# On double periodic non-homogeneous Poisson processes

Autor(en): **Garrido, José / Lu, Yi**

Objekttyp: **Article**

Zeitschrift: **Mitteilungen / Schweizerische Aktuarvereinigung = Bulletin / Association Suisse des Actuaires = Bulletin / Swiss Association of Actuaries**

Band (Jahr): **- (2004)**

Heft 2

PDF erstellt am: **27.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-967313

José Garrido and Yi Lu, Montreal

# On Double Periodic Non–Homogeneous Poisson Processes[1)]

## 1 Introduction

Homogeneous Poisson processes are commonly used in risk theory to model claim frequency. These sometimes give a crude representation since their claim intensity rate $\lambda$ is constant [see Figure 3]. A more general time–dependent model is obtained with non–homogeneous Poisson (NHP) processes, as their intensity rate $\lambda(t)$ is a function of time.

Many natural phenomena evolve in a periodic environment or under seasonal conditions. In turn, these events generate insurance claims. For example, weather factors are known to affect automobile or fire insurance claims, while seasonal snow storms in the north and hurricanes or floods in the south affect property insurance. A periodic time–dependent intensity rate is a reasonable model for the claim frequency in such situations. We show that it can also be tractable, even for the corresponding aggregate claim process.

The similarities between intensity and failure rate functions, used in reliability models, help exploring different applications of NHP process. Some characterization properties of the NHP process with (single) periodic failure rate are derived in Chukova et al. (1993) and Dimitrov et al. (1997). These properties are exploited in a risk model by Garrido et al. (1996). Berg and Haberman (1994) use a non–homogeneous Markov birth process, of which the NHP is a special case, to predict trends in life insurance claim occurences. Some ruin problems in a periodic environment are also considered by Asmussen and Rolski (1994), Rolski et al. (1999) and by Morales (2004). While Schmidli (2003) suggests a double periodic NHP process to price catastrophe PCS options.

A more practical case is when the periodic environment does not repeat itself exactly from year to year, but the short term peak changes over a relatively long period, with different levels in each year. This defines a double periodic environment, especially appropriate to model natural catastrophes, such as hurricanes, which have a peak season in the middle of the year, but with an intensity level also depending on long term climatological effects like La Niña or El Niño. A

[1)]This research was funded by the Natural Sciences and Engineering Council of Canada (NSERC) operating grant OGP0036860 and an NSERC Post-Graduate Scholarship.

corresponding Poisson process model with double periodicity is introduced in here.
Section 2 discusses the periodicity of the NHP and related characteristics. Section 3 presents some practical forms for the claim intensity periodicity. The corresponding compound NHP sums are also studied.

## 2 The NHP Process and Preliminary Resu

Let $\lambda$ be a non–negative (measurable and locally integrable) deterministic function. Consider the number of claims in the time interval $[s,\ t)$, denoted $N_{[s,\ t)}$ for $0 \leq s < t$ (and $N_t$ when $s = 0$). A NHP process is defined as follows.

**Definition 1** A counting process $\{N_t;\ t \geq 0\}$ is said to be non–homogeneous Poisson (NHP) with intensity function $\lambda$, where $\lambda(t) \geq 0$, for $t \geq 0$, if it satisfies:

(a) $N_t = 0$ at $t = 0$;

(b) $\{N_t;\ t \geq 0\}$ has independent increments;

(c) $P\{N_{t+h} - N_t = 1\} = \lambda(t)h + o(h)$, for all $t,\ h \geq 0$;

(d) $P\{N_{t+h} - N_t \geq 2\} = o(h)$, for all $t,\ h \geq 0$,

The function $\Lambda$ defined by

$$\Lambda(t) = \int_0^t \lambda(v)dv\ , \quad \text{for } t \geq 0\ , \tag{1}$$

is called the hazard function or the cumulative intensity function of the process. Consider the number, $N_{[\tau,\ \tau+t)}$, in an interval of the form $[\tau,\ \tau + t)$, where $\tau, t \geq 0$. The time parameter $\tau$, called the initial age of the process, marks the beginning of the time observation period when claims start to be counted. It is well known that for a NHP process the probability of $n$ claims occurring in a time interval of duration $t$ starting at time $\tau$ is given by

$$P\{N_{[\tau,\ \tau+t)} = n\} = \frac{e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}[\Lambda(\tau + t) - \Lambda(\tau)]^n}{n!}\ , \quad n \in \mathbb{N}\ . \tag{2}$$

That is, for a NHP process with intensity function $\lambda$, $N_{[\tau,\ \tau+t)}$ has a Poisson distribution with mean $\Lambda(\tau + t) - \Lambda(\tau) = \int_\tau^{\tau+t} \lambda(v)dv$.

A NHP process reduces to the classical homogeneous Poisson process when its intensity function does not depend on time, i.e. $\lambda(t) = \lambda$, for all $t \geq 0$, and therefore $\Lambda(t) = \lambda t$ is linear.

Now, we consider the case where the risk process evolves in a periodic environment, as when the claim arrival rate may depend on the seasons. Then the intensity function of a NHP claim counting process $\{N_t;\ t \geq 0\}$, is a periodic function, say with a period of $c > 0$ years. Consequently $t - \lfloor \frac{t}{c} \rfloor c \in [0, c)$ is the time of the season, where $\lfloor t \rfloor$ is the integer part of $t \in \mathbb{R}$. A model with double periodicity is introduced in the next section where it is illustrated by a double–beta function.

Referring to Dimitrov et al. (1997) for proofs, we list the following properties for the NHP process $\{N_t;\ t \geq 0\}$ with periodic intensity function.

**Theorem 1** Suppose that the intensity function $\lambda$ is periodic with period $c$, then

(a) The hazard function $\Lambda$ has the almost linear property

$$\Lambda(t) = \left\lfloor \frac{t}{c} \right\rfloor \Lambda(c) + \Lambda\left(t - \left\lfloor \frac{t}{c} \right\rfloor c\right), \quad t \geq 0 .$$

(b) For any integer $n \geq 0$ and $t \geq 0$

$$P\{N_{[nc,\ nc+t)} = k\} = P\{N_t = k\}, \quad k = 0, 1, \dots .$$

Moreover, the random variables $N_{nc}$ and $N_{[nc,nc+t)}$ are mutually independent.

(c) The NHP process has a periodic intensity function $\lambda$ with period $c > 0$ if and only if the random variables $N_{[0,\ c)}$ and $N_{[c,\ c+t)}$ are mutually independent and distributed as $N_c$ and $N_t$, respectively.

(d) For any $t \geq 0$ the random variable $N_t$ can be decomposed in the form

$$N_t = \begin{cases} N_{[0,\ t)}, & \text{if } t \leq c \\ M_1 + M_2 + \cdots + M_{\lfloor \frac{t}{c} \rfloor} + N_{[0,\ t - \lfloor \frac{t}{c} \rfloor c)}, & \text{if } t > c \end{cases},$$

where $\{M_i\}_{i \geq 1}$ are i.i.d. Poisson random variables distributed as $N_{[0,\ c)}$ and independent of $N_{[0,\ t-\lfloor \frac{t}{c} \rfloor c)}$, the latter being a Poisson r.v. distributed as $N_{t - \lfloor \frac{t}{c} \rfloor c}$, for $t - \lfloor \frac{t}{c} \rfloor c \in [0,\ c)$.

## 3 A Double–Beta Periodic Intensity Model

Insurance risks that are subject to seasonal conditions clearly evolve in periodic random environments. There are instances where such seasonal effects combine with social or other phenomena to produce double–periodic, or even more general environments.

Take for instance automobile insurance. In many countries, seasonal patterns affect the number of car accidents from month to month. In addition, driving and other social factors also generate a second weekly periodic pattern, that is a "day-of-the-week" effect. Figure 1 illustrates this phenomenon with the number of fatal collisions recorded in the Canadian province of Ontario, for each day of the week, in year 2002 (the total 770 fatal collisions are reported here in percentage term, see ORSAR, 2002, for a full description of this dataset).

Clearly the number of fatal accidents increases during the week–end, reaching a peak on Fridays, and then reducing progressively down to its minimum level at the beginning of the week. A beta–shape intensity function is fitted to the histogram in Figure 1 to emphasize the weekly periodic pattern.

Similarly, a "time-of-the-day" effect is also apparent in Figure 2, where these 770 fatal collisions are tallied according to the hour of occurrence. Again a beta–shape intensity function is fitted to underline the daily periodic pattern.

A double–periodic accident intensity, with daily short–term cycles, coupled to weekly longer cycles, could perhaps better predict the occurrence of future fatal collisions in each season and help in their prevention. As we will see below,

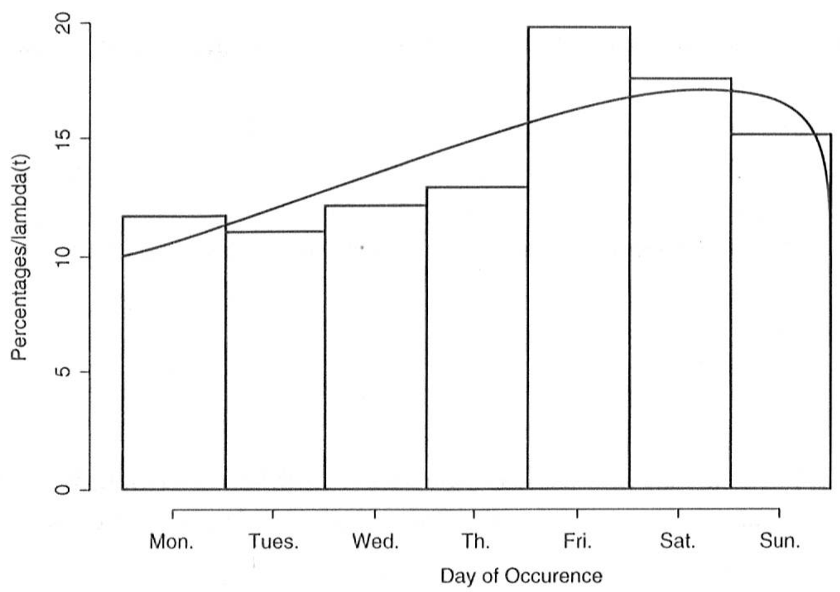


Figure 1: Histogram and fitted Beta weekly fatal collision intensities

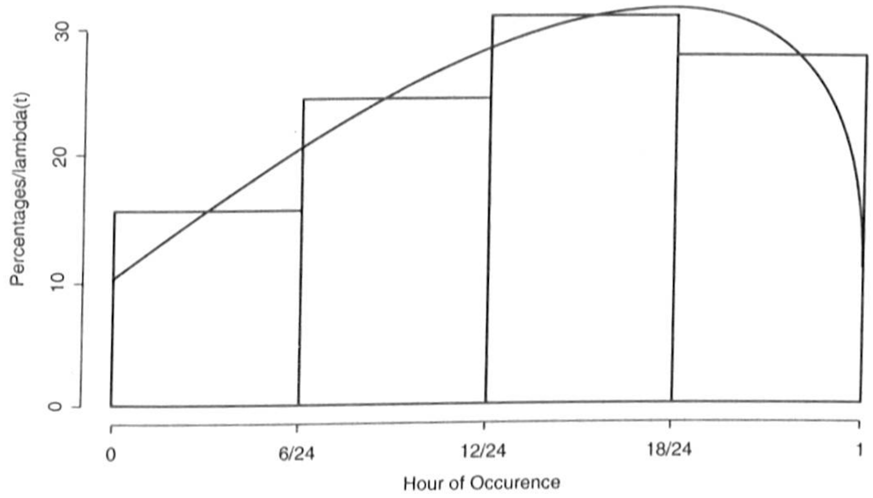


Figure 2: Histogram and fitted Beta hourly fatal collision intensities

with other insured phenomena, like hurricanes, a double–periodic claim intensity model can also better predict future insurance claims counts. With this application in mind, we define here some simple and practical beta–shaped, double–periodic claim intensity models.

First, assume that the short-term period is 1 (year). Let $\lambda_1$ be a beta–shape function, with parameters $p_1, q_1 \geq 1$, defined on $[0, 1]$, such that $\lambda_1(t_1^*) = 1$, where $t_1^* \in [0, 1]$ is the mode of the function. That is

$$\lambda_1(t) = \begin{cases} \dfrac{\left(\frac{t-m_1}{d}\right)^{p_1-1}\left(1-\frac{t-m_1}{d}\right)^{q_1-1}}{\alpha_1^*}, & 0 \leq m_1 \leq t \leq m_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}, \tag{3}$$

where $d = m_2 - m_1$ and

$$\alpha_1^* = \left(\frac{t_1^* - m_1}{d}\right)^{p_1-1}\left(1 - \frac{t_1^* - m_1}{d}\right)^{q_1-1}, \tag{4}$$

is a scale factor, while

$$t_1^* = m_1 + d\frac{p_1 - 1}{p_1 + q_1 - 2},$$

is the mode of $\lambda_1(t)$, so that at the mode $\lambda_1(t_1^*) = 1$ is the peak level.

To illustrate a NHP process with periodic intensity, consider the 155 hurricanes recorded by the National Hurricane Center along the US coastline (Texas to Maine), from 1899 through 1992, and reported by Neumann et al. (1993). If this data set is augmented by the 12 additional hurricanes occurred from 1993 to 2000, more recently reported by Landreneau (2001), we obtain a total of 167 observations. In each case we have the time (month) that the hurricane hit the US coastline, allowing us to draw a frequency histogram.

Figure 3 gives the beta intensity described above, after it was fitted to these annual hurricane frequencies. The constant intensity of a classical homogeneous Poisson process is also given for comparison. Clearly the classical model gives a crude representation of hurricane frequencies.

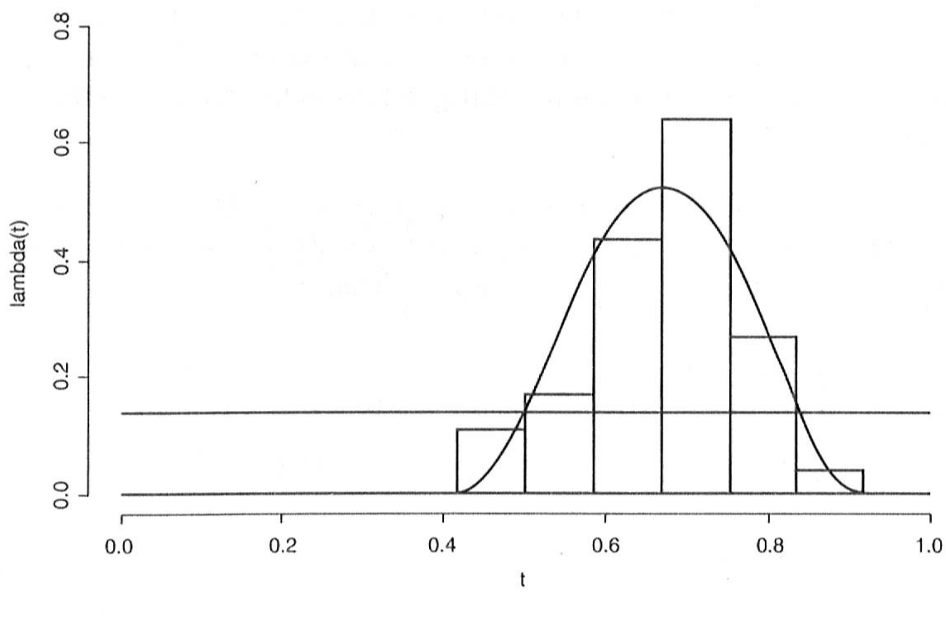


Figure 3: Histogram and fitted hurricane intensities over a 1-year cycle $\lambda(t)$

Although the beta periodic claim intensity seems to provide a better fit to hurricane frequencies, climatological studies suggest that the claim intensity does not repeat the exact same short term pattern every year. Rather, it slightly varies from year to year, as in alternating El Niño–La Niña cycles. This motivates our study of the doubly periodic NHP process presented in this section. Here the seasonality repeats a similar short term pattern every year, letting the peak intensity vary over a longer periodic cycle.

More precisely, assume that the peak value in each year follows another beta function $\lambda_c$ of period $c$ (integer number of years), so called the long term intensity function, given by

$$\lambda_c(t) = a + \frac{b-a}{\alpha_c^*}\left(\frac{t-m_c}{c} - \left\lfloor\frac{t-m_c}{c}\right\rfloor\right)^{p_c-1} \times \left[1 - \left(\frac{t-m_c}{c} - \left\lfloor\frac{t-m_c}{c}\right\rfloor\right)\right]^{q_c-1}, \tag{5}$$

where

$$\alpha_c^* = \left(\frac{t_c^*-m_c}{c}\right)^{p_c-1}\left(1 - \frac{t_c^*-m_c}{c}\right)^{q_c-1}, \tag{6}$$

is again a scale factor, so that $a$ and $b$ are the minimum and maximum amplitude of the peak values, respectively. Here $m_c$ is the starting point of the complete cycle of the second beta function and

$$t_c^* = m_c + c\left(\frac{p_c-1}{p_c+q_c-2}\right)$$

denotes the mode of $\lambda_c(t)$.
Then the double beta intensity function is given by

$$\lambda(t) = \lambda_c\left(\lfloor t - \lfloor \tfrac{t}{c}\rfloor c\rfloor + t_1^*\right)\lambda_1\left(t - \lfloor t\rfloor\right), \quad \text{for } t \geq 0\,, \tag{7}$$

where $\lambda_1$ and $\lambda_c$ are given in (3) and (5), respectively.
Figure 4 illustrates a possible shape of $\lambda(t)$ in (7) when $p_1 = 3$, $q_1 = 2$, $m_1 = \frac{5}{12}$, $d = \frac{6}{12}$, $c = 5$, $p_c = 2$, $q_c = 1.5$, $m_c = 3.75$, $a = 3$ and $b = 7$. The dotted line represents the base (long term) beta function $\lambda_c$ that serves to explain the fluctuations in the peak values of $\lambda_1$, the short term beta periodicity.
By Theorem 2, we can obtain an explicit expression for the hazard function $\Lambda$, defined by (1) in the double–beta periodic case. The corresponding claim counting process $\{N_t,\ t \geq 0\}$ is also decomposed in i.i.d. components.

**Theorem 2** Assume that the intensity function $\lambda$ is given by (7), then

(a) The hazard function $\Lambda$ has the almost linear property, given by

$$\Lambda(t) = \left\lfloor\frac{t}{c}\right\rfloor d\,B(p_1,\, q_1)\sum_{j=0}^{c-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*} + d\,B(p_1,\, q_1)\sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor -1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*} + d\,B\left(p_1,\, q_1;\, \frac{t-\lfloor t\rfloor - m_1}{d}\right)\frac{\lambda_c(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor + t_1^*)}{\alpha_1^*}, \tag{8}$$

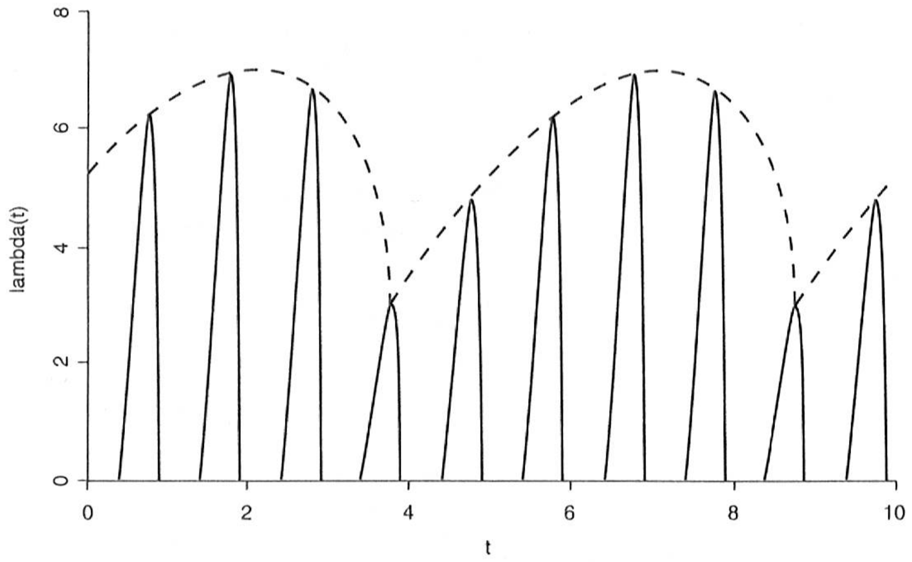


Figure 4: Double–beta intensity function $\lambda(t)$

for $t \geq m_1$, where $\lambda_c(t)$ has the form in (5) and

$$B(p,\ q) = \int_0^1 v^{p-1}(1-v)^{q-1} dv = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

is the beta function at $p,\ q > 0$, while

$$B(p,\ q;\ t) = \begin{cases} 0\ , & \text{if } t \leq 0 \\ \displaystyle\int_0^t v^{p-1}\,(1-v)^{q-1}\,dv\ , & \text{if } t \in (0,\ 1) \\ B(p,\ q)\ , & \text{if } t \geq 1 \end{cases}\quad ,$$

is the usual incomplete beta function.

(b) For any $t \geq 0$, the random variable $N_t$ is decomposed as the independent sum of $\lfloor \frac{t}{c} \rfloor$ i.i.d. Poisson variables $M_i$, for the complete periods, and a different Poisson variable for the incomplete period:

$$N_t = M_1 + \cdots + M_{\lfloor \frac{t}{c} \rfloor} + N^*_{\frac{t - \lfloor t \rfloor - m_1}{d}}\ , \tag{9}$$

where

$$M_i = \sum_{j=0}^{c-1} N^{(j)}_{[(i-1)c,\ ic)}\,, \quad i = 1, 2, \ldots, \lfloor \frac{t}{c} \rfloor \tag{10}$$

and

$$N^*_{\frac{t-\lfloor t \rfloor - m_1}{d}} = \sum_{j=0}^{\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1} N^{(j)}_c + N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}_{\frac{t-\lfloor t \rfloor - m_1}{d}}\,. \tag{11}$$

The $M_i$ are i.i.d. Poisson with mean $d\,B(p_1,\ q_1)\sum_{j=0}^{c-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$, independent of $N^{(j)}_c$, for $j = 0, 1, \ldots, \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1$, and $N^{(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor)}_{\frac{t-\lfloor t \rfloor - m_1}{d}}$, which are all Poisson random variables with mean $d\,B(p_1,\ q_1)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$, where $j = 0, 1, 2, \ldots, \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor - 1$, and $d\,B\left(p_1, q_1; \frac{t-\lfloor t \rfloor - m_1}{d}\right)\frac{\lambda_c(\lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*}$, respectively.

**Proof**

(a) By (1) and the periodicity of the intensity function $\lambda$,

$$\begin{aligned}
\Lambda(t) &= \int_0^t \frac{\lambda_c(\lfloor v \rfloor + t_1^*)}{\alpha_1^*}\left(\frac{v - \lfloor \frac{v}{c} \rfloor c - \lfloor v - \lfloor \frac{v}{c} \rfloor c \rfloor - m_1}{d}\right)^{p_1 - 1} \\
&\qquad \left(1 - \frac{v - \lfloor \frac{v}{c} \rfloor c - \lfloor v - \lfloor \frac{v}{c} \rfloor c \rfloor - m_1}{d}\right)^{q_1 - 1} dv \\
&= \left\lfloor \frac{t}{c} \right\rfloor \int_0^c \frac{\lambda_c(\lfloor v \rfloor + t_1^*)}{\alpha_1^*}\left(\frac{v - \lfloor v \rfloor - m_1}{d}\right)^{p_1 - 1} \\
&\qquad \left(1 - \frac{v - \lfloor v \rfloor - m_1}{d}\right)^{q_1 - 1} dv \\
&\quad + \int_0^{t - \lfloor \frac{t}{c} \rfloor c} \frac{\lambda_c(\lfloor v \rfloor + t_1^*)}{\alpha_1^*}\left(\frac{v - \lfloor v \rfloor - m_1}{d}\right)^{p_1 - 1} \\
&\qquad \left(1 - \frac{v - \lfloor v \rfloor - m_1}{d}\right)^{q_1 - 1} dv
\end{aligned}$$

$$= \left\lfloor \frac{t}{c} \right\rfloor \sum_{j=0}^{c-1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*} \int\limits_{j+m_1}^{j+m_1+d} \left( \frac{v-j-m_1}{d} \right)^{p_1-1} \left( 1 - \frac{v-j-m_1}{d} \right)^{q_1-1} dv$$

$$+ \sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor -1} \frac{\lambda_c(j+t_1^*)}{\alpha_1^*} \int\limits_{j+m_1}^{j+m_1+d} \left( \frac{v-j-m_1}{d} \right)^{p_1-1} \left( 1 - \frac{v-j-m_1}{d} \right)^{q_1-1} dv$$

$$+ \frac{\lambda_c(\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*} \int\limits_{\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor + m_1}^{t-\lfloor \frac{t}{c} \rfloor c} \left( \frac{v-\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor - m_1}{d} \right)^{p_1-1} \left( 1 - \frac{v-\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor - m_1}{d} \right)^{q_1-1} dv \,. \quad (12)$$

Letting $s = \frac{v-j-m_1}{d}$ in the first two integrals and $s = \frac{v-\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor - m_1}{d}$ in the last integral in (12) gives

$$\Lambda(t) = \left\lfloor \frac{t}{c} \right\rfloor \sum_{j=0}^{c-1} d \frac{\lambda_c(j+t_1^*)}{\alpha_1^*} \int\limits_0^1 s^{p_1-1}(1-s)^{q_1-1} ds$$

$$+ \sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor -1} d \frac{\lambda_c(j+t_1^*)}{\alpha_1^*} \int\limits_0^1 s^{p_1-1}(1-s)^{q_1-1} ds$$

$$+ d \frac{\lambda_c(\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*} \int\limits_0^{\frac{t-\lfloor \frac{t}{c} \rfloor c - \lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor - m_1}{d}} s^{p_1-1}(1-s)^{q_1-1} ds \,.$$

Then (8) follows by definition of the beta and incomplete beta functions.

(b) By Theorem 1–(d), $N_t$ can be decomposed as follows

$$N_t = \sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} \sum_{j=0}^{c-1} N^{(j)}_{[(i-1)c,\, ic)} + \sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor -1} N^{(j)}_{[\lfloor \frac{t}{c} \rfloor c,\, (\lfloor \frac{t}{c} \rfloor +1)c)} + N^{(\lfloor t-\lfloor \frac{t}{c} \rfloor c \rfloor)}_{[\lfloor \frac{t}{c} \rfloor c,\, (\lfloor \frac{t}{c} \rfloor +1)c)} \,, \quad (13)$$

where the first term sums over the complete period sub–sums, $M_i = \sum_{j=0}^{c-1} N^{(j)}_{[(i-1)c,\ ic)}$ in (10), while the second summation in (13) accounts for the complete years included in the last (incomplete) period. Finally the last term represents the claim count for the last (incomplete) year of the last (incomplete) period.

By periodicity of the function $\lambda$ and Theorem 1–(b), it is clear that $N^{(j)}_{[(i-1)c,\ ic)}$, $i = 1, 2, \ldots, \lfloor \frac{t}{c} \rfloor + 1$, are mutually independent and Poisson distributed random variables with mean $d\,B(p_1,\ q_1)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$, just as $N_c^{(j)}$, for $j = 0, 1, \ldots, c-1$.

As the additive property of the NHP processes, we consequently get that $M_i$, given in (10), $i = 1, 2, \ldots, \lfloor \frac{t}{c} \rfloor$ are i.i.d. Poisson random variables distributed as $N_c$, with mean $d\,B(p_1,\ q_1)\sum_{j=0}^{c-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$. Similarly, $N^{(\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor)}_{[\lfloor \frac{t}{c}\rfloor c,\ (\lfloor \frac{t}{c}\rfloor+1)c)}$ is Poisson with mean $d\,B\left(p_1,\ q_1;\ \frac{t-\lfloor t\rfloor-m_1}{d}\right) \cdot \frac{\lambda_c(\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}$, like $N^{(\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor)}_{\frac{t-\lfloor \frac{t}{c}\rfloor c-\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor-m_1}{d}}$ or $N^{(\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor)}_{\frac{t-\lfloor t\rfloor-m_1}{d}}$.

Now, setting

$$N^*_{\frac{t-\lfloor t\rfloor-m_1}{d}} = \sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor-1} N^{(j)}_{[\lfloor \frac{t}{c}\rfloor c,\ (\lfloor \frac{t}{c}\rfloor+1)c)} + N^{(\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor)}_{[\lfloor \frac{t}{c}\rfloor c,\ (\lfloor \frac{t}{c}\rfloor+1)c)}$$

$$= \sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor-1} N_c^{(j)} + N^{(\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor)}_{\frac{t-\lfloor t\rfloor-m_1}{d}}\,, \tag{14}$$

gives (11). Combining with (10), (13) leads to (9) and hence (b) holds. □

Now consider $N_{[\tau,\ \tau+t)}$, the number of claims in the time interval $[\tau,\ \tau+t)$. It is assumed to follow a NHP process with parameter $\lambda(t)$ given by (7). From Theorem 2, the probability of $n \in \mathbb{N}$ claims in the time interval $[\tau,\ \tau+t)$ is:

$$P\{N_{[\tau,\ \tau+t)} = n\} = \frac{[\Lambda(\tau+t) - \Lambda(\tau)]^n}{n!} e^{-[\Lambda(\tau+t)-\Lambda(\tau)]}\,,$$

where $\Lambda(\tau+t) - \Lambda(\tau) = \int_\tau^{\tau+t} \lambda(v)dv$ can be derived from (8).

The moment generating function (m.g.f.) of $N_{[\tau,\ \tau+t)}$ is given by

$$E(e^{rN_{[\tau,\ \tau+t)}}) = e^{[\Lambda(\tau+t)-\Lambda(\tau)](e^r-1)}\,,$$

and the expected number of claims over this time interval equals its variance and is given by

$$E(N_{[\tau,\ \tau+t)}) = V(N_{[\tau,\ \tau+t)}) = \Lambda(\tau+t) - \Lambda(\tau)\,.$$

In particular, the m.g.f. of the number of claims over one period of length $c$, with an initial age of $\tau$ (that we will denote $\delta$) equals

$$\delta = E(e^{N_{[\tau,\ \tau+c)}}) = e^{-\Lambda(c)} = e^{-d\,B(p_1,\ q_1)\sum_{j=0}^{c-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}, \tag{15}$$

where $\lambda_c(j + t_1^*)$ can be derived from (5) for $j = 1, 2, \cdots, c - 1$.
Moreover, the probability to survive the time interval $[\tau,\ \tau + t)$ without a claim is

$$P\{N_{[\tau,\ \tau+t)} = 0\} = e^{-[\Lambda(\tau+t)-\Lambda(\tau)]},$$

while the waiting time $T_1$ for the first claim in $[0,\ t)$ has an almost–lack–of–memory distribution [see Dimitrov et al. (1997)] and is given by

$$\begin{aligned} P\{T_1 \leq t\} &= 1 - P\{N_t = 0\} = 1 - e^{-\Lambda(t)} \\ &= 1 - \delta^{\lfloor\frac{t}{c}\rfloor} e^{-d\,B(p_1,\ q_1)\sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}} \\ &\quad \times e^{-d\,B\left(p_1,\ q_1;\ \frac{t-\lfloor t\rfloor-m_1}{d}\right)\frac{\lambda_c(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}}, \end{aligned}$$

where $\delta$ is given by (15). The corresponding p.d.f. is

$$\begin{aligned} f_{T_1}(t) &= \delta^{\lfloor\frac{t}{c}\rfloor} e^{-d\,B(p_1,\ q_1)\sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}} \\ &\quad \times e^{-d\,B\left(p_1,\ q_1;\ \frac{t-\lfloor t\rfloor-m_1}{d}\right)\frac{\lambda_c(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor+t_1^*)}{\alpha_1^*}} \\ &\quad \times \lambda_c\left(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor + t_1^*\right)\lambda_1\left(t-\lfloor t\rfloor\right), \end{aligned}$$

while the expectation of $T_1$ is given by

$$\begin{aligned} E(T_1) &= m_1 + \frac{c\,\delta + \sum_{j=1}^{c-1} j e^{-dB(p_1,\ q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}}\left(1-e^{-dB(p_1,\ q_1)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}}\right)}{1-\delta} \\ &\quad + \frac{d^2\int_0^1\sum_{j=0}^{c-1} e^{-dB(p_1,\ q_1)\sum_{l=0}^{j-1}\frac{\lambda_c(l+t_1^*)}{\alpha_1^*}}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}v^{p_1}(1-v)^{q_1-1}dv}{1-\delta}. \end{aligned}$$

Finally, at time $t$, the excess–life until the next claim, $T_{N_t+1} - t$ is distributed as

$$P\{T_{N_t+1} - t \leq s\} = 1 - e^{-[\Lambda(t+s)-\Lambda(t)]}\ , \quad s \geq 0\ .$$

The flexibility of the beta family of intensity functions, which depends on the value of the shape parameters $p$ and $q$, provides many possible forms of short and long term seasonal claim intensities. Other shapes, like periodic trigonometric functions can also be considered to model the long term periodicity. For example

$$\lambda_c(t) = a + b \sin 2\pi \left( \frac{t - m_c}{c} - \lfloor \frac{t - m_c}{c} \rfloor \right)\ ,$$

where $a \geq b$ and $a+b$, $a-b$ represent, respectively, the maximum and minimum amplitude of the peak values for the long term periodicity, while $m_c$ is the starting point of the periodic sine function.

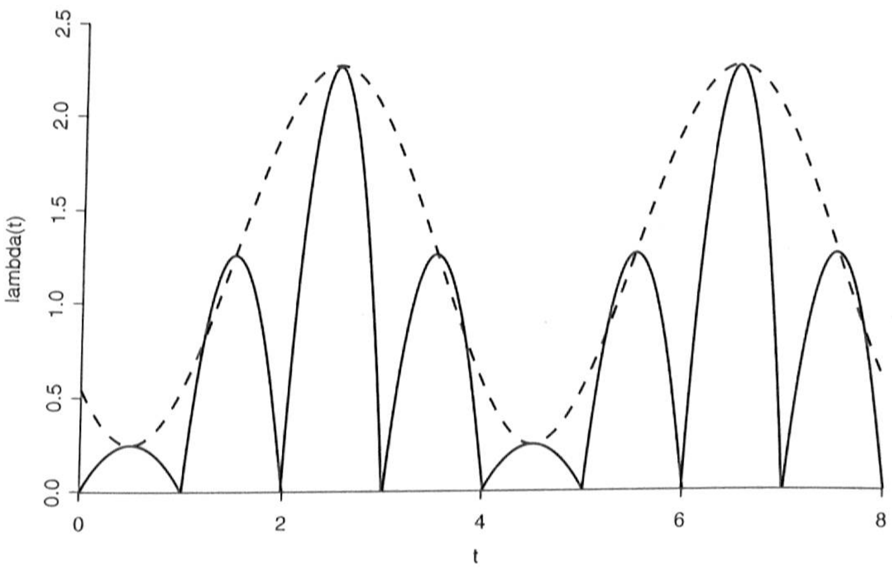


Figure 5: Sine-beta intensity function $\lambda(t)$

Figure 5 illustrates the shape of $\lambda(t)$ for $p_1 = q_1 = 2$, $m_1 = 0$, $d = 1$, $c = 4$, $m_c = \frac{3}{2}$ $a = \frac{5}{4}$ and $b = 1$. Here the short–term beta peak values vary according to the sine function (dotted line). The properties for the corresponding hazard function $\Lambda$ and claim counting process $\{N_t,\ t \geq 0\}$ can be derived analogously.

### *3.1 The Aggregate Claims Process*

The decompositions of Theorem 2 for the NHP process can be extended to compound NHP sums.

Again consider a NHP claim counting process $\{N_t;\ t \geq 0\}$. Then the corresponding aggregate claims process

$$S_t = \begin{cases} \sum_{j=1}^{N_t} X_j & \text{if } N_t > 0 \\ 0 & \text{if } N_t = 0 \end{cases},$$

is called a compound NHP process and is denoted as $S_t \sim$ C.P.$[\Lambda;\ F_X]$, for $x \geq 0$. The $X_j$ are i.i.d. claim severities, with common c.d.f. $F_X$ and finite mean $\mu$, independent of $N_t$.

Consider the claim counting process $\{N_{[\tau,\ \tau+t)},\ t \geq 0\}$, for a fixed initial age $\tau$ and periodic intensity function $\lambda$. Its corresponding hazard function has the following structure:

$$\begin{aligned}
\Lambda(\tau+t) - \Lambda(\tau) &= d\,B\left(p_1,\ q_1;\ \frac{\tau - \lfloor \tau \rfloor - m_1}{d},\ 1\right) \frac{\lambda_c(\lfloor \tau - \lfloor \frac{\tau}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*} \\
&+ d\,B(p_1,\ q_1) \sum_{j=\lfloor \tau - \lfloor \frac{\tau}{c} \rfloor c \rfloor + 1}^{c-1} \frac{\lambda_c(j + t_1^*)}{\alpha_1^*} \\
&+ \left( \left\lfloor \frac{t+\tau}{c} \right\rfloor - \left\lfloor \frac{\tau}{c} \right\rfloor - 1 \right) d\,B(p_1,\ q_1) \\
&\times \sum_{j=0}^{c-1} \frac{\lambda_c(j + t_1^*)}{\alpha_1^*} + d\,B(p_1,\ q_1) \sum_{j=0}^{\lfloor \tau + t - \lfloor \frac{\tau+t}{c} \rfloor c \rfloor - 1} \frac{\lambda_c(j + t_1^*)}{\alpha_1^*} \\
&+ d\,B\left(p_1,\ q_1;\ \frac{\tau + t - \lfloor \tau + t \rfloor - m_1}{d}\right) \frac{\lambda_c(\lfloor \tau + t - \lfloor \frac{\tau+t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*},
\end{aligned} \tag{16}$$

where for any $p, q > 0$,

$$B(p, q, t, 1) = \begin{cases} B(p,\ q)\,, & \text{if } t \leq 0 \\ B(p,\ q) - B(p,\ q;\ t)\,, & \text{if } 0 < t < 1 \\ 0\,, & \text{if } t \geq 1 \end{cases}. \tag{17}$$

The aggregate claims over $[\tau,\ \tau+t)$ is then given by $S_{[\tau,\ \tau+t)} = \sum_{n=1}^{N_{[\tau,\ \tau+t)}} X_n$, where $N_{[\tau,\ \tau+t)}$ is a NHP process with periodic intensity function $\lambda$ as in (7) and $S_{[\tau,\ \tau+t)} = 0$ if $N_{[\tau,\ \tau+t)} = 0$. Theorem 1 implies the following decomposition result.

**Corollary 1** For $t \geq 0$, independent of the initial age $\tau \geq 0$, then by Theorem 1–(d) $S_{[\tau,\, \tau+t)}$ can be decomposed as independent sum of random variables:

$$\begin{aligned} S_{[\tau,\, \tau+t)} &= S^*_{[\tau, \lfloor\tau\rfloor+1)} + \sum_{j=\lfloor \tau - \lfloor \frac{\tau}{c} \rfloor c \rfloor + 1}^{c-1} S^*_j + S_1 + \cdots + S_{\lfloor \frac{t+\tau}{c} \rfloor - \lfloor \frac{\tau}{c} \rfloor - 1} \\ &+ \sum_{j=0}^{\lfloor \tau+t-\lfloor \frac{\tau+t}{c} \rfloor c \rfloor - 1} S^*_j + S^*_{[\lfloor \tau+t \rfloor, \tau+t)} \,, \end{aligned} \tag{18}$$

where $S^*_{[\tau, \lfloor\tau\rfloor+1)}$, $S^*_j$, for $j = \lfloor \tau - \lfloor \frac{\tau}{c} \rfloor c \rfloor + 1, \ldots, c-1$, are compound Poisson sums representing claims for incomplete and complete years, in the first (incomplete) period, with means of $d\,B\left(p_1, q_1;\, \frac{\tau - \lfloor\tau\rfloor - m_1}{d},\, 1\right) \frac{\lambda_c(\lfloor \tau - \lfloor \frac{\tau}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*}$, $d\,B(p_1, q_1) \frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$, for $j = \lfloor \tau - \lfloor \frac{\tau}{c} \rfloor c \rfloor + 1, \ldots, c-1$, respectively, and $S_i$, $i = 1, \ldots, \lfloor \frac{t+\tau}{c} \rfloor - \lfloor \frac{\tau}{c} \rfloor - 1$, are i.i.d. random variables, representing claims for complete cycles, distributed as $S_1 = \sum_{n=1}^{N_c} X_n$, and $N_c$ is a Poisson r.v. with parameter $\Lambda(c)$. While the terms $S^*_j$, for $j = 0, \ldots, \lfloor \tau + t - \lfloor \frac{\tau+t}{c} \rfloor c \rfloor - 1$, and $S^*_{[\lfloor \tau+t \rfloor, \tau+t)}$ are the compound Poisson sums representing complete years and incomplete year in the last (incomplete) period with means of $d\,B(p_1, q_1) \frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$, for $j = 0, \ldots, \lfloor \tau + t - \lfloor \frac{\tau+t}{c} \rfloor c \rfloor - 1$, and $d\,B\left(p_1, q_1;\, \frac{\tau+t-\lfloor\tau+t\rfloor - m_1}{d}\right) \frac{\lambda_c(\lfloor \tau+t-\lfloor \frac{\tau+t}{c} \rfloor c \rfloor + t_1^*)}{\alpha_1^*}$, respectively. All these compound sums are mutually independent.

Moreover, the moment generating function of $S_{[\tau,\, \tau+t)}$ is obtained as

$$E(e^{r S_{[\tau,\, \tau+t)}}) = e^{[\Lambda(\tau+t) - \Lambda(\tau)][M_X(r) - 1]} \,, \tag{19}$$

where $M_X$ is the m.g.f. of the claims severity distribution. Moments of $S_{[\tau,\, \tau+t)}$ are easily obtained from (19). For instance, the total initial premium is given by

$$E(S_{[\tau,\, \tau+t)}) = [\Lambda(\tau+t) - \Lambda(\tau)] E(X_1) \,.$$

## Conclusion

Non–homogeneous Poisson processes with periodic claim intensity rate are useful in modeling risk processes under periodic environments. A double–beta periodic claim intensity model is proposed as a generalization of the classical risk model. It also serves as a more realistic alternative to periodic models with only short term (single) periodic intensity functions.
The flexible shapes of the beta function and the explicit results obtained for the risk process should make these double–periodic models as practical as the classical one. In addition, statistical methods to estimate the beta parameters of the model from real data sets are readily available and shall be illustrated in subsequent work.

## Acknowledgements

We are grateful to Glenn Meyers who, after a presentation of a single periodic model, suggested that the claim intensity peak should vary from year to year, as in alternating El Niño–La Niña cycles.

## Bibliography


[1] Asmussen, S. and Rolski, T. (1994) Risk theory in a periodic environment: the Cramér–Lundberg approximation and Lundberg inequality, *Mathematics of Operations Research*, **19**, 410–433.

[2] Berg, M. and Haberman, S. (1994) Trend analysis and prediction procedures for time nonhomogeneous claim processes, *Insurance: Mathematics and Economics*, **14**, 19–32.

[3] Chukova, S., Dimitrov, B. and Garrido, J. (1993) Renewal and non–homogeneous Poisson processes generated by distributions with periodic failure rate, *Statistics and Probability Letters*, **17**, 19–25.

[4] Dimitrov, B., Chukova, S. and Green, D., Jr. (1997) Probability distributions in periodic random environment and its applications, *SIAM Journal of Applied Mathematics*, **57**, 2, 501–517.

[5] Garrido, J., Dimitrov, B. and Chukova, S. (1996) Ruin modelling for compound non–stationary processes with periodic claim intensity rate, *Mathematics and Statistics Technical Report* No. 2/96, Concordia University, Montreal.

[6] Landreneau, D. (2001) Atlantic tropical storms and hurricanes affecting the United States: 1899–2000, *NOAA Technical Memorandum NWS SR-206 (updated through 2000)*, National Weather Service Office, Lake Charles, Louisiana.
http://www.srh.noaa.gov/lch/research/tropical.htm

[7] Morales, M. (2004) "On a periodic risk averse process: a simulation approach", *North American Actuarial Journal*, forthcoming.

[8] Neumann, C.J., Brian, R., McAdie, C.J. and Elms, J.D. (1993) Tropical cyclones of the North Atlantic ocean, 1871-1992, *Historical Climatology Series 6-2*, National Climatic Data Center, Asheville, North Carolina.

[9] ORSAR. (2002) *Ontario Road Safety Annual Report*, Ministry of Transportation, Ontario, Canada, (see web–page http://www.mto.gov.on.ca/english/safety/orsar/orsar02/)

[10] Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999) *Stochastic Processes for Insurance and Finance*, Wiley, New York.

[11] Schmidli, H. (2003) Modelling PCS options via individual indices, *CAF Working Paper*, No. 157, University of Aarhus.



Jose Garrido
Department of Mathematics and Statistics
Concordia University
7141 Sherbrooke Street West
Montreal, Qc
H4B 1R6 Canada
E-mail: garrido@mathstat.concordia.ca

## Abstract

Non-homogenous Poisson processes with periodic claim intensity rate are proposed as the claim counting process of risk theory. We introduce a doubly periodic Poisson model with short and long term trends, illustrated by a double-beta intensity function. Here periodicity does not repeat the exact same short term pattern every year, but lets its peak intensity vary over a longer period. This model reflects periodic environments like those forming hurricanes, in alternating El Niño/La Niña years. The properties of the model are discussed in detail.

## Zusammenfassung

Der Schadenzahlprozess sei ein nicht-homogener Poisson-Prozess. Dabei sei die Intensität der Schadeneintritte periodisch. Wir definieren ein doppelt-periodisches Modell mit einer Kurzzeit- und einer Langzeitperiode, dargestellt durch eine zweifach-Beta-Intensitätsfunktion.
Die Periodizität wiederholt nicht jedes Jahr das gleiche Kurzzeit-Muster, sondern sie lässt die Spitzen-Intensitäten über eine längere Periode variieren. Dadurch können periodische Bedingungen dargestellt werden, wie beispielsweise Hurrikane in sich abwechselnden El Niño/La Niña-Jahren. Die Eigenschaften des Modells werden detailliert untersucht.

## Résumé

Nous supposons que le processus de comptage des sinistres forme un processus de Poisson non-homogène, dont l'intensité d'arrivée des sinistres est périodique. Nous proposons un modèle doublement périodique, avec périodicité à court et à long terme. Ce dernier est illustré par une fonction d'intensité paramétrique, doublement-beta.
Ici la périodicité ne répète pas exactement la même tendance à court terme d'une année à l'autre, mais permet plutôt que le pic de son intensité varie sur une période à plus long terme. Ce modèle reflète un environnement périodique comme celui qui forme les ouragans, alternant les années des phénomènes El Niño et La Niña. Les propriétés du modèle sont presentées en détail.