Zeitschrift: Mitteilungen / Schweizerische Aktuarvereinigung = Bulletin / Association Suisse des Actuaires = Bulletin / Swiss Association of Actuaries

Herausgeber: Schweizerische Aktuarvereinigung

Band: - (2009)

Heft: 1-2

Artikel: Evaluation of equity-indexed annuities under transaction costs

Autor: Gaillardetz, Patrice / Lakhmiri, Joe Youssef

DOI: https://doi.org/10.5169/seals-555028

B. Wissenschaftliche Mitteilungen

PATRICE GAILLARDETZ, JOE YOUSSEF LAKHMIRI, Montréal

Evaluation of Equity-Indexed Annuities Under Transaction Costs

1 Introduction

An equity-indexed annuity (EIA) is an insurance product with benefits linked to the performance of an equity market. This retirement product is a deferred annuity that provides some participation in the financial market while protecting the policyholder's investment during the accumulation period. It guarantees a minimum rate of return as well as limited participation in the performance of an equity index (e.g. S&P 500). After the accumulation phase, policyholders may transform their investment into a fixed annuity or simply withdraw the accumulated amount. EIAs are embedded with mortality options and surrender values that protect policyholders and issuers during the accumulation period. In this article, we shall focus on the EIA accumulation phase with mortality options. See the monograph by Hardy (2003) for comprehensive discussions on this product. Introduced by Keyport Life Insurance Co in 1995, EIAs have been the most innovative annuity product over the last 10 years. They have become increasingly popular since their debut and the sales of EIAs have broken the $20 billion barrier ($23.1 billion) in 2004 and reached $27.3 billion in 2005 (see 2006 Annuity Fact Book from the National Association for Variable Annuities). It is important to point out that EIAs are similar to equity-linked insurance sold in Germany.

A fair evaluation method for equity-linked products may be obtained using the arbitrage-free theory. It is generally assumed that insurance companies can diversify the mortality risk. Working with this assumption and using the classical Black & Scholes (1973) framework, Tiong (2000) and Lee (2003) use the Esscher transform method developed in Gerber & Shiu (1994) to obtain closed-form formulas for several equity-indexed annuities. Lin & Tan (2003) and Kijima & Wong (2007) consider more general models for equity-indexed annuities, in which the external equity index and the interest rate are defined by general stochastic differential equations. In a discrete time setting, Gaillardetz & Lin (2006) and Gaillardetz & Lakhmiri (2006) propose participation rates that include security loadings based on different assumptions.

This paper considers the pricing and hedging of equity-index annuities in the presence of transaction costs. The hedging strategy underlying the fair evaluation of equity-linked products relies on a replicating portfolio that consists of money market and stock index securities. In order to reduce the financial risk, the fraction of each security needs to be readjusted at each time interval. Because of the transaction costs, there is a cost linked to each revision. This additional charge should be considered when pricing equity-linked products since the no-transaction cost assumption underestimates the value of embedded financial options. In this paper, we subject equity positions to bid/ask spreads; this means that the purchasing price of a share no longer coincides with its selling price. Particularly, spreads that give rise to proportional, constant and mixed transaction costs are considered. It is important to point out that policy fees, which are yearly or monthly constant charges, are similar to constant transaction costs and accordingly could be treated as such.

Most equity-indexed annuities use the S&P 500 as reference index. There is a liquid market for this particular index that should decrease the revision cost of hedging portfolios. Nonetheless, insurance expenses should always be considered in the evaluation of such contracts. Moreover, pricing EIAs under transaction costs using the arbitrage-free theory should allow insurance companies to introduce other indices that might not be as liquid as the S&P 500.

In a discrete time framework, financial guarantees are evaluated under transaction costs using the approach of Boyle & Vorst (1992). Similarly to Leland (1985), the self-financing strategy for financial contingent claim revises the underlying replicating portfolio at each time interval. Because of the portfolio rebalancing at fixed time intervals, it could sometimes be cheaper to dominate, rather than perfectly replicate the financial contingent claim. Bensaid et al. (1992), Edirsinghe et al.(1993), and Boyle & Tan (1994) replace the hedging strategy by a super-replicating strategy leading to a cheaper cost for financial options. In such a case, linear programming needs to be used to minimize the value of the financial option while making sure that the hedging strategy is greater than or equal to its payoff. Because of the nature of financial guarantees embedded into EIAs the linear programming approach is really tedious. Moreover, in the case of European call options, Edirsinghe et al. (1993) show that the difference between both approaches is unsignificant when proportional transaction costs are lower than 2%. Hence, the paper focuses on the natural approach of Boyle & Vorst (1992) that revises the replicating portfolio at each time to obtain the underlying replicating portfolio and the risk-neutral price for equity-indexed annuity contracts.

The aim of this paper is to evaluate EIAs under transaction costs and policy fees. The bid/ask model from Boyle & Vorst (1992) is generalized to proportional, constant, and mixed transaction costs. The pricing of equity-linked products needs to be extracted from the hedging strategy. In the case of EIAs, the participation rate, which is the pricing element, is obtained by seeking the

hedging strategy such that the replicating portfolio value is equal to the policyholder initial investment. This is an improvement over the traditional approach, under which no transaction cost is assumed and the fair evaluation is obtained using the expected discounted payoff under the martingale measure. The proposed approach leads to a challenging programming algorithm and calculations are derived such that readers can implement the approach.
This paper is organized as follows. The next section presents a discrete financial model for the equity index and introduces actuarial notation. We then present replicating portfolios for standard financial contingent claims. In Section 4, we extract the hedging strategy underlying the risk-neutral evaluation of equity-linked products. Finally, we examine the implications of the proposed approaches on the EIAs by conducting a detailed numerical analysis in Section 5.

2 Financial Model and Actuarial Notations

In this section, we present a lattice model that describes the dynamic of a stock index. These lattice models have been intensively used to model stocks, stock indices, interest rates, and other financial securities due to their flexibility and tractability; see Panjer et al. (1998) and Lin (2006) for example. Moreover, as it often happens when working in a continuous framework, it becomes necessary to resort to simulation methods in order to obtain a solution to the problem considered. The goal is to evaluate equity-index annuities under transaction costs and this task shall be clearer under a discrete financial model. Among others, Bacinello (2003) and Costabile et al. (2008) evaluate equity-linked insurance using a discrete financial framework. Furthermore, the premiums obtained from discrete models converge rapidly to the premiums obtained with the corresponding continuous models when considering equity-indexed annuities. We conclude this section by introducing the standard actuarial notation and mortality probabilities.
Let δ be the force of interest, i.e. δ is a nominal annual rate of interest compounded continuously. It is assumed that δ is constant. For each year, assume that there are N trading periods, each with the length of $\Delta = 1/N$. The (stock) index process is denoted as $S(t), t = 0, \Delta, 2\Delta, \cdots$, a real-valued function where $S(0)$ is the initial level of the index. At time Δ, the index process can take exactly 2 distinct values denoted $S(\Delta, \{0\})$, $S(\Delta, \{1\})$. Indeed, $S(\Delta, \{0\})$ represents a down move from the index and $S(\Delta, \{1\})$ indicates the value of the index after a up move. For notational purposes, let

$$\mathbf{i}_t = \{i_0, i_\Delta, i_{2\Delta}, \cdots, i_t\},$$

which shall represent the index's realization up to time t and where $i_t \in \{0, 1, \cdots, tN\}$ is the number of up moves up to time t with $i_0 = 0$. Hence, $S(t, \mathbf{i}_t)$ represents the index level at time t that has followed the path $\mathbf{i}_t$. For the time period $[t, t+\Delta]$, $t = 0, \Delta, 2\Delta, \cdots$, the index $S(t, \mathbf{i}_t)$ has two possible outcomes: $S(t+\Delta, \{\mathbf{i}_t, i_t\})$ and $S(t+\Delta, \{\mathbf{i}_t, i_t+1\})$. Hence, the index process can move up from $S(t)$ to $S(t+\Delta, \{\mathbf{i}_t, i_t+1\})$, or down to $S(t+\Delta, \{\mathbf{i}_t, i_t\})$. Without loss of generality, let us also assume that the time-0 index value is one unity. Because of the constant assumption of the interest rates, the time-t value $B(t)$, $B(0) = 1$, of the money-market account is given by

$$B(t) = e^{\delta t},$$

for $t = 0, \Delta, 2\Delta, \cdots$. Figure 1 presents the dynamic of the index process between $[t, t+2\Delta]$.

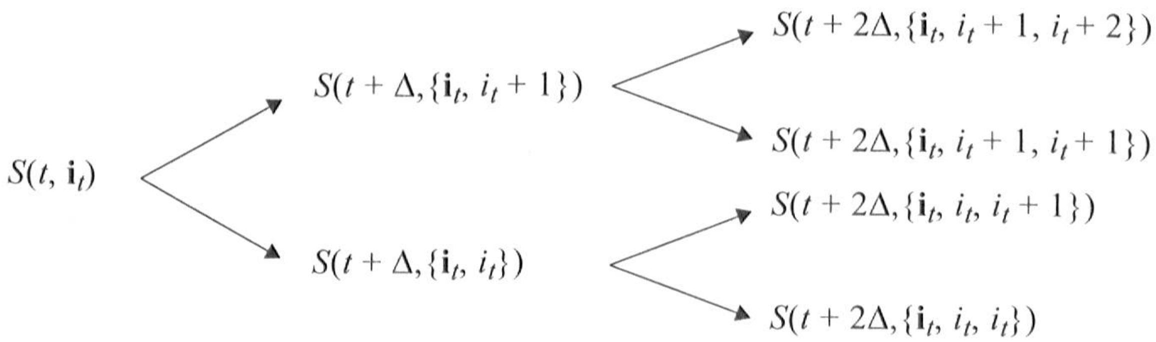

Figure 1: The probability tree of the index between $[t, t+2\Delta]$

The isolated market consisting of money market securities and stock index securities is arbitrage-free if and only if

$$S(t+\Delta, \{\mathbf{i}_t, i_t\}) < S(t, \mathbf{i}_t)e^{\delta\Delta} < S(t+\Delta, \{\mathbf{i}_t, i_t+1\}), \tag{2.1}$$

for all t and $\mathbf{i}_t$. That is, the return on bond instruments is not allowed to dominate the return of the stock index and vice-versa. Apart from transaction costs, the model assumes the usual frictionless market: no tax, short sale restrictions, etc. The filtration associated with the index process is the one generated by the process.

We now introduce the standard actuarial notation, as described in Bowers et al. (1997). Let $T(x)$ be the future lifetime of insured (x) at time $t = 0$ and the modified curtate-future-lifetime

$$K(x) = \lfloor NT(x) \rfloor \Delta, \tag{2.2}$$

the lifetime of (x), computed up to multiples of Δ periods. Here, $\lfloor . \rfloor$ is the floor function.

Let ${}_{t|\Delta}q_x$ denote the probability that (x) survives t years and die within the next Δ year, i.e.

$${}_{t|\Delta}q_x = \Pr[t \le T(x) < t + \Delta] = \Pr[K(x) = t].$$

Define the probability that (x) survives to $x + t$ by

$${}_t p_x = \Pr[T(x) \ge t].$$

Throughout the remainder of the paper, we assume independence between the index process and $K(x)$.

3 Evaluation of Financial Contingent Claims with Transaction Costs

This section develops pricing and hedging formulas for financial contingent claims. To include transaction costs in our analysis, we employ a method that generalizes the approach of Boyle & Vorst (1992). Namely, we introduce arbitrary bid/ask spreads to the index fund and ignore other restrictions (e.g. lot size). This means that an agent of the financial market must provide $S^A(t)$ for the purchase of an index share at time t, whereas only $S^B(t)$ is received when selling it.[1] These price levels are interpreted here as transformations of the underlying value of the asset $S(t)$. On the other hand, holdings in the money market account are not subject to any such transaction costs. Later in this paper, we develop examples of spreads that arise naturally in financial markets; they include:

- Proportional transaction costs, where $S^A(t) = (1 + k_1)S(t)$ and $S^B(t) = (1 - k_1)S(t)$, for some constant k_1;
- Constant transaction costs, where $S^A(t) = S(t) + k_2$ and $S^B(t) = S(t) - k_2$, for some constant k_2;
- Mixed transaction costs, where $S^A(t) = (1 + k_1)S(t) + k_2$ and $S^B(t) = (1 - k_1)S(t) - k_2$, for some constants k_1 and k_2.

Before going any further, let $D(n)$ denote the payoff at time n of the financial contingent claim. Since embedded financial guarantees offered by equity-linked products are usually path dependent, we extend our earlier notation to include all intermediary totals of up moves. Hence, the payoff $D(n, \mathbf{i}_n)$ is the outcome of $D(n)$ associated to the path $\mathbf{i}_n$.

The issuer is interested in the replication of this contingent claim, that is, the insurance company desires to obtain a self-financing hedging strategy that

1 Evidently, $S^A(t) \ge S^B(t)$.

completely eliminates its exposure to the randomness of the payoff $D(n)$. Let $\phi_n(t, \mathbf{i}_t) = \{a_n(t, \mathbf{i}_t), b_n(t, \mathbf{i}_t)\}$ be a stochastic process representing the hedging strategy. The replicating portfolio consists of $a_n(t, \mathbf{i}_t)$ index shares and $b_n(t, \mathbf{i}_t)$ invested in the money market account at time t in provision of a financial claim. Let $V_n(t, \mathbf{i}_t)$ denote the cost of maintaining (or obtaining) such a hedging strategy $\phi_n(t, \mathbf{i}_t)$ at time t. That is,

$$\begin{aligned} V_n(t, \mathbf{i}_t) = {} & a_n(t-\Delta, \mathbf{i}_{t-\Delta})S(t, \mathbf{i}_t) + b_n(t, \mathbf{i}_t) + \big\{1_{\{a_n(t,\mathbf{i}_t) \geq a_n(t-\Delta, \mathbf{i}_{t-\Delta})\}} S^A(t, \mathbf{i}_t) \\ & + 1_{\{a_n(t,\mathbf{i}_t) < a_n(t-\Delta, \mathbf{i}_{t-\Delta})\}} S^B(t, \mathbf{i}_t)\big\}(a_n(t, \mathbf{i}_t) - a_n(t-\Delta, \mathbf{i}_{t-\Delta})), \end{aligned} \tag{3.3}$$

for $t = 0, \Delta, 2\Delta, \cdots, n$, where $1_{\{.\}}$ is the indicator function and $a_n(-\Delta, \mathbf{i}_{-\Delta})$ is equal to 0. Note that,with transaction costs, more funds are required to extend an equity position and less funds are available when liquidating such a position; this makes the overall hedging portfolio more expensive to maintain. Moreover, transaction costs associated with maintaining the hedging strategy are directly influenced by the evolution of index share positions. Therefore, contrarily to frictionless markets, the evaluation of a given portfolio here is dependent on a previously held equity position. The hedging portfolio includes transaction costs at the origin, since the original outlay of $V_n(0, \mathbf{i}_0)$ incorporates bid/ask spreads.
Let $V_n((t+\Delta)^-, \mathbf{i}_{t+\Delta})$ denote the hedge portfolio from t that has accumulated to $t+\Delta$

$$V_n((t+\Delta)^-, \mathbf{i}_{t+\Delta}) = a_n(t, \mathbf{i}_t)S(t+\Delta, \mathbf{i}_{t+\Delta}) + b_n(t, \mathbf{i}_t)e^{\delta\Delta}, \tag{3.4}$$

for $t = 0, \Delta, 2\Delta \cdots, n-\Delta$.
The dynamic nature of the hedging strategy must not require any external infusion of money, which means it should be self-financing. That is

$$V_n((t+\Delta)^-, \mathbf{i}_{t+\Delta}) = V_n(t+\Delta, \mathbf{i}_{t+\Delta}). \tag{3.5}$$

In other words, the investment $V_n(t, \mathbf{i}_t)$ may be transacted to obtain $V_n(t+\Delta, \{\mathbf{i}_t, i_t+1\})$ in case of an up move and $V_n(t+\Delta, \{\mathbf{i}_t, i_t\})$ in case of a down move during the period $[t, t+\Delta]$. Using (3.3) and (3.4), the self-financing conditions that must be satisfied by the replicating portfolio in the presence of transaction costs can be expressed as

$$\begin{aligned} & b_n(t, \mathbf{i}_t)e^{\delta\Delta} - b_n(t+\Delta, \{\mathbf{i}_t, i_t+j\}) = \\ & \quad 1_{\{a_n(t+\Delta, \{\mathbf{i}_t, i_t+j\}) \geq a_n(t,\mathbf{i}_t)\}} S^A(t, \{\mathbf{i}_t, i_t+j\})(a_n(t+\Delta, \{\mathbf{i}_t, i_t+j\}) - a_n(t, \mathbf{i}_t)) \\ & \quad + 1_{\{a_n(t+\Delta, \{\mathbf{i}_t, i_t+j\}) < a_n(t,\mathbf{i}_t)\}} S^B(t, \{\mathbf{i}_t, i_t+j\})(a_n(t+\Delta, \{\mathbf{i}_t, i_t+j\}) - a_n(t, \mathbf{i}_t)), \end{aligned} \tag{3.6}$$

for $j = 0, 1$. These conditions must hold for all $t = 0, \cdots, n-\Delta$ and all possible paths of the stock price. Equation (3.6) suggests that the (potentially negative) left-over funds from the rebalancing of the money market account at time $t+\Delta$ should coincide with the funds required to change equity positions with transaction fees. An alternative and more intuitive formulation of these conditions can be made by expressing the indicator function in terms of absolute values, i.e. by noting that

$$x1_{\{x\geq 0\}} = \frac{x+|x|}{2} \qquad \text{and} \qquad x1_{\{x<0\}} = \frac{x-|x|}{2}.$$

Making the appropriate substitutions in (3.6) yields;

$$\begin{aligned} b_n(t,\mathbf{i}_t)e^{\delta\Delta} + a_n(t,\mathbf{i}_t)S^*(t+\Delta,\{\mathbf{i}_t,i_t+j\}) &= \\ & b_n(t+\Delta,\{\mathbf{i}_t,i_t+j\}) + a_n(t+\Delta,\{\mathbf{i}_t,i_t+j\})S^*(t+\Delta,\{\mathbf{i}_t,i_t+j\}) \\ & + k(t+\Delta,\{\mathbf{i}_t,i_t+j\})|a_n(t+\Delta,\{\mathbf{i}_t,i_t+j\}) - a_n(t,\mathbf{i}_t)|, \end{aligned} \tag{3.7}$$

for $j=0,1$. Here we define

$$S^*(t,i_t) = \frac{S^A(t,\mathbf{i}_t)+S^B(t,\mathbf{i}_t)}{2} \qquad \text{and} \qquad k(t,\mathbf{i}_t) = \frac{S^A(t,\mathbf{i}_t)-S^B(t,\mathbf{i}_t)}{2},$$

as the average stock price and the half-bid/ask spread respectively. This formulation has the merit of making explicit the allowance for transaction costs, given by $k(t+\Delta,\{\mathbf{i}_t,i_t+j\})|a_n(t+\Delta,\{\mathbf{i}_t,i_t+j\}) - a_n(t,\mathbf{i}_t)|$. Note that (3.7) entails that, under transaction costs, the equity position should be valued according to transformed price process S^*. With that understanding in mind, we see that the value of the replicating portfolio before rebalancing at time $t+\Delta$ must equal the value after rebalancing, plus the allowance for transaction fees. Later, we show how these conditions reduce to the Boyle & Vorst (1992) equations when the bid/ask prices are defined by proportional transaction costs.

In order to replicate the contingent claim, the portfolio process $\{\phi_n(t,\mathbf{i}_t), t=0,\Delta,\cdots,n\}$ must satisfy the following endpoint constraints:

$$\phi_n(n,\mathbf{i}_n) = \{0, D(n,\mathbf{i}_n)\}, \tag{3.8}$$

for all $\mathbf{i}_n$. Note that, at maturity, all holdings should be invested in the money market account, since the transaction between the two parties cannot be resolved in shares. Moreover, these funds should coincide with the incurred liability $D(n,\mathbf{i}_n)$. Because any position in shares is eventually liquidated with transaction costs, any other target portfolio would be considered sub-optimal. This is an important distinction from the frictionless market setting, where no preference is made as to the composition of the portfolio at maturity.

It follows from (3.7) and (3.8) that

$$\begin{aligned} D(n,\{\mathbf{i}_{n-\Delta},i_{n-\Delta}+j\}) &= b_n(n-\Delta,\mathbf{i}_{n-\Delta})e^{\delta\Delta} + a_n(n-\Delta,\mathbf{i}_{n-\Delta})S^*(n,\{\mathbf{i}_{n-\Delta},i_{n-\Delta}+j\}) \\ & - k(n,\{\mathbf{i}_{n-\Delta},i_{n-\Delta}+j\})\big|a_n(n-\Delta,\mathbf{i}_{n-\Delta})\big|, \end{aligned} \tag{3.9}$$

for $j=0,1$. As expected, this means that the hedging strategy requires the position in stock to be eliminated at maturity in order to provide for the appropriate payoff since the issuer needs to reimburse the policyholder. Furthermore, we may deduce the following by substituting (3.8) into (3.3),

$$V_n(n, \mathbf{i}_n) \geq D(n, \mathbf{i}_n).$$

That is, the terminal cost of the hedging portfolio exceeds its associated payoff; the realized difference is given by the incurred transaction costs. By extension, the entire $V_n(t)$ process will overcompensate its intrinsic value.
An explicit expression for the hedging strategy at time t can be obtained in terms of potential portfolio weights at time $t+\Delta$ by solving the self-financing conditions. We choose to solve (3.7) for $\{a_n(t,\mathbf{i}_t), b_n(t,\mathbf{i}_t)\}$; this formulation of the problem involves the use of absolute values, but they can be circumvented by using the relationship

$$|x| = x\mathrm{Sig}(x), \text{where } \mathrm{Sig}(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

In that respect, we define the following vector

$$\left(I_n^{(1)}(t,\mathbf{i}_t), I_n^{(2)}(t,\mathbf{i}_t)\right) =$$
$$(\mathrm{Sig}(a_n(t+\Delta,\{\mathbf{i}_t, i_t+1\}) - a_n(t,\mathbf{i}_t)), \mathrm{Sig}(a_n(t+\Delta,\{\mathbf{i}_t, i_t\}) - a_n(t,\mathbf{i}_t))), \quad (3.10)$$

in order to lighten the notation. Now, (3.7) may be rewritten as

$$\begin{bmatrix} e^{\delta\Delta} & C_1 \\ e^{\delta\Delta} & C_2 \end{bmatrix} \begin{bmatrix} b_n(t,\mathbf{i}_t) \\ a_n(t,\mathbf{i}_t) \end{bmatrix} = \begin{bmatrix} b_n(t+\Delta,\{\mathbf{i}_t,i_t+1\}) + C_1\ a_n(t+\Delta,\{\mathbf{i}_t,i_t+1\}) \\ b_n(t+\Delta,\{\mathbf{i}_t,i_t\}) + C_2\ a_n(t+\Delta,\{\mathbf{i}_t,i_t\}) \end{bmatrix}, \quad (3.11)$$

where

$$C_1 = S^*(t+\Delta,\{\mathbf{i}_t,i_t+1\}) + k(t+\Delta,\{\mathbf{i}_t,i_t+1\})I_n^{(1)}(t,\mathbf{i}_t),$$

and

$$C_2 = S^*(t+\Delta,\{\mathbf{i}_t,i_t\}) + k(t+\Delta,\{\mathbf{i}_t,i_t\})I_n^{(2)}(t,\mathbf{i}_t),$$

for $t = 0, \Delta, \cdots, n-\Delta$.
Note that $(I_n^{(1)}(t,\mathbf{i}_t), I_n^{(2)}(t,\mathbf{i}_t))$ is limited to only four possible values: $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$. Although $(I_n^{(1)}(t,\mathbf{i}_t), I_n^{(2)}(t,\mathbf{i}_t))$ is functionally dependent on $a_n(t,\mathbf{i}_t)$, we treat it above as a fixed, but unknown vector. That is, if a solution to the system exists, $(I_n^{(1)}(t,\mathbf{i}_t), I_n^{(2)}(t,\mathbf{i}_t))$ is already defined and simply acts as a constant. If $S^B(t+\Delta,\{\mathbf{i}_t,i_t+1\}) > S^A(t+\Delta,\mathbf{i}_t)$, the solution of (3.11) is given by

$$a_n(t,\mathbf{i}_t) =$$
$$\frac{b_n(t+\Delta,\{\mathbf{i}_t,i_t+1\}) + a_n(t+\Delta,\{\mathbf{i}_t,i_t+1\})C_1 - b_n(t+\Delta,\{\mathbf{i}_t,i_t\}) - a_n(t+\Delta,\{\mathbf{i}_t,i_t\})C_2}{C_1 - C_2}, \quad (3.12)$$

and

$$b_n(t, \mathbf{i}_t) = e^{-\delta\Delta}\Big(b_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) + C_1(a_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) - a_n(t, \mathbf{i}_t))\Big), \quad (3.13)$$

for $t = 0, \Delta, \cdots, n-\Delta$.
Therefore, we fix a particular vector for $\big(I_n^{(1)}(t, \mathbf{i}_t), I_n^{(2)}(t, \mathbf{i}_t)\big)$ and compute its associated solution. The relevant range of parameters, for which the obtained solution is consistent with the fixed vector chosen *a priori*, is then determined. For instance, suppose that the vector $\big(I_n^{(1)}(t, \mathbf{i}_t), I_n^{(2)}(t, \mathbf{i}_t)\big)$ is set to $(1, -1)$, which means that

$$\mathrm{Sig}(a_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) - a_n(t, \mathbf{i}_t)) = 1,$$

and

$$\mathrm{Sig}(a_n(t+\Delta, \{\mathbf{i}_t, i_t\}) - a_n(t, \mathbf{i}_t)) = -1.$$

The previous constraints are respected if

$$\begin{aligned}
&a_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) - a_n(t, \mathbf{i}_t) \\
&= \frac{C_2\,\xi - (b_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) - b_n(t+\Delta, \{\mathbf{i}_t, i_t\}))}{C_1 - C_2} \\
&= \frac{(S^*(t+\Delta, \{\mathbf{i}_t, i_t\}) - k(t+\Delta, \{\mathbf{i}_t, i_t\}))\cdot\xi - (b_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) - b_n(t+\Delta, \{\mathbf{i}_t, i_t\}))}{(S^*(t+\Delta, \{\mathbf{i}_t, i_t+1\}) + k(t+\Delta, \{\mathbf{i}_t, i_t+1\})) - (S^*(t+\Delta, \{\mathbf{i}_t, i_t\}) - k(t+\Delta, \{\mathbf{i}_t, i_t\}))} \\
&\geq 0,
\end{aligned}$$

and

$$\begin{aligned}
&a_n(t+\Delta, \{\mathbf{i}_t, i_t\}) - a_n(t, \mathbf{i}_t) \\
&= \frac{C_1\,\xi - (b_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) - b_n(t+\Delta, \{\mathbf{i}_t, i_t\}))}{C_1 - C_2} \\
&= \frac{(S^*(t+\Delta, \{\mathbf{i}_t, i_t+1\}) + k(t+\Delta, \{\mathbf{i}_t, i_t+1\}))\cdot\xi - (b_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) - b_n(t+\Delta, \{\mathbf{i}_t, i_t\}))}{(S^*(t+\Delta, \{\mathbf{i}_t, i_t+1\}) + k(t+\Delta, \{\mathbf{i}_t, i_t+1\})) - (S^*(t+\Delta, \{\mathbf{i}_t, i_t\}) - k(t+\Delta, \{\mathbf{i}_t, i_t\}))} \\
&< 0,
\end{aligned}$$

where

$$\xi = a_n(t+\Delta, \{\mathbf{i}_t, i_t\}) - a_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}).$$

Then, the solution given by (3.12) and (3.13) is valid with $\big(I_n^{(1)}(t, \mathbf{i}_t), I_n^{(2)}(t, \mathbf{i}_t)\big) = (1, -1)$ if and only if

$$\frac{b_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) - b_n(t+\Delta, \{\mathbf{i}_t, i_t\})}{S^A(t+\Delta, \{\mathbf{i}_t, i_t+1\})} > \xi \geq \frac{b_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) - b_n(t+\Delta, \{\mathbf{i}_t, i_t\})}{S^B(t+\Delta, \{\mathbf{i}_t, i_t\})}. \quad (3.14)$$

More generally, the functional form of $\left(I_n^{(1)}(t,\mathbf{i}_t), I_n^{(2)}(t,\mathbf{i}_t)\right)$, expressed in terms of time $t+\Delta$ parameters, is given by

$$
\left(I_n^{(1)}(t,\mathbf{i}_t), I_n^{(2)}(t,\mathbf{i}_t)\right) =
$$

$$
\begin{cases}
(\ \ 1,\ \ 1), \text{if } \xi \geq \max\left(\frac{b_n(t+\Delta,\{\mathbf{i}_t,i_t+1\})-b_n(t+\Delta,\{\mathbf{i}_t,i_t\})}{S^A(t+\Delta,\{\mathbf{i}_t,i_t+1\})}, \frac{b_n(t+\Delta,\{\mathbf{i}_t,i_t+1\})-b_n(t+\Delta,\{\mathbf{i}_t,i_t\})}{S^B(t+\Delta,\{\mathbf{i}_t,i_t\})}\right) \\
(-1,\ \ 1), \text{if } \frac{b_n(t+\Delta,\{\mathbf{i}_t,i_t+1\})-b_n(t+\Delta,\{\mathbf{i}_t,i_t\})}{S^A(t+\Delta,\{\mathbf{i}_t,i_t+1\})} > \xi \geq \frac{b_n(t+\Delta,\{\mathbf{i}_t,i_t+1\})-b_n(t+\Delta,\{\mathbf{i}_t,i_t\})}{S^B(t+\Delta,\{\mathbf{i}_t,i_t\})} \\
(\ \ 1,-1), \text{if } \frac{b_n(t+\Delta,\{\mathbf{i}_t,i_t+1\})-b_n(t+\Delta,\{\mathbf{i}_t,i_t\})}{S^B(t+\Delta,\{\mathbf{i}_t,i_t\})} > \xi \geq \frac{b_n(t+\Delta,\{\mathbf{i}_t,i_t+1\})-b_n(t+\Delta,\{\mathbf{i}_t,i_t\})}{S^A(t+\Delta,\{\mathbf{i}_t,i_t+1\})} \\
(-1,-1), \text{if } \xi < \min\left(\frac{b_n(t+\Delta,\{\mathbf{i}_t,i_t+1\})-b_n(t+\Delta,\{\mathbf{i}_t,i_t\})}{S^A(t+\Delta,\{\mathbf{i}_t,i_t+1\})}, \frac{b_n(t+\Delta,\{\mathbf{i}_t,i_t+1\})-b_n(t+\Delta,\{\mathbf{i}_t,i_t\})}{S^B(t+\Delta,\{\mathbf{i}_t,i_t\})}\right)
\end{cases}
\tag{3.15}
$$

It is worth mentioning that the above solution is a generalization to the well-known formulas for hedging strategies in the no-transaction cost setting.
Therefore, the self-financing portfolio process $\{\phi_n(t,\mathbf{i}_t), t=0,\Delta,\cdots,n\}$ may be obtained recursively using (3.12), (3.13), and (3.15). First, the payoff at time n ($D(n,\mathbf{i}_n)$) is determined for all possible outcomes $\mathbf{i}_n$. The hedging strategy during the interval $[n-\Delta, n]$ is uncovered using (3.12), (3.13), and (3.15) with starting values based on the terminal condition given by (3.8). Then, we may extract the hedging strategy by applying (3.12), (3.13), and (3.15) recursively from time $n-2\Delta$ to 0.
Naturally, facing the financial contingent claim with payoff $D(n)$ at time n, an individual standing at time 0 may completely eliminate his exposure to the financial risk with $V_n(0,\mathbf{i}_0)$ dollars. He purchases $a_n(0,\mathbf{i}_0)$ index shares and invests $b_n(0,\mathbf{i}_0)$ in the money market account, which worth $V_n(0,\mathbf{i}_0)$, and eliminates the financial risk completely by following the hedging strategy ϕ_n, implied by (3.12), (3.13), and (3.15).
Note that the individual at time t that stands in front of the payoff $D(n)$ at time n may eliminate the financial risk using the hedging strategy ϕ_n. However, the cost of beginning this replicating portfolio is not given by $V_n(t,\mathbf{i}_t)$ since the latter represents the maintaining cost. Thus, the complete transaction fees at time t should be added and the cost will be given by

$$
a_n(t,\mathbf{i}_t)S^*(t,\mathbf{i}_t) + k(t,\mathbf{i}_t)|a_n(t,\mathbf{i}_t)| + b_n(t,\mathbf{i}_t).
$$

Then, the hedging strategy defined by (3.12), (3.13), and (3.15) may be performed.

Moreover, the value of maintaining or beginning the replicating portfolio may be expressed as an expectation using a risk-neutral measure Q. That is, under the risk-neutral[2] condition, the time-t value of the replicating portfolio may be written using

$$V_n(t, \mathbf{i}_t) = E\Big[D(n)e^{-\delta(n-t)}|\mathbf{i}_t, K(x) \geq t\Big], \tag{3.16}$$

where $E[.]$ represents expectation with respect to a risk-neutral measure Q. Let $\pi(t, \mathbf{i}_t)$ denote the risk-neutral probability that the index goes up during the period $[t, t+\Delta]$ given that the index path has followed $\mathbf{i}_t$. That is

$$\pi(t, \mathbf{i}_t) = Q\big[V_n(t+\Delta, \mathbf{i}_{t+\Delta}) = a_n(t, \mathbf{i}_t)S(t+\Delta, \{\mathbf{i}_t, i_t+1\}) + b_n(t, \mathbf{i}_t)e^{\delta\Delta}\big], \tag{3.17}$$

for $t = 0, \Delta, \cdots, n-\Delta$. Under transaction costs, the financial is not complete, which means there is an infinity of risk-neutral measures. The risk-neutral measure defined from the replicating portfolio (3.12), (3.13), and (3.15) is given by

$$\pi(t, \mathbf{i}_t) = \frac{e^{\delta\Delta}V(t, \mathbf{i}_t) - V_n(t+\Delta, \{\mathbf{i}_t, i_t\})}{V_n(t+\Delta, \{\mathbf{i}_t, i_t+1\}) - V_n(t+\Delta, \{\mathbf{i}_t, i_t\})}. \tag{3.18}$$

This measure has the particularity of being directly influenced by the equity positions underlying the hedging strategy, a radical departure from the no-transaction case. Note that the risk-neutral measure (3.18) may be used to evaluate the price at time 0 of a financial contingent claim with payoff $D(n)$ or to extract the value of maintaining the time-t replicating portfolio.

4 Evaluation of Equity-Linked Products under Transaction Costs

In this section, we evaluate equity-linked products using the replication portfolio theory developed in the previous section. These contracts are particular in incorporating both financial and mortality risks. In addition to provide insurance coverage, the level of the benefits is linked to the financial market performance and an equity index in particular. Hence, the hedging strategy underlying the fair evaluation of those contracts relies heavily on the financial assumption. In the previous section, we obtained the price of a financial contingent claim with transaction costs and extracted the hedging strategy. We now extend this evaluation to equity-linked contracts. We assume that the insurance company

2 The risk-neutral condition does not necessarily imply the no-arbitrage condition. It is sometimes cheaper to super-replicate a financial contingent claim when transaction costs are considered. However, the difference in the value appears only with high transaction cost (see Edirsinghe et al., 1993).

issued the contract at time 0 and there is no surrender option embedded in the equity-linked contract.

For every unit invested in an equity-linked contract with maturity n, the issuer is responsible for providing the payoff $D(t, \mathbf{i}_t)$ at time t if death occurs in the period $[t - \Delta, t)$[3], granted that the maturity of the contract has not yet been reached. Otherwise, a payment of $D(n, \mathbf{i}_n)$ is effectuated at maturity. In other words, the insurance company pays

$$\begin{cases} D(K(x) + \Delta), \text{if } K(x) = 0, \cdots, n - 2\Delta \\ \quad D(n), \text{if } K(x) = n - \Delta, n, \cdots \end{cases}.$$

Here, it is assumed that the functional form of the payoff is the same at maturity. This is not always the case, but we adopt this view for the sake of simplicity.

Let $W_n(x, t, \mathbf{i}_t)$ denote the cost of starting when $t = 0$ or the cost of maintaining $(0 < t \leq n)$ the equity-linked contract, which is issued to (x). This value may be obtained using the expected discounted payoff of the equity-linked contract

$$\begin{aligned} W_n(x, t, \mathbf{i}_t) = E[&D(K(x) + \Delta) 1_{\{K(x) < n - \Delta\}} e^{-\delta(K(x) + \Delta - t)} \\ &+ D(n) 1_{\{K(x) \geq n - \Delta\}} e^{-\delta(n-t)} | \mathbf{i}_t, K(x) \geq t], \text{ for } t = 0, \Delta, \cdots, n. \end{aligned} \tag{4.19}$$

It is also natural to assume independence between the policyholder and the financial market under the risk-neutral measure. We also assume that insurance companies may diversify their equity-linked portfolio. It is important to point out that the accumulation period for EIAs is ranging between 5 to 15 years. In this time interval, the mortality risk will not play a major role compared to the index. Under these assumptions, the value process given in (4.19) becomes

$$\begin{aligned} W_n(x, t, \mathbf{i}_t) = &\sum_{l \in A_{n-t-2\Delta}} E[D(t + l + \Delta) e^{-\delta(l + \Delta)} | \mathbf{i}_t]_{l|\Delta} q_{x+t} \\ &+ E[D(n) e^{-\delta(n-t)} | \mathbf{i}_t]_{n-t-\Delta} p_{x+t}, \end{aligned} \tag{4.20}$$

where $A_t = \{0, \Delta, \cdots, t\}$. Now the underlying hedging strategy, using (3.16) in (4.20) leads to

$$W_n(x, t, \mathbf{i}_t) = \sum_{l \in A_{n-t-2\Delta}} V_{t+l+\Delta}(t, \mathbf{i}_t)_{l|\Delta} q_{x+t} + V_n(t, \mathbf{i}_t)_{n-t-\Delta} p_{x+t} \,. \tag{4.21}$$

The risk-neutral evaluation of the contract is given by $W_n(x, 0, \mathbf{i}_0)$, which represents the single benefit premium of the equity-linked contract with transaction costs. Equation (4.21) can be simplified if a mortality assumption is

3 That is, if $K(x) = t$.

made. If mortality can only take into effect at year-end, most terms then vanish from the sum. Also, by substituting (3.3) in $W_n(x,0,\mathbf{i}_0)$, we obtain further insight in the evaluation of the contract:

$$\begin{aligned}W_n(x,0,\mathbf{i}_0) &= \sum_{l=0}^{n-2}\left[a_{l+1}(0,\mathbf{i}_0)S^*(0,\mathbf{i}_0)+k(0,\mathbf{i}_0)|a_{l+1}(0,\mathbf{i}_0)|+b_{l+1}(0,\mathbf{i}_0)\right]_{l|}q_x \\ &\quad +\left[a_n(0,\mathbf{i}_0)S^*(0,\mathbf{i}_0)+k(0,\mathbf{i}_0)|a_n(0,\mathbf{i}_0)|+b_n(0,\mathbf{i}_0)\right]_{n-1}p_x \\ &= S^*(0,\mathbf{i}_0)a_n(x,0,\mathbf{i}_0)+b_n(x,0,\mathbf{i}_0)+k(0,\mathbf{i}_0)\{\sum_{l=0}^{n-2}|a_{l+1}(0,\mathbf{i}_0)|_{l|}q_x \\ &\quad +|a_n(0,\mathbf{i}_0)|_{n-1}p_x\},\end{aligned} \tag{4.22}$$

where

$$a_n(x,t,\mathbf{i}_t) = \sum_{l=0}^{n-\lfloor t\rfloor-2} a_{l+\lfloor t\rfloor+1}(t,\mathbf{i}_t)_{l|}q_{x+\lfloor t\rfloor} + a_n(t,\mathbf{i}_t)_{n-\lfloor t\rfloor-1}p_{x+\lfloor t\rfloor}, \tag{4.23}$$

and

$$b_n(x,t,\mathbf{i}_t) = \sum_{l=0}^{n-\lfloor t\rfloor-2} b_{l+\lfloor t\rfloor+1}(t,\mathbf{i}_t)_{l|}q_{x+\lfloor t\rfloor} + b_n(t,\mathbf{i}_t)_{n-\lfloor t\rfloor-1}p_{x+\lfloor t\rfloor}. \tag{4.24}$$

The premium can be expressed as the sum of two components: a weighted average of positions in index (4.23) and a weighted average of positions in a money market fund (4.24). Here the weights are mortality probabilities, and the respective positions emanate from the various replicating portfolios. This suggests a dynamic method of investing the single benefit premium. That is

$$\phi_n(x,t,\mathbf{i}_t) = \{a_n(x,t,\mathbf{i}_t), b_n(x,t,\mathbf{i}_t)\}.$$

However, in (4.22), these positions are established individually rather than collectively, resulting in potentially unnecessary transaction costs throughout the investment period. In fact, by rebalancing the various portfolios, the efficiency of our hedging strategy is undermined if stock is bought and sold simultaneously. Therefore, an algorithm is required here to optimize the performance of the strategy. Note that, by assuming year-end mortality, $\phi_n(x,t,\mathbf{i}_t)$ becomes a mixture of only n replicating portfolios, making this problem more tractable.
Furthermore, if the mortality experience of a group of individuals is deterministic, then the claims are matched exactly by the various replicating portfolios, leaving no risk to the insurer. This strategy is therefore reasonable if the mortality risk is diversified. Additionally, if we denote the cost process of this dynamic portfolio by $V(x,t,\mathbf{i}_t)$, we may show, using (3.3) and (4.21), that

$$V(x,t,\mathbf{i}_t) \leq W_n(x,t,\mathbf{i}_t).$$

This is because the obtention (and maintain) of $\phi_n(x,t,\mathbf{i}_t)$ is not subject to the above-mentioned inefficiencies inherent in the fair evaluation of the contract. However, this is not to say that the contract should be valued using the $V(x,t,\mathbf{i}_t)$ process. Whereas its underlying dynamic portfolio is based on an allocation of the premium across $V_1(0,\mathbf{i}_0),\cdots,V_n(0,\mathbf{i}_0)$, the fact of the matter is that only one of these amounts are necessary for replication, namely $V_{\min(K(x)+\Delta,n)}(0,\mathbf{i}_0)$. This, of course, cannot be known at time 0, so that any attempt at replication inevitably gives rise to hedging errors[4]. Therefore, $\phi_n(x,t,\mathbf{i}_t)$ does not constitute a hedging portfolio *per se* and we must resort to risk-neutral pricing to determine the premium.

5 Evaluation of Equity-Indexed Annuities under Transaction Costs

This section concentrates on the pricing of equity-indexed annuities under different types of transaction costs. EIAs appeal to investors because they offer protection against the financial losses and also provide participation in the equity market. From Lin & Tan (2003) and Tiong (2000), EIA designs may be generally grouped in two broad classes: Annual Reset and Point-to-Point. The index growth on an EIA with the former is measured and locked in each year. Particularly, the index growth with a term-end point design is calculated using the index value at the beginning and the end of each year. The EIA return grows with point-to-point class is based on the growth between two time points over the entire term of the annuity. Particularly, the index growth with a high-water mark feature is calculated to the highest index anniversary value. In other hands, the point-to-point EIA with a term-end point design is based on the ending index value.
The contingent claim $D(.)$ of an equity-indexed annuity contract is dependent on a set of parameters Υ. These usually include the participation rate α, the minimum guaranteed rate g, the guaranteed fraction of premium β, etc. That is

$$\Upsilon = \{\alpha, g, \beta, \cdots\}. \tag{5.25}$$

Therefore, the premium of the contract $W_n(x,0,i_0)$ is a function of the parameter set Υ.
As explained in Lin & Tan (2003), an EIA is usually priced through its participation rate α. Without loss of generality, we suppose that the initial value of EIA contracts is one monetary unit. The present value of the EIA is a function

4 For a complete treatment of dynamic hedging errors and loading using risk measures, see Gaillardetz and Lakhmiri (2006).

of the participation rate through the payoff function. Thus, fixing all other parameters constant, we solve for α, the fair participation rate, such that

$$W_n(x, 0, \mathbf{i}_0) = 1, \tag{5.26}$$

where $W_n(x, 0, i_0)$ is obtained using (4.22). Since the cost of the EIA contract is reflected through the participation rate, it is expected to be lower for expensive designs.

The next sub-sections are devoted specifically to each of the three spreads. In order to demonstrate the stated theory, we shall give concrete examples on a particular EIA. Namely, we shall price a five-year, point-to-point EIA with term-end point design. This implies the following payoff:

$$D(t, \mathbf{i}_t) = \max\left[\min\left\{1 + \alpha R(t, \mathbf{i}_t), (1+\zeta)^t\right\}, \beta(1+g)^t\right], \tag{5.27}$$

where

$$R(t, \mathbf{i}_t) = \frac{S(t, \mathbf{i}_t)}{S(0)} - 1. \tag{5.28}$$

The minimum interest rate guarantee shall consist of either 3% on 100% of the premium or 3% on 90% of the premium; concurrently, the cap rate will be set to 12%, 15%, 20%, and ∞[5]. This EIA will be issued to an individual aged 55, with year-end mortality specified by the $1979 - 1981$ U.S. Life Table (see Bowers et al., 1997, Table 3.3.1). We also assume that the force of interest δ is constant over time and is equal to 6 %. For simplification purposes, the index will be governed by the Cox, Ross, & Rubinstein (1979) model where $S(0) = 1$ and the number of trading dates N is 6. In this recombining model, the index at time t $S(t, \mathbf{i}_t)$ has two possible outcomes at time $t + \Delta$: it is either increasing to $S(t+\Delta, \{\mathbf{i}_t, i_t + 1\}) = uS(t, \mathbf{i}_t)$ or decreasing to $S(t+\Delta, \{\mathbf{i}_t, i_t\}) = dS(t, \mathbf{i}_t)$. The increasing and decreasing factors u and d are supposed to be constant and are obtained from the volatility of the index σ. In other words, $u = e^{\sigma/\sqrt{N}}$ $(\sigma = 0.2, 0.3)$ and $d = u^{-1}$.

5.1 *Proportional Transaction Costs*

Proportional transaction costs are the most common type bid/ask spread in the market. The ask-price $S^A(t) = (1 + k_1)S(t)$, and bid-price $S^B(t) = (1 - k_1)S(t)$, is subject to a proportional cost, which increases with the value of the underlying stock. Therefore, a change in equity position is more heavily penalized when a favorable stock outcome occurs.

5 $\zeta = \infty$ represents the case without cap.

In this setting, we have $k(t, \mathbf{i}_t) = k_1 S(t, \mathbf{i}_t)$ and $S^*(t, \mathbf{i}_t) = S(t, \mathbf{i}_t)$. Furthermore, the self-financing conditions of (3.7) become

$$\begin{aligned} b_n(t, \mathbf{i}_t)e^{\delta\Delta} + a_n(t, \mathbf{i}_t)S(t+\Delta, \{\mathbf{i}_t, i_t + j\}) = \\ b_n(t+\Delta, \{\mathbf{i}_t, i_t + j\}) + a_n(t+\Delta, \{\mathbf{i}_t, i_t + j\})S(t+\Delta, \{\mathbf{i}_t, i_t + j\}) \\ + k_1 S(t+\Delta, \{\mathbf{i}_t, i_t + j\})|a_n(t+\Delta, \{\mathbf{i}_t, i_t + j\}) - a_n(t, \mathbf{i}_t)|, \end{aligned}$$

for $j = 0, 1$. These are, in fact, the Boyle & Vorst (1992) equations; our methods therefore coincide in this particular case.

		k_1-%								
σ	ζ	0.0	0.1	0.2	0.3	0.4	0.5	1.0	2.0	5.0
3% Minimum Guarantee on 100% Premium										
20%	12%	83.98	82.15	80.84	79.56	78.31	77.10	71.39	61.08	41.29
	15%	72.46	71.46	70.49	69.54	68.62	67.71	63.43	56.76	42.39
	20%	67.24	66.46	65.71	64.97	64.26	63.56	60.30	54.75	43.15
	∞	65.94	65.22	64.52	63.84	63.17	62.52	59.50	54.39	43.59
30%	12%	90.12	88.27	86.44	84.62	82.81	81.02	72.12	62.72	41.67
	15%	66.65	65.61	64.58	63.56	62.55	61.55	58.05	51.76	38.07
	20%	56.54	55.90	55.28	54.67	54.08	53.49	50.73	45.85	36.22
	∞	51.45	50.97	50.50	50.04	49.59	49.15	47.07	43.47	35.52
3% Minimum Guarantee on 90% Premium										
20%	12%	115.13	112.63	110.21	107.84	105.53	103.28	92.75	77.28	50.47
	15%	91.66	90.15	88.67	87.22	85.80	84.41	78.43	69.96	51.49
	20%	81.40	80.47	79.55	78.66	77.79	76.94	72.94	66.35	52.05
	∞	78.27	77.50	76.74	75.99	75.27	74.56	71.23	65.50	53.03
30%	12%	191.90	183.34	174.93	166.63	158.44	150.37	117.39	94.77	55.53
	15%	98.39	95.63	93.04	91.59	90.17	88.79	82.26	70.85	49.79
	20%	75.16	74.21	73.28	72.37	71.48	70.61	66.45	59.69	46.63
	∞	64.05	63.51	62.98	62.46	61.95	61.45	59.08	54.90	45.43

Table 1: Fair Participation Rates under Proportional Transaction Costs

Table 1 gives the fair participation rates based on Equation (4.22) for a range of parameter values. Here $k_1 = 0$ corresponds to the no-transaction costs case and serves as a benchmark for our results. Naturally, the fair participation rate declines with increasing cap rates. This is because a higher cap rate renders the contract more expensive for all values of α. The only column where this phenomenon is not always observed is when $k_1 = 5\,\%$. In this case, we can note that fair participation rates increase with the ceilings. Generally, the cost of transactions proportionally increase with the amount invested in the index and this investment increases with cap rates. Hence, the participation rates increase under high transaction costs since the issuer is more exposed to the price index fluctuations.

The effects of these parameters on β, when set to $\zeta = 12\,\%$ and $\beta = 90\,\%$, are dramatic (e.g. $\alpha = 191.90\,\%$ for the benchmark). In such a case, the minimum guarantee and the ceiling on the return capture almost all outcomes of the stock process. Since α acts only on a few outcomes, this parameter must be overcompensated to restore the balance.

Furthermore, participation rates decline as we move to the right on any given row. For instance, when setting $\beta = 100\,\%$, $\sigma = 20\,\%$ and $\zeta = 20\,\%$, the participation rate declines from $67.24\,\%$ ($k_1 = 0$) to $65.71\,\%$, $64.26\,\%$, $63.56\,\%$ and $60.30\,\%$ when k_1 is increased to $0.2\,\%$, $0.4\,\%$, $0.5\,\%$ and $1.0\,\%$ respectively. High transaction costs force us to inflate the positions of our portfolio, this observation is also consistent with our intuition. Notice that these results suggest a convex relationship between α and k_1; that is, *ceteris paribus*, the decrease in the fair participation rate is less pronounced as we consider higher transaction costs. However, these changes in α are not uniform across rows; when cap rates are low, participation rates are more sensitive to k_1. Between $k_1 = 0$ and $k_1 = 0.2\,\%$, these changes are $-4.92\,\%$ ($\zeta = 12\,\%$), $-2.99\,\%$ ($\zeta = 15\,\%$), $-1.85\,\%$ ($\zeta = 20\,\%$) and $-1.53\,\%$ ($\zeta = \infty$), for parameter values $\beta = 90\,\%$ and $\sigma = 20\,\%$. This behavior is directly related to the general above-mentioned effect of caps.

5.2 *Constant Transaction Costs*

Constant transaction costs are also manifest in capital markets in the form of commission fees. In the case of insurance contracts it could refer to policy fees. The ask-price $S^A(t) = S(t) + k_2$ and bid-price $S^B(t) = S(t) - k_2$ are simply translations of the underlying value of the asset. Unlike proportional transaction costs, its bid/ask spread is insensitive to the value of the stock process and therefore remains constant in time.

In this setting we have $k(t, \mathbf{i}_t) = k_2$ and $S^*(t, \mathbf{i}_t) = S(t, \mathbf{i}_t)$. Furthermore, the self-financing conditions of (3.7) become

$$\begin{aligned} b_n(t, \mathbf{i}_t)e^{\delta\Delta} + a_n(t, \mathbf{i}_t)S(t+\Delta, \{\mathbf{i}_t, i_t + j\}) = {} \\ b_n(t+\Delta, \{\mathbf{i}_t, i_t + j\}) + a_n(t+\Delta, \{\mathbf{i}_t, i_t + j\})S(t+\Delta, \{\mathbf{i}_t, i_t + j\}) \\ + k_2|a_n(t+\Delta, \{\mathbf{i}_t, i_t + j\}) - a_n(t, \mathbf{i}_t)|, \end{aligned}$$

for $j = 0, 1$. Here, transaction costs depend only on the change of equity positions; compared to our first model, this constitutes an advantage for high stock prices and conversely for low stock prices. This also means that a constant charge of k_2 is deducted for expenses every time the insurance company rebalances the replicating portfolio.
Because our algorithm hold if and only if $S(t+\Delta, i_t+1) - k_2 > S(t+\Delta, i_t) + k_2$ for all $t = 0, \Delta, \cdots, n-\Delta$, there is considerably less freedom in choosing values for k_2. For $\sigma = 20\,\%$ and $\sigma = 30\,\%$, this condition reduces to $k_2 < 0.766$ and $k_2 < 0.352$, respectively.

		k_2								
σ	ζ	0.00	0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.75
3% Minimum Guarantee on 100% Premium										
20%	12%	83.98	83.09	82.39	81.85	81.32	80.27	79.26	78.26	75.89
	15%	72.46	72.04	71.64	71.23	70.83	70.05	69.28	68.54	66.74
	20%	67.24	66.91	66.58	66.26	65.95	65.32	64.72	64.12	62.71
	∞	65.94	65.63	65.32	65.02	64.72	64.13	63.55	62.99	61.65
30%	12%	90.12	89.37	88.61	87.87	87.13	85.67	n/a	n/a	n/a
	15%	66.65	66.24	65.83	65.42	65.02	64.22	n/a	n/a	n/a
	20%	56.54	56.28	56.03	55.78	55.53	55.04	n/a	n/a	n/a
	∞	51.45	51.25	51.05	50.85	50.65	50.26	n/a	n/a	n/a
3% Minimum Guarantee on 90% Premium										
20%	12%	115.13	114.06	113.01	111.98	110.96	108.97	107.03	105.14	100.65
	15%	91.66	91.02	90.39	89.77	89.15	87.94	86.75	85.60	82.81
	20%	81.40	81.00	80.60	80.20	79.81	79.04	78.28	77.55	75.78
	∞	78.27	77.92	77.58	77.23	76.90	76.23	75.58	74.95	73.42
30%	12%	191.90	188.13	184.40	180.72	177.09	169.93	n/a	n/a	n/a
	15%	98.39	97.25	96.12	95.00	93.89	92.38	n/a	n/a	n/a
	20%	75.16	74.77	74.38	74.00	73.62	72.88	n/a	n/a	n/a
	∞	64.05	63.81	63.57	63.33	63.10	62.64	n/a	n/a	n/a

Table 2: Fair Participation Rates under Constant Transaction Costs

Table 2 gives the fair participation rates in the constant transaction cost setting based on (4.22). Again, $k_2 = 0$ corresponds to the no-transaction costs case and serves as a benchmark for our results. Also, our previous comments regarding the general effects of β and ζ on the fair participation rate remain valid.
For $\sigma = 30\,\%$, the fair participation rate is unavailable for $k_2 = 0.40$, 0.50 and 0.75 since the algorithm fails for constant transaction costs larger than 0.352. In

other words there is no α such that $W_n(x, 0, i_0)$ is equal to 1 (the single premium).
Just as before, α declines as the parameter k_2 increases; for $\beta = 100\,\%$, $\sigma = 20\,\%$ and $\zeta = 20\,\%$, we obtain participation rates of $67.24\,\%$ ($k_2 = 0$), $66.58\,\%$ ($k_2 = 0.10$), $65.95\,\%$ ($k_2 = 0.20$), $64.72\,\%$ ($k_2 = 0.40$) and $62.71\,\%$ ($k_2 = 0.75$). In order to compare these results to Table 1, we equate time-0 transaction cost under both methods. In that sense, a proportional cost of $k_1 = 0.30\,\%$ is equivalent to a constant cost of $k_2 = 0.30$. When this is done, we note that the proportional cost always yield a smaller participation rate; however, the difference in α never exceeds $1\,\%$. This is understandable, since the advantage of constant costs over proportional costs for high stock price offsets its disadvantage for low stock prices.
Furthermore, a (weak) convex relationship between α and k_2 is also observed here. As before, changes in participation rates are sensitive to the cap rate: between $k_2 = 0$ and $k_2 = 0.20$, these changes are $-14.81\,\%$ ($\zeta = 12\,\%$), $-4.50\,\%$ ($\zeta = 15\,\%$), $-1.54\,\%$ ($\zeta = 20\,\%$) and $-0.95\,\%$ ($\zeta = \infty\,\%$), for parameter values $\beta = 90\,\%$ and $\sigma = 30\,\%$. Again, these changes are similar to the proportional transaction costs case.

5.3 *Mixed Transaction Costs*

Mixed transaction costs represent a generalization of the two previous bid/ask spreads and, as such, have features emanating from both methods. The ask-price $S^A(t) = (1 + k_1)S(t) + k_2$, and bid-price $S^B(t) = (1 - k_1)S(t) - k_2$, is obtained by translating a proportion of the stock price. In that way, transaction costs are subject to the same explosive growth in high stock prices but are now kept above a given bound for low stock prices.
In this setting, we have $k(t, \mathbf{i}_t) = k_1 S(t, \mathbf{i}_t) + k_2$ and $S^*(t, \mathbf{i}_t) = S(t, \mathbf{i}_t)$. Furthermore, the self-financing conditions of (3.7) become

$$\begin{aligned}
&b_n(t, \mathbf{i}_t)e^{\delta\Delta} + a_n(t, \mathbf{i}_t)S(t + \Delta, \{\mathbf{i}_t, i_t + j\}) = \\
&\qquad b_n(t + \Delta, \{\mathbf{i}_t, i_t + j\}) + a_n(t + \Delta, \{\mathbf{i}_t, i_t + j\})S(t + \Delta, \{\mathbf{i}_t, i_t + j\}) \\
&\qquad\quad + (k_1 S(t + \Delta, i_t + j) + k_2)|a_n(t + \Delta, \{\mathbf{i}_t, i_t + j\}) - a_n(t, \mathbf{i}_t)|,
\end{aligned}$$

for $j = 0, 1$. Note that the lower bound on transaction costs is equal to k_2. For comparative purposes, we fix $k_2 = 0.15$ and allow k_1 to range as in Table 1.

σ	ζ	0.0	0.1	0.2	0.3	0.4	0.5	1.0	2.0	5.0
						$k_2 = 0.15, k_1$ %				
3% Minimum Guarantee on 100% Premium										
20%	12%	81.85	80.55	79.29	78.05	76.85	75.67	70.11	60.08	40.74
	15%	71.23	70.27	69.33	68.41	67.51	66.63	62.44	56.10	41.98
	20%	66.26	65.51	64.79	64.08	63.38	62.71	59.54	54.17	42.78
	∞	65.02	64.32	63.65	62.99	62.34	61.71	58.79	53.82	43.25
30%	12%	87.87	86.05	84.25	82.45	80.68	78.91	70.16	61.84	41.26
	15%	65.42	64.40	63.39	62.38	61.39	60.52	57.34	51.11	37.76
	20%	55.78	55.16	54.56	53.96	53.38	52.81	50.12	45.33	35.95
	∞	50.85	50.38	49.92	49.48	49.04	48.61	46.59	43.07	35.27
3% Minimum Guarantee on 90% Premium										
20%	12%	111.98	109.58	107.24	104.96	102.74	100.57	90.33	76.04	50.05
	15%	89.77	88.30	86.86	85.46	84.08	82.74	77.30	69.04	51.10
	20%	80.20	79.29	78.41	77.55	76.70	75.87	71.99	65.63	51.55
	∞	77.23	76.48	75.75	75.03	74.32	73.64	70.41	64.82	52.62
30%	12%	180.72	172.40	164.20	156.11	148.13	140.27	114.36	92.30	54.76
	15%	95.00	92.71	91.27	89.87	88.49	87.14	80.80	69.64	49.28
	20%	74.00	73.08	72.17	71.29	70.42	69.56	65.51	59.07	46.23
	∞	63.33	62.81	62.29	61.79	61.29	60.80	58.49	54.40	45.10

Table 3: Fair Participation Rates under Mixed Transaction Costs

Table 3 gives the fair participation rates based on (4.22). Here, $k_1 = 0$ corresponds to constant transaction costs with $k_2 = 0.15$; this benchmark allows us to isolate the effect of increased proportional costs. Furthermore, the general effects of β and ζ on the fair participation rate, as well as the convexity of α relative to k_1, remain true.

Our results have a striking resemblance with those obtained in Table 1: save for a few basis points, the effect of k_1 on the fair participation rate is unchanged. This seems to indicate that the influence of proportional transaction costs is largely independent of the prevailing lower bound on such costs. To demonstrate this, consider the changes in α when k_1 is raised from 0 to $0.20\,\%$: we obtain $-4.74\,\%$ ($\zeta = 12\,\%$), $-2.91\,\%$ ($\zeta = 15\,\%$), $-1.79\,\%$ ($\zeta = 20\,\%$) and $-1.48\,\%$ ($\zeta = \infty$), for parameter values$\beta = 90\,\%$ and $\sigma = 20\,\%$. These results are virtually identical to those outlined in Table 1. The full change in α, relative to the no-transaction cost case, can be obtained therefore by adding the cost of constant transaction costs and proportional transaction costs separately.

Moreover, by equating initial transaction costs, we may also determine the most expensive bid/ask design. For instance, taking $(k_1, k_2) = (0.30\,\%, 0)$, $(0, 0.30)$ or $(0.15\,\%, 0.15)$ renders the stock equally expensive at time 0. However, for $\beta = 90\,\%$, $\sigma = 30\,\%$ and $\zeta = 20\,\%$, the fair participation rate associated with these cost structures are $72.37\,\%$, $72.88\,\%$ and $72.62\,\%$, respectively. For mixed transaction costs, α lies mid-way between the more expensive, proportional cost design and the less expensive, constant cost design. This is expected, since the mixed cost design is equally derived from them.

6 Conclusions

The purpose of this paper is to introduce the transaction costs in the evaluation of equity-linked products. To this end, we employ a general method similar to the approach of Boyle & Vorst (1992) that allows arbitrary bid/ask spreads. The underlying replicating portfolio is obtained for financial contingent claims as well as equity-linked products. The participation rates for EIAs are extracted through the hedging strategy. It is important to point-out that the proposed approach includes transaction costs at the origin and the ending points. A detailed numerical analysis is then performed, for various transaction cost designs, on an EIA existing in the North-American market.

Acknowledgments

This research was supported by the Natural Sciences and Engineering Research Council of Canada.

References

[1] Bacinello, A.R. (2003). Pricing Guaranteed Life Insurance Participating Policies with Annual Premiums, and Surrenders Option. *North American Actuarial Journal,* 7(3): 1–17.

[2] Bensaid, B., Lesne, J.P., Pages, H.F. & Scheinkman, J.A. (1992). Derivative Asset Pricing with Transaction Costs. *Mathematical Finance*, 2(2), 63–86.

[3] Black, F. & Scholes, M.S. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(1), 637–654.

[4] Boyle, P.P. & Tan, K.S. (1994). Lure of the Linear. *Risk*, 7(4), 43–46.

[5] Boyle, P.P. & Vorst, T. (1992). Option Replication in Discrete Time with Transaction Costs. *The Journal of Finance*, 47(1), 271–293.

[6] Bowers, N.L., Jr., Gerber, H.U., Hickman, J.C., Jones, D.A & Nesbitt, C. (1997). *Actuarial Mathematics, 2nd ed.* Society of Actuaries, Schaumburg, Illinois.

[7] Costabile, M., Massabo, I. & Russo, E. (2008). A binomial model for valuing equity-linked policies embedding surrender options. *Insurance: Mathematics and Economics*, 42(3), 873–886.

[8] Cox, J., Ross, S.A. & Rubinstein, M. (1979). Option Pricing: A Simplified Approach. *Journal of Financial Economics*, 7(3), 229–263.

[9] Edirsinghe, C., Naik, V. & Uppal, R. (1993). Optimal Replication of Option with Transaction Cost and Trading Restrictions. *Journal of Financial and Quantitative Analysis*, 28(1), 117–128.

[10] Gaillardetz, P. & Lakhmiri, J.Y. (2006). Loaded Participation Rates for Equity-indexed Annuities. *ARCH Proceeding of the Actuarial Research Conference* (Montréal, Canada).

[11] Gaillardetz, P. & Lin, X.S. (2006). Valuation of Equity-linked Insurance and Annuity Products with Binomial Models. *North American Actuarial Journal*, 10(4), 117–144.

[12] Gerber, H.U. & Shiu, E.S.W. (1994). Option Pricing by Esscher Transforms. *Transactions of the Society of Actuaries*, 46, 99–140.

[13] Hardy, M.R. (2003). *Investment Guarantees: Modeling and Risk Management for Equity-linked life Insurance*. Wiley, New York.

[14] Kijima, M. & Wong, T. (2007). Pricing of Ratchet equity-indexed annuities under stochastic interest rates. *Insurance: Mathematics and Economics*, 41(3), 317–338.

[15] Lee, H. (2003). Pricing Equity-indexed Annuities with Path-dependent Options. *Insurance: Mathematics and Economics*, 33(3), 677–690.

[16] Leland, H.E. (1985). Option Pricing and Replication with Transactions Costs. *Journal of Finance*, 40(5), 1283–1301.

[17] Lin, X.S. (2006). *Introductory Stochastic Analysis for Finance and Insurance*. Wiley, New York.

[18] Lin, X.S. & Tan, K.S. (2003). Valuation of Equity-indexed Annuities under Stochastic Interest Rate. *North American Actuarial Journal*, 7(3), 72–91.

[19] National Association of Variable Annuities (2006). *2006 Annuity Fact Book, 5th ed.* Online at http://www.navanet.org.

[20] Panjer, H.H. et al. (1998). *Financial Economics with Applications to Investment, Insurance and Pensions*. The Actuarial Foundation, Schaumburg, Illinois.

[21] Tiong, S. (2000). Valuing Equity-indexed Annuities. *North American Actuarial Journal*, 4(4), 149–163.

Patrice Gaillardetz
Department of Mathematics and Statistics
Concordia University
1455 de Maisonneuve Blvd. West
Montreal, QC, H3G 1M8

gaillard@mathstat.concordia.ca

Abstract

In this paper, we will evaluate equity-indexed annuities under transaction costs. The hedging strategy underlying the fair evaluation of equity-linked products is critical in the evaluation of such embedded financial guarantees. The proportional and constant transaction costs play a crucial role in the revision of this replicating portfolio since it represents extra cost for issuers. Moreover, the pricing of equity-linked products needs to be extracted from the hedging strategy that leads to challenging programming issues. A detailed numerical analysis is performed for an existing equity-indexed annuity in the North American market.

Résumé

Cet article évalue les rentes variables liées aux valeurs boursières lorsque l'on considère les coûts de transaction. Déterminer la stratégie de couverture sous-jacente à l'évaluation de ces produits d'assurance est essentielle afin réduire le risque lié aux garanties financières. Des coûts de transaction constants et proportionnels représentent des dépenses additionnelles pour l'émetteur et doivent absolument être considérés lors de l'ajustement de la stratégie de couverture. Pour ce type de produit, le prix doit être déterminé basé sur cette stratégie, ce qui augmente considérablement la complexité de l'algorithme informatique. Nous concluons en présentant un exemple numérique impliquant des rentes variables liées aux valeurs boursières disponibles en Amérique du Nord.

Zusammenfassung

In diesem Paper werden Renten bewertet, deren Höhe von einem Börsenindex abhängen. Dabei werden Transaktionskosten berücksichtigt. Die Bestimmung der Hedging-Strategie ist wesentlich, um das Risiko, das mit den abgegebenen finanziellen Garantien verbunden ist, zu reduzieren. Konstante und proportionale Transaktionskosten stellen zusätzliche Ausgaben für den Emittenten dar und müssen unbedingt bei der Adjustierung der Hedging-Strategie berücksichtigt werden. Für solche Produkte muss der Preis unter Berücksichtigung dieser Strategie bestimmt werden, was die Komplexität der Programmierung deutlich erhöht. Anschliessend präsentieren wir ein numerisches Beispiel, das auf entsprechenden Produkten im nordamerikanischen Markt basiert.