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A Note on the Maximum Severity of Ruin in an Erlang(n) Risk Process

1 Introduction

Consider a Sparre Andersen surplus process

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i\,, \quad t \geq 0\,, \tag{1}$$

where $u \geq 0$ is the initial reserve, $\{X_i\}_{i=1}^{\infty}$ are i.i.d. positive random variables with common probability distribution (d.f.) P and density p, representing claim amounts. Denote by $\mu_k = E[X^k]$ the k-th moment of X and by $\hat{p}(s) = \int_0^\infty e^{-sx}\, p(x)\, dx$ its Laplace transform. The counting process $\{N(t);\, t \geq 0\}$ denotes the number of claims up to time t and is defined as $N(t) = \max\{k : W_1 + W_2 + \cdots + W_k \leq t\}$, where the inter-claim times W_i's are assumed to be i.i.d. random variables with common density function k. Further assume that $\{W_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ are independent.
Define

$$T = \inf\{t > 0 : U(t) < 0\} \qquad (\infty,\ \text{otherwise})$$

to be the time of ruin,

$$\Psi(u) = \mathbb{P}(T < \infty \,|\, U(0) = u), \qquad u \geq 0\,,$$

to be the ruin probability, and $\Phi(u) = 1 - \Psi(u)$ to be the non-ruin probability. In this paper, we consider the risk model in which the inter-claim times are Erlang(n) distributed, e.g.,

$$k(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}\,, \quad t \geq 0\,, \lambda > 0\,, n \in \mathbb{N}^+\,.$$

We also assume that $c\,E(W_1) > E(X_1)$ so $cn > \lambda\,\mu_1$, providing a positive safety loading factor.
The Sparre Andersen risk model with Erlang inter-claim times has been widely studied recently. See Dickson (1998), Dickson and Hipp (1998, 2001), Cheng and Tang (2003), Sun and Yang (2004), Li and Garrido (2004a, b), Gerber and Shiu

(2005), and references therein for details. The focus of this paper is on finding the distributions of the maximum surplus before ruin and the maximum severity of ruin for Erlang(n) risk model which extends the results in Picard (1994) and Li and Dickson (2006). Li and Lu (2008) study the distributions of the maximum surplus before ruin and the maximum severity of ruin in a Markov-modulated risk model which is an extention of the classical compound Poisson risk model in which premium rates, the rate for the Poisson claim arrivals and the distribution of the claim amounts vary in time depending on the state of an underlying Markov jump process.

2 The probability of the surplus attaining a certain level before ruin

Define

$$\tau_b := \inf\{t > 0 : U(0) = u, U(t) \geq b\}, \qquad 0 \leq u \leq b,$$

to be the first time that surplus process upcrosses the level b, and

$$\chi(u,b) = P(T > \tau_b \,|\, U(0) = u)$$

to be the probability that the surplus process attains a given level b from initial surplus u without first falling below zero. Define

$$\xi(u,\, b) = 1 - \chi(u\,, b), \qquad 0 \leq u \leq b\,.$$

Since eventually either ruin occurs without the surplus process attaining b or the surplus attains level b under the assumption that the loading factor is positive, then $\xi(u; b)$ is the probability that ruin occurs from initial surplus u without the surplus ever reaching level b. A mathematical definition is

$$\xi(u,\, b) = P\Big\{ \sup_{0 \leq t \leq T} U(t) < b\,,\; T < \infty \,\Big|\, U(0) = u \Big\}\,. \tag{2}$$

It follows from Li and Dickson (2006) that $\chi(u,b)$ satisfies the following homogeneous integro-differential equation for $0 \leq u < b$:

$$\sum_{k=0}^{n} \frac{\partial^k \chi(u,b)}{\partial u^k} \left(\frac{-c}{\lambda}\right)^k \binom{n}{n-k} = \int_0^u \chi(u-y,\, b)\, p(y)\, dy\,, \tag{3}$$

with the boundary conditions

$$\chi(b,b) = 1\,, \tag{4}$$

$$\left.\frac{\partial^k \chi(u,b)}{\partial u^k}\right|_{u=b} = 0\,, \qquad k = 1,2,\dots,n-1\,. \tag{5}$$

Li and Dickson (2006) show that the solution to Eq. (3) with boundary conditions (4) and (5) is of the form

$$\chi(u;b) = \sum_{j=1}^{n} \eta_j(b)\, v_j(u)\,, \qquad 0 \le u \le b\,, \tag{6}$$

where $v_j(u)$, $j = 1,2,\dots,n$, are n linearly independent particular solutions of the following integro-differential equation

$$B(\mathcal{D})\, v(u) = \int_0^u v(u-y)\, p(y)\, dy\,, \qquad u \ge 0\,, \tag{7}$$

with $\mathcal{D}$ denoting the *differentiation operator* and

$$B(\mathcal{D}) = [\mathcal{I} - (c/\lambda)\mathcal{D}]^n = \sum_{k=0}^{n} (-1)^k \left(\frac{c}{\lambda}\right)^k \binom{n}{k} \mathcal{D}^k = \sum_{k=0}^{n} B_k\, \mathcal{D}^k$$

being an n-th order linear differentiation operator. The n particular solutions $v_1(u), v_2(u), \dots, v_n(u)$ are analyzed in the next section. We can use boundary conditions (4) and (5) to determine coefficients $\eta_1(b), \eta_2(b), \dots, \eta_n(b)$ as follows :

$$\sum_{j=1}^{n} \eta_j(b) v_j(b) = 1\,,$$

$$\sum_{j=1}^{n} \eta_j(b) v_j^{(k)}(b) = 0\,, \quad k = 1,2,\dots,n-1\,.$$

In matrix form,

$$\vec{\boldsymbol{\eta}}^\top(b) = [\mathbf{V}(b)]^{-1} \vec{\mathbf{e}}^\top\,, \tag{8}$$

where $\vec{\boldsymbol{\eta}}(b) = (\eta_1(b), \eta_2(b), \dots, \eta_n(b))$ and $\vec{\mathbf{e}} = (1,0,0,\dots,0)$ are two $1 \times n$ row vectors and $\mathbf{V}(b)$ is an $n \times n$ matrix with (i,j) element being $v_j^{(i-1)}(b)$. Then

$$\chi(u;b) = \vec{\mathbf{v}}(u) \vec{\boldsymbol{\eta}}^\top(b) = \vec{\mathbf{v}}(u) [\mathbf{V}(b)]^{-1} \vec{\mathbf{e}}^\top\,, \tag{9}$$

where $\vec{\mathbf{v}}(u) = (v_1(u), v_2(u), \dots, v_n(u))$ is a $1 \times n$ row vector.
In the following section, we will show that $v_1(u), v_2(u), \dots, v_n(u)$ can be expressed in terms of the non-ruin probability $\Phi(u)$.

3 The n particular solutions

The solution to the homogenous equation (7) is uniquely determined by the initial conditions $v^{(k)}(0)$, for $k = 0, 1, \dots, n-1$, and can be solved by Laplace transforms.
Taking Laplace transform on both sides of (7) yields

$$\hat{v}(s) = \frac{d(s)}{B(s) - \hat{p}(s)}, \qquad s \in \mathbb{C}, \tag{10}$$

where

$$d(s) := \sum_{j=0}^{n-1} s^j \sum_{k=j+1}^{n} B_k \, v^{(k-1-j)}(0) = \sum_{j=0}^{n-1} d_j \, s^j$$

is a polynomial of degree $n-1$ and $B(s) = (1 - c\,s/\lambda)^n$.
It follows from Li and Garrido (2004a, Eq. (3)) or Gerber and Shiu (2005, Eq. (5.9)) that the non-ruin probability $\Phi(u)$ satisfies the homogenous Eq. (7). Further, setting $\delta = 0$ in Eq. (20) of Li and Garrido (2004a), we obtain the following Laplace transform of $\Phi(u)$:

$$\hat{\Phi}(s) = -\Phi(0)\, \frac{c^n}{\lambda^n} \frac{\prod_{i=1}^{n-1}(\rho_i - s)}{B(s) - \hat{p}(s)}, \tag{11}$$

where $\rho_1, \rho_2, \dots, \rho_{n-1}$ are all roots of the Lundberg's equation

$$B(s) = \hat{p}(s), \qquad s \in \mathbb{C},$$

with positive real parts, and

$$\Phi(0) = 1 - \frac{\lambda^{n-1}(n\,c - \lambda\,\mu_1)}{c^n \prod_{i=1}^{n-1} \rho_i},$$

see Li and Garrido (2004a, p. 400) for a detailed derivation of $\Phi(0)$.
In the following Theorem, we assume that $\rho_1, \rho_2, \dots, \rho_{n-1}$ are distinct and we can express the n linearly independent particular solutions $v_1(u), v_2(u), \dots, v_n(u)$ in terms of the non-ruin probability $\Phi(u)$ and ρ_i's.

Theorem 1 If $\rho_1, \rho_2, \dots, \rho_{n-1}$ are distinct, then we have the following expressions for $v_i(u)$'s :

$$v_1(u) = \Phi(u), \tag{12}$$

$$v_j(u) = \sum_{i=1}^{j-1} a_{i,j} \int_0^u \Phi(u-y) e^{\rho_i\, y} dy, \qquad j = 2, 3, \dots, n, \tag{13}$$

where

$$a_{i,j} = -\frac{1}{\prod_{k=1,k\neq i}^{j-1}(\rho_k - \rho_i)}, \quad i = 1, 2, \ldots, j-1.$$

Proof For $j = 1, 2, \ldots, n$, it follows from Eq. (10) that $v_j(u)$ has the following Laplace transform,

$$\hat{v}_j(s) = \int_0^\infty e^{-s\,u} v_j(u)\, du = \frac{d_j(s)}{B(s) - \hat{p}(s)}, \tag{14}$$

where $d_j(s)$ is a polynomial of degree $n-1$ or less. Then we can employ

$$d_j(s) = -\Phi(0)\,\frac{c^n}{\lambda^n} \prod_{i=j}^{n-1}(\rho_i - s), \quad j = 1, 2, \ldots, n-1,$$

$$d_n(s) = -\Phi(0)\,\frac{c^n}{\lambda^n}.$$

By comparing $\hat{\Phi}(s)$ in (11) with $\hat{v}_j(s)$ in (14), we have that

$$\hat{v}_1(s) = \hat{\Phi}(s), \tag{15}$$

$$\hat{v}_j(s) = \frac{\hat{\Phi}(s)}{\prod_{i=1}^{j-1}(\rho_i - s)}, \qquad j = 2, 3, \ldots, n. \tag{16}$$

Inverting (15) we obtain that $v_1(u) = \Phi(u)$. For $j = 2, 3, \ldots, n$, since we assume that $\rho_1, \rho_2, \ldots, \rho_{n-1}$ are distinct, by using partial fractions, we can rewrite (16) as

$$\hat{v}_j(s) = \sum_{i=1}^{j-1} a_{i,j}\,\frac{\hat{\Phi}(s)}{(s - \rho_i)}, \qquad j = 2, 3, \ldots, n.$$

$$v_j(u) = \sum_{i=1}^{j-1} a_{i,j} \int_0^u \Phi(u-y) e^{\rho_i\, y}\, dy, \qquad j = 2, 3, \ldots, n. \tag{17}$$

To prove that the above chosen $v_1(u), v_2(u), \ldots, v_n(u)$ are linearly independent, we assume that there are constants $c_1, c_2, \ldots, c_n$ such that $\sum_{j=1}^n c_j v_j(u) \equiv 0$ for all $u \geq 0$. Then we have $\sum_{j=1}^n c_j \hat{v}_j(s) \equiv 0$ for all $s \in \mathbb{C}$. That is

$$\sum_{j=1}^{n-1} c_j \prod_{i=j}^{n-1}(\rho_i - s) + c_n \equiv 0, \qquad \forall s \in \mathbb{C}.$$

Setting s to be $\rho_{n-1}, \rho_{n-2}, \ldots, \rho_1$, respectively, we obtain $c_n = c_{n-1} = \cdots = c_1 = 0$. This shows that $v_1(u), v_2(u), \ldots, v_n(u)$ are linearly independent. □

Remarks

1. For the classical risk model, e.g., $n = 1$, then formula (9) simplifies to

$$\chi(u;b) = \frac{\Phi(u)}{\Phi(b)}, \qquad 0 \le u \le b,$$

which was first given in Dickson and Gray (1984).

2. When $n = 2$, the Lundberg's equation $B(s) = \hat{p}(s)$ has a unique positive root, say ρ, and

$$\mathbf{V}(b) = \begin{pmatrix} \Phi(b) & \int_0^b e^{\rho x}\, \Phi(b-x)\, dx \\ \Phi'(b) & \Phi(b) + \rho \int_0^b e^{\rho x}\, \Phi(b-x)\, dx \end{pmatrix},$$

$$[\mathbf{V}(b)]^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\begin{pmatrix} \Phi(b) + \rho \int_0^b e^{\rho x}\, \Phi(b-x)\, dx \\ -\Phi'(b) \end{pmatrix}}{\Phi^2(b) + (\rho\Phi(b) - \Phi'(b)) \int_0^b e^{\rho x}\, \Phi(b-x)\, dx},$$

and formula (9) simplifies to

$$\begin{aligned} \chi(u,b) &= \left(\Phi(u), \int_0^u \Phi(u-x) e^{\rho x}\right) [\mathbf{V}(b)]^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{\left[\Phi(b) + \rho \int_0^b \Phi(b-x) e^{\rho x}\, dx\right] \Phi(u) - \Phi'(b) \int_0^u \Phi(u-x) e^{\rho x}\, dx}{\Phi^2(b) + [\rho\, \Phi(b) - \Phi'(b)] \int_0^b \Phi(b-x) e^{\rho x}\, dx}, \end{aligned}$$

which is given in Li and Dickson (2006).

4 The maximum severity of ruin

In this section, we allow the surplus process to continue if ruin occurs, and consider the insurer's maximum severity of ruin from the time of ruin until the time that the surplus next returns to level 0. Since we assume that $cn > \lambda\mu_1$, it is certain that the surplus process attains this level.

Define T' to be the first upcrossing of the surplus process through level 0 after ruin occurs, i.e.,

$$T' = \inf\{t : t > T\,,\ U(t) \geq 0\}\,,$$

and define

$$M_u = \sup\{|U(t)| : U(0) = u\,,\ T \leq t \leq T'\}\,,\quad u \geq 0\,,$$

to be the maximum severity of ruin. Let

$$J(z;\,u) = P(M_u \leq z \,|\, T < \infty)\,,\qquad u,\, z \geq 0\,,$$

denote the distribution function of the maximum severity of ruin given that ruin occurs.

As shown in Dickson (2005, p. 164),

$$J(z;\,u) = \frac{1}{\Psi(u)} \int_0^z g(u,y)\,\chi(z-y,\, z)\,dy\,, \tag{18}$$

where $g(u,y) = \partial G(u,y)/\partial y$, with

$$G(u,y) = P(T < \infty,\, U(T) \geq -y)$$

being the probability that ruin occurs and that the deficit at ruin is at most y.

Substituting (9) into (18) gives

$$J(z;\,u) = \frac{\int_0^z g(u,y)\big(v_1(z-y), v_2(z-y), \ldots, v_n(z-y)\big)\,dy\,[\mathbf{V}(z)]^{-1}\vec{\mathbf{e}}^{\top}}{\Psi(u)}. \tag{19}$$

It follows from (8.7) of Dickson (2005) that

$$\Psi(u+z) = \int_z^\infty g(u,y)\,dy + \int_0^z g(u,\,y)\Psi(z-y)\,dy\,,$$

then

$$\int_0^z g(u,y)v_1(z-y)\,dy = \int_0^z g(u,y)\Phi(z-y)\,dy$$
$$= \int_z^\infty g(u,\,y)\,dy + \int_0^z g(u,\,y)\,dy - \Psi(u+z)$$
$$= \Psi(u) - \Psi(u+z)$$
$$= \Phi(u+z) - \Phi(u)\,, \tag{20}$$

and for $j = 2, 3, \ldots, n$,

$$\int_0^z g(u,y)v_j(z-y)\,dy = \sum_{i=1}^{j-1} a_{i,j} \int_0^z g(u,y) \int_0^{z-y} \Phi(z-y-x)e^{\rho_i\,x}\,dx\,dy$$
$$= \sum_{i=1}^{j-1} a_{i,j} \int_0^z g(u,z-y) \int_0^y \Phi(y-x)e^{\rho_i\,x}\,dx\,dy$$
$$= \sum_{i=1}^{j-1} a_{i,j} \int_0^z e^{\rho_i\,x} \int_x^z g(u,z-y)\Phi(y-x)\,dy\,dx$$
$$= \sum_{i=1}^{j-1} a_{i,j} \int_0^z e^{\rho_i\,x} \int_0^{z-x} g(u,t)\Phi(z-x-t)\,dt\,dx$$
$$= \sum_{i=1}^{j-1} a_{i,j} \int_0^z e^{\rho_i\,x}[\Psi(u) - \Psi(u+z-x)]\,dx$$
$$= \sum_{i=1}^{j-1} a_{i,j} \int_0^z e^{\rho_i\,x}[\Phi(u+z-x) - \Phi(u)]\,dx\,.$$

Let $h_1(z,u) = \Phi(u+z) - \Phi(u)$ and $h_j(z,u) = \sum_{i=1}^{j-1} a_{i,j} \int_0^z e^{\rho_i\,x}[\Phi(u+z-x) - \Phi(u)]\,dx$ for $j = 2, 3, \ldots, n$. Then we have

$$J(z;\,u) = \frac{\vec{\mathbf{h}}(z,u)\,[\mathbf{V}(z)]^{-1}\vec{\mathbf{e}}^\top}{\Psi(u)}\,, \qquad u \geq 0\,,\ z \geq 0\,, \tag{21}$$

where $\vec{\mathbf{h}}(z,u) = \big(h_1(z,u), h_2(z,u), \ldots, h_n(z,u)\big)$ is a $1 \times n$ row vector, $\mathbf{V}(z) = \big(v_j^{(i-1)}(z)\big)_{i,j=1}^n$ is an $n \times n$ matrix, and $\vec{\mathbf{e}} = (1, 0, \ldots, 0)$ is a $1 \times n$ row vector.

In the classical risk model ($n=1$), $\vec{\mathbf{h}}(z,u) = \Phi(u+z) - \Phi(u)$, $\mathbf{V}^{-1}(z) = \frac{1}{\Phi(z)}$, and $\vec{\mathbf{e}} = 1$, then (21) simplifies to

$$J(z;\,u) = \frac{\Phi(u+z) - \Phi(u)}{\Psi(u)\,\Phi(z)} = \frac{\Psi(u) - \Psi(u+z)}{\Psi(u)\,[1-\Psi(z)]}\,,$$

which was first given in Picard (1994).
When $n=2$, $\vec{\mathbf{h}}(z,u) = \left(\Phi(u+z) - \Phi(u)\,,\int_0^z e^{\rho\,x}[\Phi(u+z-x) - \Phi(u)]\,dx\right)$, and (21) simplifies to

$$\begin{aligned} J(z;\,u) = {} & \frac{\Phi(z) + \rho\int_0^z \Phi(z-y)e^{\rho\,y}\,dy}{\Phi^2(z) + [\rho\,\Phi(z) - \Phi'(z)]\int_0^z \Phi(z-y)e^{\rho\,y}\,dy}\left[\frac{\Phi(u+z) - \Phi(u)}{1-\Phi(u)}\right] \\ & - \frac{\Phi'(z)\int_0^z e^{\rho\,x}[\Phi(u+z-x) - \Phi(u)]\,dx}{\Phi^2(z) + [\rho\,\Phi(z) - \Phi'(z)]\int_0^z \Phi(z-y)e^{\rho\,y}\,dy}\,\frac{1}{[1-\Phi(u)]}\,. \end{aligned} \tag{22}$$

Eq. (22) was given in Li and Dickson (2006).

5 Explicit results for exponential claims

In this section, we aim at calculating the moments of M_u and the probability that the maximum deficit occurs at ruin given that ruin occurs. As will be seen, it is not easy to achieve explicit expressions for these quantities despite of knowing the distribution function of M_u. We only focus on finding the explicit results for the risk model with exponential claims and Erlang(2) inter-claim times.

5.1 *The moments of M_u*

Let us suppose that $P(x) = 1 - e^{-\eta\,x}$, $x > 0$, and $c = (1+\theta)\lambda/(2\eta)$ with $\theta > 0$ being the relative safety loading factor. It follows from Dickson (1998) that

$$\Phi(u) = 1 - \left(1 - \frac{R}{\eta}\right)e^{-R\,u}, \qquad u \geq 0\,,$$

where $R > 0$ is the adjustment coefficient and $-R$ is the solution of the equation :

$$\left(\frac{\lambda}{c} - s\right)^2 (s+\eta) - \frac{\lambda^2\,\eta}{c^2} = 0\,. \tag{23}$$

Eq. (23) has another positive root ρ. Then (22) gives

$$1 - J(z;u) = \frac{\alpha e^{-Rz}}{1 - \beta e^{-Rz} - \gamma e^{-(\rho+R)z}},$$

where

$$\alpha = \frac{R(\rho+R)}{\eta(\rho+\eta)}, \quad \gamma = \frac{R(R-\eta)}{\rho(\rho+\eta)}, \quad \beta = 1 - \alpha - \gamma .$$

The r-th moment of M_u given that ruin occurs is

$$\begin{aligned}
&E[M_u^r | T < \infty] \\
&= r \int_0^\infty z^{r-1}[1 - J(z;u)]\, dz \\
&= r\alpha \int_0^\infty z^{r-1} e^{-Rz} \sum_{k=0}^{\infty} \left(\beta e^{-Rz} + \gamma e^{-(\rho+R)z}\right)^k dz \\
&= r\alpha \int_0^\infty z^{r-1} e^{-Rz} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \binom{k}{j} \beta^j \gamma^{k-j} e^{-Rjz} e^{-(\rho+R)(k-j)z}\, dz \\
&= r\alpha \sum_{k=0}^{\infty}\sum_{j=0}^{k} \binom{k}{j} \beta^j \gamma^{k-j} \int_0^\infty z^{r-1} e^{-z[R(k+1)+\rho(k-j)]}\, dz \\
&= \alpha r! \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{\binom{k}{j}\beta^j \gamma^{k-j}}{[R(k+1)+\rho(k-j)]^r} .
\end{aligned} \tag{24}$$

Table 1: The values for $E(M_u)$ and $s.d.(M_u)$ for $n = 1$ and $n = 2$

θ	$n = 1$ $E(M_u)$	$s.d.(M_u)$	$n = 2$ $E(M_u)$	$s.d.(M_u)$
0.05	3.197	7.324	2.474	5.532
0.1	2.638	5.007	2.063	3.805
0.15	2.342	4.015	1.848	3.069
0.2	2.150	3.443	1.709	2.646
0.25	2.012	3.064	1.611	2.368
0.3	1.906	2.792	1.536	2.169

We remark that, like the classical risk model (Dickson (2002)), both $J(z;u)$ and the moments of M_u are independent of u. Table 1 gives the mean and standard deviation of M_u for different values of the loading factor θ for the cases $n=1$ and $n=2$ with exponential claims of mean 1. For $n=1$, we set $\lambda=1$ and so $c=1+\theta$. For $n=2$, we set $\lambda=2$ and so $c=1+\theta$. The values for the case $n=1$ are from Dickson (2002). As we would expect, the mean and the standard deviation of M_u are decreasing as θ increases for both the classical risk model and the Sparre Andersen model with Erlang(2) inter-claim times. Moreover, for fixed loading factor θ, the mean and the standard deviation of M_u in the Erlang(2) risk model are slightly smaller than those in the compound Poisson risk model.

5.2 *The probability that the maximum deficit occurs at ruin*

It follows from Picard (1994) that the probability that the maximum deficit occurs at ruin given that ruin occurs is

$$P(M_u = |U(T)| \,|\, T<\infty) = \frac{\int_0^\infty g(u,y)\chi(0,y)\,dy}{\Psi(u)}.$$

When $p(y)=P'(y)=\eta e^{-\eta y}$, $y>0$, then the lack of memory property of the exponential distribution gives $G(u,y)=P(|U(T)|\le y\,,T<\infty)=\Psi(u)\,P(y)$ and $g(u,y)=\Psi(u)\,p(y)$. Furthermore,

$$\chi(0,y) = \frac{\left[\Phi(y)+\rho\int_0^y \Phi(y-x)e^{\rho x}\,dx\right]\Phi(0)}{\Phi^2(y)+[\rho\,\Phi(y)-\Phi'(y)]\int_0^y \Phi(y-x)e^{\rho\,x}\,dx}$$

$$= \left(\frac{R}{\eta}\right)^2 \frac{1+(\rho\gamma/R)e^{-y(\rho+R)}}{1-\beta e^{-Ry}-\gamma e^{-(\rho+R)y}}.$$

Then

$$P(M_u = |U(T)| | T < \infty)$$
$$= \int_0^\infty p(y)\,\chi(0,y)\,dy$$
$$= \frac{R^2}{\eta}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\beta^j\gamma^{k-j}\int_0^\infty e^{-\eta y}\left[1+(\rho\gamma/R)e^{-y(\rho+R)}\right]e^{-y[Rk+\rho(k-j)]}\,dz$$
$$= \frac{R^2}{\eta}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\beta^j\gamma^{k-j}\left[\frac{1}{\eta+k(\rho+R)-j\rho}+\frac{\gamma\,\rho/R}{\eta+(k+1)(\rho+R)-j\,\rho}\right].$$

Table 2 gives the probabilities that the maximum deficit occurs at ruin for different values of the loading factor θ for the cases $n = 1$ and $n = 2$ with exponential claims of mean 1. For $n = 1$, we set $\lambda = 1$ and so $c = 1 + \theta$. For $n = 2$, we set $\lambda = 2$ and so $c = 1 + \theta$. As expected, as θ increases, the probability that the maximum deficit occurs at ruin increases for both models. For a fixed θ, the probability that the maximum deficit occurs at ruin for the classical risk model is slightly bigger than that for the Erlang(2) risk model.

Table 2: The probability that the maximum deficit occurs at ruin for $n = 1$ and $n = 2$

$n\backslash\theta$	0.05	0.1	0.15	0.2	0.25	0.3
1	0.0476	0.0909	0.1304	0.1667	0.2000	0.2308
2	0.0441	0.0857	0.1249	0.1619	0.1967	0.2296

Concluding Remarks

This note shows that, same as the classical risk model, the distributions of the maximum surplus before ruin and the maximum severity of ruin in the Sparre Andersen model with Erlang(n) inter-claim times depend on the ruin probability only which has been studied in Dickson (1998) for Erlang(2) inter-claim times, Li and Garrido (2004a) for Erlang(n) inter-claim times, and Gerber and Shiu (2005) for generalized Erlang(n) inter-claim times. These papers show that when the claim amounts distribution has a rational Laplace transform, explicit expression exists for the ruin probability, so explicit expressions are available for the distributions for the maximum surplus before ruin and the maximum severity of ruin. Further results on these two quantities can be obtained through their distributions.

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Abstract

We study the distributions of the maximum severity of ruin in a Sparre Andersen risk process with the inter-claim times being Erlang(n) distributed. The distribution can be analyzed through the probability that the surplus process attains a given level from the initial surplus without first falling below zero. This note extends the results in Li and Dickson (2006) from Erlang(2) inter-claim times to Erlang(n) inter-claim times and shows that the distribution of the maximum severity of ruin can be expressed explicitly in terms of the non-ruin probability. When the claim amounts are exponentially distributed, explicit expressions for the moments and the probability that the maximum deficit occurs at ruin are derived.

Zusammenfassung

Wir untersuchen die Verteilungsfunktion der maximalen Schwere des Ruins in einem Sparre Andersen Risikoprozess, wobei die Schadenzwischenzeiten einer Erlang(n)-Verteilung folgen. Dazu bestimmen wir die Wahrscheinlichkeit, dass der Surplus-Prozess, ausgehend von einem Anfangsniveau, eine bestimmte Höhe erreicht, ohne vorher je negativ zu sein. Diese Arbeit verallgemeinert das Ergebnis von Li und Dickson (2006) von Erlang(2)- zu Erlang(n)-Schadenzwischenzeiten und zeigt, dass die Verteilungsfunktion der maximalen Schwere des Ruins explizit mit Hilfe der Nicht-Ruin-Wahrscheinlichkeit ausgedrückt werden kann. Für exponential verteilte Schadenhöhen werden explizite Ausdrücke hergeleitet, einerseits für die Momente, andererseits für die Wahrscheinlichkeit, dass bei Ruin das maximale Defizit auftritt.

Résumé

On étudie la fonction de distribution de la ruine de sévérité maximale du processus de risque de Sparre Andersen, dans le cas où la durée entre deux sinistres est décrite par une distribution d'Erlang(n). On détermine la probabilité que le processus du surplus, partant d'un niveau donné, atteigne un certain montant, sans auparavant avoir atteint une valeur négative. Ce travail est une extension des résultats de Li et Dickson (2006) pour une distribution d'Erlang(2) à une distribution d'Erlang(n). Il montre que la fonction de distribution de la ruine de sévérité maximale peut être exprimée comme fonction de la probabilité de non ruine. On donne des formules explicites pour les moments et pour la probabilité que la ruine soit de sévérité maximale, dans le cas où les montants des sinistres sont décrits par une distribution exponentielle.