slowly varying function $L_1(x)$ such that

$$\lim_{x\to\infty}\frac{L_1(x)}{x\bar{F}(x)}=\infty$$

and

$$0\leq 1-\frac{Var[g_0(N)]}{Var[Z_2(u)]}\leq\frac{L_1(u)}{u}.$$

For the proof, we shall need the following two lemmata:

Lemma 2.6 For all $u>0$ and $n>1$, $1-\left(\frac{\bar{F}^{n*}(u)}{n\bar{F}(u)}\right)^2\leq 2(n-1)\bar{F}(u)$.

Proof. Consider the inequality

$$\begin{aligned}\bar{F}^{n*}(x)&=\mathbb{P}\left(\sum_{i=1}^{n}X_i>x\right)\geq\mathbb{P}\left(\bigcup_{i=1}^{n}\{X_i>x,\max_{j\neq i}(X_j)\leq x\}\right)\\&=\sum_{i=1}^{n}\mathbb{P}(X_i>x)\mathbb{P}(\max_{j\neq i}(X_i)\leq x)=n\bar{F}(x)F(x)^{n-1}.\end{aligned}\tag{5}$$

From (5), we have

$$1-\frac{\bar{F}^{n*}(u)^2}{n^2\bar{F}(u)^2}\leq 1-\frac{n^2\bar{F}(u)^2 fF(u)^{2(n-1)}}{n^2\bar{F}(u)^2}=1-F(u)^{2(n-1)}\leq 2(n-1)\bar{F}(u).$$

□

Remark 2.2 Observe that, by $1-\left(\frac{\bar{F}^{n*}(u)}{n\bar{F}(u)}\right)^2 = \left(1+\frac{\bar{F}^{n*}(u)}{n\bar{F}(u)}\right)\left(1-\frac{\bar{F}^{n*}(u)}{n\bar{F}(u)}\right)$, the bound in Lemma 2.6 is closely related to results on the second order tail behaviour of subordinated regularly varying distributions, see e.g. (OW86; GL96). However, Lemma 2.6 gives a bound that is appropriate for the whole range of interest for the parameters u and α and furthermore is uniform in n.

Lemma 2.7 Let $\bar{F}(x)$ be regularly varying with index $\alpha > 0$ and regularly varying density $f(x)$, then for every $\delta > 0$ there exists a u_1 and a k_2 such that for all $u > u_1$

$$\frac{E[\bar{F}(M_{n-1} \vee (u-S_{n-1}))^2]}{\bar{F}(u)^2} - 1 \leq \begin{cases} k_2 n^{3(\alpha+\delta)+1} u^{-1} & if\ E[Y_1] < \infty \\ k_2 n^{3(\alpha+\delta)+1} \bar{F}(u) & if\ \alpha < 1 \end{cases}.$$

Further if $\alpha = 1$ and $E[Y_1] = \infty$ then there exists a slowly varying function $L_1(x)$ such that

$$\lim_{x\to\infty} \frac{L_1(x)}{x\bar{F}(x)} = \infty$$

and

$$\frac{E[\bar{F}(M_{n-1} \vee (u-S_{n-1}))^2]}{\bar{F}(u)^2} - 1 \leq n^{3(\alpha+\delta)+1} \frac{L_1(u)}{u}$$

Proof. At first we provide some well known bounds for the functions $\bar{F}(x)$ and $f(x)$. Note that $(1+x)^\alpha \bar{F}(x)$ is slowly varying and bounded away from 0 and ∞ for every compact subset of $[0,\infty)$. We get from (BGT87, Theorem 1.5.6) that for every $\delta > 0$ there exists a c_1 such that for all $x \geq y > 0$

$$\frac{\bar{F}(y)}{\bar{F}(x)} = \left(\frac{1+x}{1+y}\right)^\alpha \frac{(1+y)^\alpha \bar{F}(y)}{(1+x)^\alpha \bar{F}(x)} \leq c_1 \left(\frac{1+x}{1+y}\right)^\alpha \left(\frac{x}{y}\right)^\delta \leq c_1 \left(\frac{x}{y}\right)^{\alpha+\delta}. \tag{6}$$

Further we get from (BGT87, Theorem 1.5.6) that for every $\delta > 0$ and $c_2 > 0$ there exists a u_2 such that for all $x \geq y > u_2$ and

$$\frac{f(y)}{f(x)} \leq c_2 \left(\frac{x}{y}\right)^{\alpha+\delta+1}. \tag{7}$$

From Karamata's Theorem (BGT87, Proposition 1.5.10) we get that there exists an u_3 and a c_3 such that for all $u > u_3$

$$\frac{f(u)}{\bar{F}(u)} \leq \frac{c_3}{u}. \tag{8}$$

At last note that from (BGT87, Theorem 1.5.3) it follows that for every $u_4 > 0$ there exists a c_4 such that for all $u > u_2$

$$\sup_{t>u} f(t) \leq c_4 f(u). \tag{9}$$

For every $u > 0$ we have

$$\frac{E[\bar{F}(M_{n-1} \vee (u - S_{n-1}))^2]}{\bar{F}(u)^2} - 1$$
$$= \sum_{k=1}^{\left\lceil \frac{u}{2(n-1)} \right\rceil} \left(\frac{E[\bar{F}(M_{n-1} \vee (u - S_{n-1}))^2; k-1 < M_{n-1} \leq k \wedge \frac{u}{2(n-1)}]}{\bar{F}(u)^2} \right.$$
$$\left. -\mathbb{P}\left(k-1 < M_{n-1} \leq k \wedge \frac{u}{2(n-1)}\right)\right)$$
$$+\frac{E[\bar{F}(M_{n-1} \vee (u - S_{n-1}))^2; M_{n-1} > \frac{u}{2(n-1)}]}{\bar{F}(u)^2} - \mathbb{P}\left(M_{n-1} > \frac{u}{2(n-1)}\right).$$

Note that it follows from $M_{n-1} \vee (u - S_{n-1}) \geq u/n$ that

$$\frac{E[\bar{F}(M_{n-1} \vee (u - S_{n-1}))^2; M_{n-1} > \frac{u}{2(n-1)}]}{\bar{F}(u)^2} \leq \frac{\bar{F}(u/n)^2}{\bar{F}(u)^2} \mathbb{P}\left(M_{n-1} > \frac{u}{2(n-1)}\right)$$

With Bernoulli inaquality and (6) we get

$$\frac{\bar{F}(u/n)^2}{\bar{F}(u)^2} \mathbb{P}\left(M_{n-1} > \frac{u}{2(n-1)}\right) \leq (n-1) \frac{\bar{F}(u/n)^2}{\bar{F}(u)^2} \bar{F}\left(\frac{u}{2(n-1)}\right)$$
$$\leq c_1^3 2^{\alpha+\delta} n^{2(\alpha+\delta)} (n-1)^{\alpha+\delta+1} \bar{F}(u) \leq c_1^3 2^{\alpha+\delta} n^{3(\alpha+\delta)+1} \bar{F}(u)$$

With Taylor formula and (9) we get that there exists an u_4 such that for $u > 2u_4$

$$\sum_{k=1}^{\left\lceil \frac{u}{2(n-1)} \right\rceil} \left(E[\bar{F}(M_{n-1} \vee (u - S_{n-1}))^2; k-1 < M_{n-1} \leq k \wedge \frac{u}{2(n-1)}] \right.$$
$$\left. -\mathbb{P}\left(k-1 < M_{n-1} \leq k \wedge \frac{u}{2(n-1)}\right)\right)$$
$$\leq \sum_{k=1}^{\left\lceil \frac{u}{2(n-1)} \right\rceil} \frac{\bar{F}\left(u - (n-1)k \wedge \frac{u}{2(n-1)}\right)^2 - \bar{F}(u)^2}{\bar{F}(u)^2} \mathbb{P}\left(k-1 < M_{n-1} \leq k \wedge \frac{u}{2(n-1)}\right)$$
$$\leq c_a \frac{2(n-1) f(u/2) \bar{F}(u/2)}{\bar{F}(u)^2}$$

$$\times \sum_{k=1}^{\left\lceil \frac{u}{2(n-1)} \right\rceil} \left(k \wedge \frac{u}{2(n-1)}\right) \mathbb{P}\left(k-1 < M_{n-1} \leq k \wedge \frac{u}{2(n-1)}\right).$$

By (6), (7) and (8) we get that there exists a u_5 such that for $u > 2u_5$

$$\frac{f(u/2)\bar{F}(u/2)}{\bar{F}(u)^2} \leq \frac{c_3 c_2 2^{2(\alpha+\delta)+1}}{u}$$

Further note that

$$\sum_{k=1}^{\left\lceil \frac{u}{2(n-1)} \right\rceil} \left(k \wedge \frac{u}{2(n-1)}\right) \mathbb{P}\left(k-1 < M_{n-1} \leq k \wedge \frac{u}{2(n-1)}\right)$$

$$\leq (n-1) \int_0^{\frac{u}{2(n-1)}} (x+1) F(x)^{n-2} f(x) dx \leq (n-1) \int_0^{\frac{u}{2(n-1)}} (x+1) f(x) dx. \tag{10}$$

For $E[Y_1] < \infty$ the Lemma follows with

$$\int_0^{\frac{u}{2(n-1)}} (x+1) f(x) dx \leq \int_0^{\infty} (x+1) f(x) dx = E[Y_1 + 1].$$

In the case $\alpha = 1$ and $E[Y_1] = \infty$ the Lemma follows from (BGT87, Proposition 1.5.9a). In the case $\alpha < 0$ the Lemma follows from Karamata' s Theorem (BGT87, Proposition 1.5.8)

$$\int_0^{\frac{u}{2(n-1)}} (x+1) f(x) dx = \int_0^{\frac{u}{2(n-1)}} \bar{F}(x) dx - \frac{u}{2(n-1)} \bar{F}\left(\frac{u}{2(n-1)}\right) + 1$$

$$\leq \frac{c_5 u}{(1-\alpha) 2(n-1)} \bar{F}\left(\frac{u}{2(n-1)}\right) + 1 \leq \frac{c_5 c_1}{1-\alpha} (2(n-1))^{\alpha+\delta-1} u \bar{F}(u) + 1.$$

□

Proof of Theorem 2.5. We will only consider the case $E[Y_1] < \infty$, since the proof for the other cases is analogous. Let $P(N = n) = p_n$, $A(u) = \sum_{n=1}^{\infty} p_n n^2 \bar{F}(u)^2 - z(u)^2$ and $c_1 = 2E[N^3]$. Then,

$$\begin{aligned}
Var[g_0(N)] &= \sum_{n=0}^{\infty} p_n \bar{F^{n*}}(u)^2 - z(u)^2 = A(u) + \sum_{n=1}^{\infty} p_n \left(\bar{F^{n*}}(u) - n^2 \bar{F}(u)^2\right) \\
&= A(u) + \sum_{n=1}^{\infty} p_n n^2 \bar{F}(u)^2 \left(\frac{\bar{F^{n*}}(u)^2}{n^2 \bar{F}(u)^2} - 1\right) \\
&\geq A(u) - c_1 \bar{F}(u)^3.
\end{aligned}$$

Furthermore, for $c_2 = k_2 E[N^{3(\alpha+\delta+1)}]$ (k_2 of Lemma 2.7), we have

$$\begin{aligned}
Var[Z_2(u)] &= \sum_{n=0}^{\infty} p_n n^2 E[\bar{F}(M_{n-1} \vee (u - S_{n-1}))^2] - z(u)^2 \\
&= A(u) + \sum_{n=1}^{\infty} p_n (n^2 E[\bar{F}(M_{n-1} \vee (u - S_{n-1}))^2] - n^2 \bar{F}(u)^2) \\
&= A(u) + \sum_{n=1}^{\infty} p_n n^2 \bar{F}(u)^2 \left(\frac{E[\bar{F}(M_{n-1} \vee (u - S_{n-1}))^2]}{\bar{F}(u)^2} - 1\right) \\
&\leq A(u) + c_2 u^{-1} \bar{F}(u)^2.
\end{aligned}$$

From $\mathbb{P}(S_N > u) \sim E[N]\bar{F}(u)$ and $E[N^2] - E[N]^2 = Var[N] > 0$ follows the existence of a constant $c_3 > 0$ together with an $u_0 > 0$ such that for $u > u_0$

$$A(u) = \sum_{n=1}^{\infty} p_n n^2 \bar{F}(u)^2 - z(u)^2 = E[N^2]\bar{F}(u)^2 - \mathbb{P}(S_N > u)^2 \geq c_3 \bar{F}(u)^2.$$

Thus,

$$\begin{aligned}
1 - \frac{Var[g_0(N)]}{Var[Z_2(u)]} &\leq 1 - \frac{A(u) - c_1 \bar{F}(u) \bar{F}(u)^2}{A(u) + c_2 u^{-1} \bar{F}(u)^2} \\
&\leq \frac{(c_2 u^{-1} + c_1 \bar{F}(u)) \bar{F}(u)^2}{(c_3 + c_2 u^{-1}) \bar{F}(u)^2} \leq k u^{-1}.
\end{aligned}$$

□

3 Numerical results

In this part we present numerical illustrations for the proposed algorithms, comparing effects for Monte Carlo and QMC techniques.

Observe, that QMC integration of $Z_2(u)$ and $Z_3(u)$ is not directly applicable for two reasons. Formally, the integrands have infinite dimension as N has infinite support. For practical purposes, this problem has been solved by cutting off the integrand after a suitable large number of claims as the contributions of these

ρ	$A(u)$	MC	stH	shH	shS	MC	stH	shH	sS
0.25	0.01	0.052	0.015	0.016	0.015	0.031	0.015	0.016	0.014
0.25	1e-05	0.031	0.0021	0.0026	0.0024	0.0014	0.0012	0.0012	0.0012
0.25	1e-08	0.031	0.0018	0.0022	0.0021	3.2e-05	4.8e-05	3.0e-05	3.3e-05
0.25	1e-11	0.031	0.0017	0.0023	0.0021	2.6e-07	3.0e-07	1.2e-06	3.6e-07
0.5	0.01	0.077	0.027	0.030	0.029	0.052	0.028	0.029	0.028
0.5	1e-05	0.044	0.0026	0.0032	0.0031	0.0015	0.0014	0.0013	0.0013
0.5	1e-08	0.044	0.0022	0.0029	0.0026	5.4e-05	5.6e-05	2.7e-05	0.00013
0.5	1e-11	0.044	0.0022	0.0029	0.0026	4.2e-07	3.5e-07	3.2e-07	2.2e-07
0.75	0.01	0.11	0.055	0.059	0.058	0.091	0.054	0.058	0.057
0.75	1e-05	0.054	0.0044	0.0066	0.0062	0.0020	0.0015	0.0015	0.0015
0.75	1e-08	0.054	0.0041	0.0063	0.0060	3.7e-05	2.7e-05	2.4e-05	2.3e-05
0.75	1e-11	0.054	0.0040	0.0064	0.0060	2.8e-07	3.8e-07	3.2e-07	3.8e-07

Table1: Half-length of a 95 % confidence interval expressed as a percentage of the estimated value for the estimators Z_2 (first 4 columns) and Z_3 (last 4 columns) for different randomized QMC methods; Y is Pareto distributed with $\alpha = 1.5$.

large claim numbers are negligible. Furthermore, the integrands are not of bounded variation in the sense of Hardy and Krause due to cusps induced by the max-function, see e. g. (OW05). Here, we propose to apply the Chen-Harker-Kanzov-Smale function $f(u, v, t) = (\sqrt{(u-v)^2 + t^2} + u + v)/2$ widely used in the literature for approximations of the max-function. Observe, that $\lim_{t\to 0} f(u, v, t) = \max\{u, v\}$. For fixed precision one can choose t large enough such that the error induced by this approximation is negligible, but the variation of the integrand is bounded. Asymptotically (i. e. for $M \to \infty$) this does not lead to efficient error estimates. (The order obtained by a straight-forward 3-epsilon argument is $\mathcal{O}((M^{-1/s} log\, M))$.) We give a brief numerical illustration (a thourough numerical analysis for a whole range of randomized QMC methods and rare event estimators may be found in (K05)). As in (AK06), we consider Pareto-distributions with $\bar{F}(x) = (1+x)^{-\alpha}, \alpha \in \{0.5, 1.5\}$ for the Y. The number of summands was chosen geometrically, i. e. $P(N = n) = p_n = p^n(1-\rho), \rho \in \{0.25, 0.5, 0.75\}$. The threshold u is picked, such that the asymptotic approximation $A(u) = \rho/(1-\rho)\bar{F}(u)$ of $P(S_N > u)$ has the form $z(u) \in \{10^{-k} | k \in \{2, 5, 8, 11\}\}$. As in (AK06) we used a variance reduction technique to avoid realisations with $N = 0$. For every setting, we compare MC methods and three randomized QMC methods: Halton (shH) resp. Sobol (sHS) sequences with random shift, cf. (CP76; TU96), and random start Halton (stH)

sequences, cf. (WH00). For the generation of pseudorandom-numbers we use Mersenne Twister, see (MN98). In Table1, 10^7 iterations for the MC estimates and 10^4 random QMC sequences with length 10^3 were used. Figure1 shows a log-log plot of the length of a 95 %-CI in procentage of the estimated value and the number of iterations for MC and random start Halton estimators.
We see in Table 1 as well as in the Figure 1 that the confidence intervals are significantly smaller in all quasi-Monte Carlo methods compared to MC for the estimator $Z_2(u)$. The effect of QMC in the variance reduction is increasing far out in the tails. These effects were expected by the results in Section 2.1 For the estimator $Z_3(u)$ in particular for large u, this effect is not as strong as for $Z_2(u)$, but observe that one gets relative errors less than 1 % with just 100 iterations for $A(u) = 10^{-5}$.

4 Conclusion

In this paper we have investigated the applicability of Quasi Monte Carlo techniques to a recently proposed rare event estimator $Z_2(u)$ for the tail probability of aggregate claim distributions. Further we studied its control variate version $Z_3(u)$. We have seen that the effective dimension of the estimator $Z_1(u)$ tends to one as $u \to \infty$, which means that asymptotically the variance of the estimator is determined by the variance of a single dimension, namely the number of claims N. This gives a strong indication that Quasi Monte Carlo should perform well for this estimator. On the other hand the estimator $Z_3(u)$ uses a control variate to reduce the variance induced by the number of claims N. The numerical example shows that especially for the estimator $Z_2(u)$, Quasi Monte

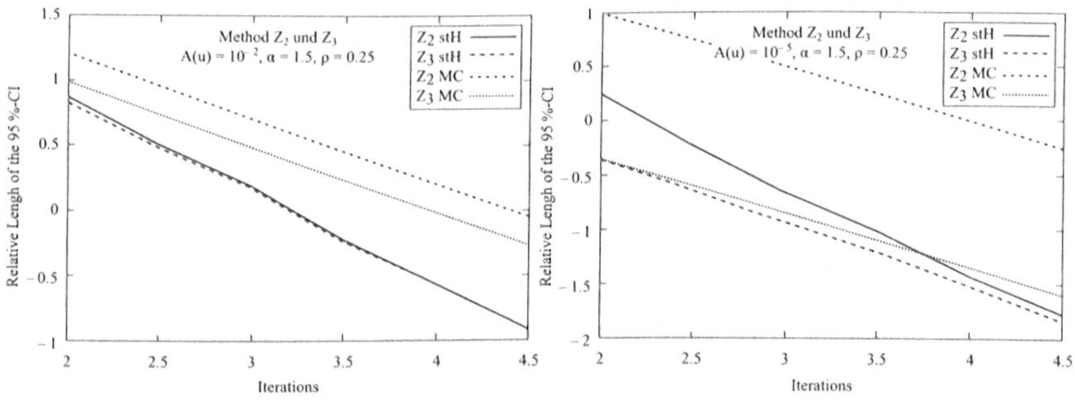

Figure 1: Log-Log-Plot: Comparison of Monte and quasi-Monte Carlo rare event techniques; $Y_i \sim$ Pareto (1.5), $N \sim$ Geometric (0.25) and $A(u) = 10^{-2}$ (left) and $A(u) = 10^{-5}$ (right).

Carlo improves significantly over Monte Carlo. For the estimator $Z_3(u)$, the improvement is only significant when the value u is moderate, but one should note that also this case is of relevance in practice and hence one should combine Quasi Monte Carlo techniques with the estimator $Z_3(u)$.

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Jürgen Hartinger
Kärntner Landesversicherung auf Gegenseitigkeit
Faculty of Buisness and Economics
Domgasse 21
A-9020 Klagenfurt

Juergen.Hartinger@klv.at

Dominik Kortschak
Department of Actuarial Science
Faculty of Buisness and Economics
University of Lausanne

dominik.kortschak@unil.ch

Abstract

In the last decade considerable practical interest, e.g. in credit and insurance risk applications, as well as methodological challenges caused intensive research on estimation of rare event probabilities. This article aims to show that recently developed rare event estimators are especially well-suited for a quasi-Monte Carlo framework. We establish limit relations for the so-called effective dimension and propose smoothing methods to overcome problems with cusps of the integrand.

Keywords:

Rare event, relative error, effective dimension, randomized quasi-Monte Carlo, heavy-tailed distributions