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D. Kurzmitteilungen

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On the non-optimality of proportional reinsurance according to the dividend criterion

In a recent paper in this journal, Beveridge et al. (2008) considered the classical compound Poisson risk model. They studied the effect of (static) proportional and excess-of-loss reinsurance on the expected difference between the discounted dividends until ruin and the discounted penalty at ruin. According to this criterion, and based on their numerical investigations, they conjecture that no reinsurance is better than proportional reinsurance. The goal of this note is to give a theoretical background to this conjecture and to show that it is true in a more general setting and under fairly general conditions.

Without reinsurance, the surplus of the company at time t is

$$U(t;u) = u + ct - S(t),$$

where u is the initial surplus, $S(t)$ the aggregate claims up to time t, and $c > S(1)$ is the constant rate at which the premiums are received. The maximal value function $W(u)$ is defined as follows. For given u, $W(u)$ is the maximal expected difference between the discounted dividends until ruin and the discounted penalty at ruin; the maximum is taken with respect to all dividend strategies.

Proportional reinsurance is available; the retained fraction of the claims is denoted by a, $0 < a < 1$. First we assume that the relative loading contained in the reinsurance premium is the same as in c. Then, with proportional reinsurance corresponding to the parameter a, the surplus of the company at time t is

$$U_a(t;u) = u + a \cdot ct - a \cdot S(t) = a \cdot U\Big(t;\frac{u}{a}\Big).$$

With proportional reinsurance, the maximal expected difference between the discounted dividends and the discounted penalty is denoted as $W_a(u)$. By a change of scale, we see that

$$W_a(u) = a \cdot W\Big(\frac{u}{a}\Big).$$

As a consequence, $W_a(u)$ can be obtained by the geometric construction that is shown in Figure 1. Suppose now that the function $W(u)$ satisfies the following Condition C: for any $x > 0$, the ray between the origin and the point $(x, W(x))$ is below the graph of the function. Then $W_a(u) < W(u)$ for all a and u, and we conclude that for any u no reinsurance $(a = 1)$ is better than any proportional reinsurance. We note that Condition C is for instance satisfied in the particular case, where the graph of $W(u)$ is concave (see the recent paper of Loeffen and Renaud (2010) for general results on the shape of $W(u)$). Finally, if the relative loading of the reinsurance premium exceeds the one contained in c, the conclusion is a fortiori the same.

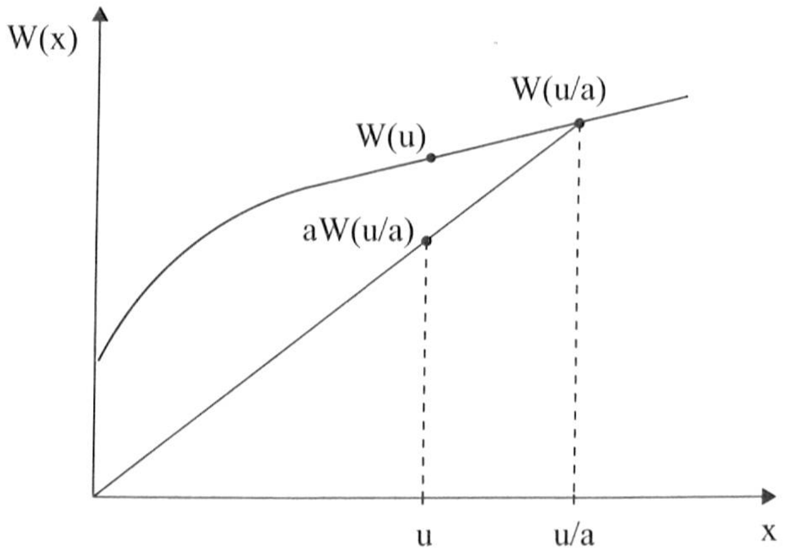

Figure 1: Comparison of $W_a(u)$ and $W(u)$

References

[1] Beveridge, C.J., Dickson, D.C.M., and Wu, X (2008). Optimal Dividends under Reinsurance. Bulletin of the Swiss Association of Actuaries, Heft 1 & 2, 149–166.

[2] Loeffen, R., and Renaud, J.-F. (2010). De Finetti's optimal dividends problem with an affine penalty function at ruin. Insurance: Mathematics & Economics, Vol. 46 (1), to appear.

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Crossing Time of Annuities with Exponential Payment Rates

This note is motivated by a question raised by Dr. Olivier Deprez, a pension consulting actuary in Zurich. In his practice, he examines and compares life annuities with exponential payment rates.

Consider a continuous annuity to a life age x, where the rate of payment at time t is

$$r(t) = re^{ct}, \qquad t \geq 0. \tag{1}$$

Here $r = r(0)$ is the initial rate, and c can be positive or negative. At the force of interest δ, the net single premium of such an annuity is

$$\int_0^\infty r(t)e^{-\delta t}{}_tp_x dt = r\int_0^\infty e^{-(\delta-c)t}{}_tp_x dt = r\bar{a}_x(\delta - c), \tag{2}$$

where

$$\bar{a}_x(\delta) = \int_0^\infty e^{-\delta t}{}_tp_x dt. \tag{3}$$

For the following, it is useful to adapt two notions in the theory of immunization (which is for deterministic cash flows); see Chapter 3 of Panjer (1998). The expression

$$D(\delta) = -\frac{d}{d\delta}\ln \bar{a}_x(\delta) = \frac{\int_0^\infty te^{-\delta t}{}_tp_x dt}{\int_0^\infty e^{-\delta t}{}_tp_x dt} \tag{4}$$

is the Macaulay duration (see Macaulay (1938)), and

$$V(\delta) = \frac{d^2}{d\delta^2}\ln \bar{a}_x(\delta) = \frac{\int_0^\infty [t - D(\delta)]^2 e^{-\delta t}{}_tp_x dt}{\int_0^\infty e^{-\delta t}{}_tp_x dt} \tag{5}$$

is what Fong and Vasicek (1983, 1984) would call M^2. Actuaries recognize that $D(\delta)$ is the mean and $V(\delta)$ the variance of an Esscher transform of the probability density function that is proportional to ${}_tp_x$, $t \geq 0$. To see this, note that the function

$$z \rightarrow \ln \bar{a}_x(\delta - z) - \ln \bar{a}_x(\delta)$$

is the cumulant generating function.

Now we consider two annuities to the same life with different exponential payment rates $r_0(t)$ and $r_1(t)$, given by the pairs (r_0, c_0) and (r_1, c_1). We assume that the annuities are actuarially equivalent, that is, that their net single premiums are the same. Hence, the assumption is that

$$r_0 \bar{a}_x(\delta - c_0) = r_1 \bar{a}_x(\delta - c_1). \tag{6}$$

We are interested in the crossing time $T = T(c_0, c_1)$ when the payment rates are equal. From the condition $r_0(T) = r_1(T)$ we obtain

$$T(c_0, c_1) = \frac{\ln r_0 - \ln r_1}{c_1 - c_0}, \tag{7}$$

and, because of (6),

$$T(c_0, c_1) = \frac{\ln \bar{a}_x(\delta - c_1) - \ln \bar{a}_x(\delta - c_0)}{c_1 - c_0}. \tag{8}$$

Thus $T(c_0, c_1)$ can be interpreted as the slope of a secant of the graph of the function $z \longrightarrow \ln \bar{a}_x(\delta - z)$. It follows that in the limit $c_1 \longrightarrow c_0$, T has the limiting value

$$T(c_0, c_0) = -\frac{d}{d\delta} \ln \bar{a}_x(\delta - c_0) = D(\delta - c_0), \tag{9}$$

which is the Macaulay duration at the force of interest $\delta - c_0$. If $c_1 = c$ is near c_0, we can use the first few terms of the Taylor expansion of $f(c) = T(c_0, c)$ around $c = c_0$ to obtain an approximation. The Taylor series of $\ln \bar{a}_x(\delta - c) - \ln \bar{a}_x(\delta - c_0)$ around $c = c_0$ is

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} [\frac{d^n}{d\delta^n} \ln \bar{a}_x(\delta - c_0)] \cdot (c - c_0)^n.$$

After a division by $(c - c_0)$ we obtain the Taylor series of $f(c) = T(c_0, c)$:

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!} [\frac{d^n}{d\delta^n} \ln \bar{a}_x(\delta - c_0)] \cdot (c - c_0)^{n-1}.$$

If we use the first two terms, we obtain the approximation

$$T(c_0, c) \approx D(\delta - c_0) + \frac{1}{2} V(\delta - c_0) \cdot (c - c_0). \tag{10}$$

This rule of thumb is a refinement of formula (9). If c_0 were an inflation rate and δ a nominal interest rate, the difference $\delta - c_0$ would be the "real" interest rate. Then D and V would be the duration and M^2 with respect to the real interest rate.

Example. We choose de Moivre's law, so that the reader can reproduce the results. Thus ${}_tp_x = \dfrac{\omega - x - t}{\omega - x}$, $\quad 0 \le t \le \omega - x$. After some calculations, we find that $\bar{a}_x(0) = \dfrac{1}{2}(\omega - x)$, $D(0) = \dfrac{1}{3}(\omega - x)$, $V(0) = \dfrac{1}{18}(\omega - x)^2$, and for $\delta \neq 0$, $\bar{a}_x(\delta) = \dfrac{1}{\delta} - \dfrac{1}{\delta^2}\dfrac{1 - e^{-\delta(\omega - x)}}{\omega - x}$, from which $D(\delta)$ and $V(\delta)$ can be determined as derivatives by using Mathematica. For a numerical illustration, we set $x = 65, \omega = 100, \delta = 0.05$. For c_0 between 0.01 and 0.05 and $c = c_1 = 0$ (Table 1), 0.04 (Table 2), 0.08 (Table 3) the estimated values according to formula (10) are compared to the exact values. We conclude that the approximation (10) gives excellent results.

	$D(\delta - c_0)$	$V(\delta - c_0)$	T estimated	T exact	Error
$c_0 = 0.01$	9.21864	54.0667	8.94831	8.95429	0.00598
$c_0 = 0.02$	9.77737	57.6809	9.19928	9.22464	0.02536
$c_0 = 0.03$	10.3722	61.2694	9.46316	9.50702	0.03386
$c_0 = 0.04$	11.0024	64.7560	9.70728	9.80136	0.09408
$c_0 = 0.05$	11.6667	68.0555	9.96532	10.1074	0.14208

Table 1: Values of D, V, T estimated and T when $c = 0$

	$D(\delta - c_0)$	$V(\delta - c_0)$	T estimated	T exact	Error
$c_0 = 0.01$	9.21864	54.0667	10.0296	10.0837	0.05406
$c_0 = 0.02$	9.77737	57.6809	10.3542	10.3781	0.02393
$c_0 = 0.03$	10.3722	61.2694	10.6786	10.6844	0.00585
$c_0 = 0.04$	11.0024	64.7560	11.0024	11.0024	0
$c_0 = 0.05$	11.6667	68.0555	11.3264	11.3318	0.01538

Table 2: Values of D, V, T estimated and T when $c = 0.04$

	$D(\delta - c_0)$	$V(\delta - c_0)$	T estimated	T exact	Error
$c_0 = 0.01$	9.21864	54.0667	11.1110	11.3967	0.28580
$c_0 = 0.02$	9.77737	57.6809	11.5078	11.7137	0.10600
$c_0 = 0.03$	10.3722	61.2694	11.9039	12.0421	0.03817
$c_0 = 0.04$	11.0024	64.7560	12.2975	12.3815	0.08400
$c_0 = 0.05$	11.6667	68.0555	12.6875	12.7314	0.04387

Table 3: Values of D, V, T estimated and T when $c = 0.08$

Remark 1. There is an unexpected mathematical connection between the time T and the premium P that is determined by the exponential premium calculation principle. If S is a random variable (the claim to be paid by the insurer), the premium according to the exponential premium calculation principle is

$$P = \frac{1}{a} \ln E[e^{aS}], \tag{11}$$

where the parameter $a > 0$ is the constant risk aversion of the insurer; see for example Chapter 5 of Gerber (1979). For small values of a, we have the approximation

$$P \approx E[S] + \frac{a}{2} Var[S]. \tag{12}$$

Now suppose $c_1 > c_0$ and consider a random variable S with probability density function that is proportional to $e^{-(\delta - c_0)t} {}_tp_x$, $t \geq 0$. If we set $a = c_1 - c_0$, P calculated by (11) is the same as T according to (8). Furthermore, formulas (10) and (12) are mathematically equivalent.

Remark 2. The methods and results can be extended to the situation where net single premiums are determined with time-dependent interest rates. The trick is to write the force of interest at time t in the form $\delta + \gamma(t)$, where δ is a parameter and $\gamma(t)$ is a given function. Thus the discount factor from t to 0 is $e^{-\delta t} f(t)$ with $f(t) = e^{-\int_0^t \gamma(s)ds}$. To adapt formulas (2)–(10) to this new situation, it suffices to replace ${}_tp_x$ by $f(t){}_tp_x$ everywhere.

Remark 3. The analysis can be easily adapted to discrete annuities with m-$thly$ payments in geometric progression. For an annuity-due, the payment at time $t = \frac{k}{m}$, $k = 0, 1, 2, ...$, is $r(t)/m$, where $r(t)$ is given by formula (1). It suffices to replace t in the differential dt by $\lceil mt \rceil / m$, where $\lceil \cdot \rceil$ denotes the ceiling function. The resulting integrals are to be understood in the Riemann-Stieltjes sense and become summations. For example, instead of (3), we have

$$\ddot{a}_x^{(m)}(\delta) = \frac{1}{m} \sum_{k=0}^{\infty} e^{-\delta k/m} {}_{k/m}p_x \, .$$

This way, formulas (2)–(10) can be readily adapted. Of course T will not be a multiple of $1/m$ in general. Nevertheless, it contains the information that the sign of $r_1(\frac{k}{m}) - r_0(\frac{k}{m})$ changes between $k = \lfloor mT \rfloor$ and $k = \lfloor mT \rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes the floor function.

For an annuity-immediate, the t in dt would be replaced by $\lfloor mt \rfloor / m$.

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References

Fong, H. G., and O. Vasicek (1983). Return Maximization for Immunized Portfolio. In Kaufman, G.G., G.O. Bierwag and A. Toevs (eds.) (1983), *Innovations in Bond Portfolio Management: Duration Analysis and Immunization*. JAI Press Inc., Greenwich, Connecticut, 227–238.

Fong, H. G. and O. Vasicek (1984). A risk minimizing strategy for portfolio immunization, *Journal of Finance*, **39**: 1541–1546.

Gerber, H. U. (1979). *An Introduction to Mathematical Risk Theory*, S.S. Huebner Foundation Monograph Series No. 8. Irwin, Homewood, Illinois.

Macaulay, F. R. (1938). *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1856*, New York: Bureau of Economic Research.

Panjer, H. H. (editor) (1998). *Financial Economics: With Applications to Investments, Insurance and Pensions*, The Actuarial Foundation, Schaumburg, Illinois.

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Premium Formulas for general Drop Down Excess of Loss Covers

1 The Treaty.

Let the random variables $X_1, X_2, X_3, \ldots$ be the claims of a collective of risks and denote with N the random variable of the number of claims of the collective of risks.
Now let the claims be ordered in nonincreasing size, what shall be described by the random variables:

$$X_{N:1} \geq X_{N:2} \geq \cdots \geq X_{N:N}.$$

Furthermore let $(f_i, i \geq 1)$ be a family of mappings such that the random variable

$$R = \sum_{i=1}^{N} f_i(X_{N:i})$$

can be interpreted as the part of the total claims amount taken by a reinsurer. The family $(f_i, i \geq 1)$ defines a reinsurance treaty. In about 20 years the author developed for this type of treaties a comprehensive risk theory (see for a survey Kremer (2004)). One such treaty is the socalled **Drop-Down-Excess-of-Loss** (in short **DDXL) cover**. For that one has with given priorities $\pi_i, i \geq 1$:

$$f_i(x) = \max(x - \pi_i, 0).$$

Suppose there exists a sequence $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = \infty$ in $\mathbb{N}$ and priorities $P_1 \geq \ldots \geq P_{k+1} \geq 0$ such that

$$\pi_i = P_j, \text{for all } i \in \{t_{j-1}+1, \ldots, t_j\}$$
$$\text{and } j \in \{1, \ldots, k+1\}.$$

The R for this cover is denoted by R_{DDXL}.

Example:
Take the case $k = 2$ (that is of certain practical interest). One gets here:

$$R_{\mathrm{DDXL}} = \sum_{i=1}^{t_1} \max(X_{N:i} - P_1, 0) + \sum_{i=t_1+1}^{t_2} \max(X_{N:i} - P_2, 0) + \sum_{i=t_2+1}^{N} \max(X_{N:i} - P_3, 0).$$

Note that case $k = 1$ is already in Kremer (2005). □

2 Most Basic Result.

The **net premium** of the most general DDXL is just:

$$m = E(R_{\text{DDXL}}).$$

For deriving most elegant results on m, one can use results of Kremer (2003). There was considered a mixture between the (classical) excess-of-loss cover and the (classical) largest claims reinsurance treaty. With a given number t and a priority $\pi \geq 0$ its reinsurers claims amount is defined as:

$$R_1(\pi, t) = \sum_{i=1}^{t} \max(X_{N:i} - \pi, 0).$$

Denote this treaty in short with **XLLC** (t, π) and its net premium by

$$\nu(\pi, t) = E(R_1(\pi, t)).$$

Now remember the (classical) **excess-of-loss cover** with given priority $\pi \geq 0$ (in short **XL**(π)). Its reinsurers claims amount is defined as:

$$R_2(\pi) = \sum_{i=1}^{N} \max(X_i - \pi, 0).$$

Denote its net premium by:

$$\mu(\pi) = E(R_2(\pi)).$$

With certain longer thinking one concludes that it holds true:

$$R_{\text{DDXL}} = R_2(P_{k+1}) - \sum_{j=1}^{k} R_1(P_{j+1}, t_j) + \sum_{j=1}^{k} R_1(P_j, t_j).$$

what is most basic for the present paper, since it implies at once the basic formula:

$$m = \mu(P_{k+1}) - \sum_{j=1}^{k} \nu(P_{j+1}, t_j) + \sum_{j=1}^{k} \nu(P_j, t_j). \tag{1}$$

3 Main Result

In the above context let Y_{ji}, $i = 1, \ldots, N_j$ be the excess claims $(X_i - P_j)$ with $X_i > P_j$. N_j is the claims number of these excess claims.
Now assume for this section that:

(A) $X_1, X_2, X_3, \ldots$ are identically distributed.
(B) $N, X_1, X_2, \ldots$ are independent.
(C) $N_j, Y_{j1}, Y_{j2}, \ldots$ are independent (for each j).

Furthermore suppose that the reinsurer knows

(i) the mean claims number $\lambda = E(N)$ of the collective.

(ii) for a given $a \in (0, P_{k+1})$ the probability

$$q = P(X_i > a).$$

(iii) the distribution function G of the conditional distribution of the X_i, given the event $\{X_i > a\}$:

$$G(x) = P(X_i \leq x | X_i > a).$$

It shall hold:

$$q > 0 \text{ and } G(P_1) < 1.$$

One has the following result:

Theorem

In the above context with assumptions (A)–(C) and (i)–(iii) one gets for the net premium of the DDXL:

$$\begin{aligned} m = (\lambda \cdot q) \cdot & \int_{[P_{k+1},\infty)} (x - P_{k+1}) G(dx) - \\ & - \sum_{i=1}^{t_k} \left(\frac{1}{q_{k+1}^i \Gamma(i)} \right) \cdot \int_0^{q_{k+1}} G^{-1}(1-t) \cdot t^{i-1} \cdot M_{k+1}^{(i)}(1 - t/q_{k+1}) dt + \\ & + \sum_{j=1}^{k} \sum_{i=t_{j-1}+1}^{t_j} \left(\frac{1}{q_j^i \Gamma(i)} \right) \cdot \int_0^{q_j} G^{-1}(1-t) \cdot t^{i-1} \cdot M_j^{(i)}(1 - t/q_j) dt + \\ & + P_{k+1} \cdot \sum_{i=1}^{t_k} \frac{1}{\Gamma(i)} \cdot \int_0^1 t^{i-1} \cdot M_{k+1}^{(i)}(1-t) dt - \\ & - \sum_{j=1}^{k} P_j \cdot \sum_{i=t_{j-1}+1}^{t_j} \frac{1}{\Gamma(i)} \cdot \int_0^1 t^{i-1} M_j^{(i)}(1-t) dt \end{aligned}$$

with:

$$q_j = 1 - G(P_j)$$

$$\Gamma(i) = (i-1)!$$

and the i-th derivative $M_j^{(i)}$ of the probability generating function:

$$M_j(t) = \sum_{m=0}^{\infty} P(N_j = m) \cdot t^m \quad (j = 1, \ldots, k+1).$$

G^{-1} denotes the pseudo-inverse of G:

$$G^{-1}(u) = \inf\{x : G(x) \geq u\}.$$

Proof. It is well-known that

$$\mu(P_{k+1}) = (\lambda \cdot q) \cdot \int_{[P_{k+1},\infty)} (x - P_{k+1}) G(dx).$$

Furthermore one knows from theorem 1 in Kremer (2003) that:

$$\nu(P_j, t) = \sum_{i=1}^{t} \left(\frac{1}{q_j^i \Gamma(i)} \right) \cdot \int_0^{q_j} G^{-1}(1-t) \cdot t^{i-1} M_j^{(i)}(1 - t/q_j) dt$$

$$-P_j \cdot \sum_{i=1}^{t} \left(\frac{1}{\Gamma(i)} \right) \cdot \int_0^1 t^{i-1} \cdot M_j^{(i)}(1-t) dt.$$

One inserts these formulas into the rhs of (1) and gets with routine the statement. □

Certainly this result can be specialized easily to the cases $k = 1$ and $k = 2$ etc. Furthermore one can assume more special that N_j is Poisson-distributed and G is (generalized) Pareto-distributed (see on this Kremer (1998)). The details are left to the interested reader (they were given in Kremer (2008)).

References

[1] Kremer, E. (1998): Die verallgemeinerte Paretoverteilung in der Rückversicherung, Transactions of the International Congress of Actuaries. Birmingham.

[2] Kremer, E. (2003): A combination of the excess-of-loss and largest claims reinsurance treaties. Blätter der deutschen Gesellschaft für Versicherungsmathematik.

[3] Kremer, E. (2004): Einführung in die Risikotheorie der verallgemeinerten Höchstschadenrückversicherung. Verlag Versicherungswirtschaft, Karlsruhe.

[4] Kremer, E. (2005): Most elegant premium formulas for the most important drop down excess-of-loss cover. Blätter der deutschen Gesellschaft für Versicherungsmathematik.

[5] Kremer, E. (2008): Most elegant premium formulas for the most general drop-down-excess-of-loss cover. Lecture at the international ASTIN-colloquium at Manchester.

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