Optimal quota share reinsurance for dependent lines of business

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Optimal Quota Share Reinsurance for Dependent Lines of Business

1 Introduction

The problem of optimal quota share reinsurance for a heterogeneous portfolio has been addressed repeatedly in the literature. One of the earliest contributions to this problem is the famous work of De Finetti [1940], and recent publications like those of Mack [2002], Schmitter [1987, 2001] and Schnieper [2000] indicate that the problem is still of interest in actuarial practice.

The problem of optimal quota share reinsurance for a heterogeneous portfolio is interesting since it allows for individual quotas for the different subportfolios. From the point of view of a primary insurer, different sets of quotas for the subportfolios may lead either

- to the same expected return but to different values of the variance of the retention or
- to the same value of the variance of the retention but to different expected returns.

Therefore, there is a trade–off between the expected return and the variance of the retention which leads to the following optimization problems:

- Maximization Problem: *Choose quotas which maximize the expected return under the condition that the variance of the retention is fixed in advance.*
- Minimization Problem: *Choose quotas which minimize the variance of the retention under the condition that the expected return is fixed in advance.*

Until now, these problems have been studied mainly in the case where the losses of the subportfolios are independent or at least uncorrelated; a notable exception is De Finetti [1940].

Before stating these optimization problems in a more precise way, let us fix some notation: For $\mathbf{x}, \mathbf{z} \in \mathbf{R}^n$, we write $\mathbf{x} \leq \mathbf{z}$ if $x_i \leq z_i$ holds for all $i \in \{1, \ldots, n\}$, and in this case we define $[\mathbf{x}, \mathbf{z}] := \{\mathbf{y} \in \mathbf{R}^n \mid \mathbf{x} \leq \mathbf{y} \leq \mathbf{z}\}$. Also, we denote by $\mathbf{0}$ the n–dimensional vector with all coordinates being equal to 0 and by $\mathbf{1}$ the n–dimensional vector with all coordinates being equal to 1. We define $\mathbf{R}^n_+ := \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{x} \geq \mathbf{0}\}$.

Throughout this paper, we consider a random vector $\mathbf{S}$ with values in $\mathbf{R}^n$ and a parameter vector $\boldsymbol{\nu} \in \mathbf{R}^n$. We assume that $\mathbf{1}'\boldsymbol{\nu} > 0$ and that the matrix

$$\boldsymbol{\Sigma} := \mathrm{var}[\mathbf{S}]$$

is regular.

The coordinates of the random vector $\mathbf{S}$ are interpreted as the losses of the n subportfolios of a heterogeneous portfolio of a primary insurer and the coordinates of the vector $\boldsymbol{\nu}$ are interpreted as the contributions of the subportfolios to the expected return of the portfolio. Therefore, $\mathbf{S}$ is called the *vector of losses* and $\boldsymbol{\nu}$ is called the *vector of expected returns*.

Let us first consider the case without reinsurance. The random variable

$$\mathbf{1}'\mathbf{S}$$

is called as the (*total*) *loss* of the primary insurer and the variance of the loss satisfies

$$\mathrm{var}[\mathbf{1}'\mathbf{S}] = \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}$$

We assume that the expected return of the portfolio can be written as

$$\varrho + \mathbf{1}'\boldsymbol{\nu}$$

with some $\varrho \in \mathbf{R}$.

Let us now consider a quota share reinsurance contract with a *vector of quotas* $\mathbf{q} \in [\mathbf{0}, \mathbf{1}]$. The random variable

$$\mathbf{q}'\mathbf{S}$$

is called the *retention* of the primary insurer and the variance of the retention satisfies

$$\mathrm{var}[\mathbf{q}'\mathbf{S}] = \mathbf{q}'\boldsymbol{\Sigma}\mathbf{q}$$

We assume that the expected return of the primary insurer is in this case equal to

$$\varrho + \mathbf{q}'\boldsymbol{\nu}$$

with the same ϱ considered before.

In the special case where the vector of quotas satisfies

$$\mathbf{q} = q\mathbf{1}$$

for some *general quota* $q \in [0, 1]$, we have

$$\mathrm{var}[\mathbf{q}'\mathbf{S}] = q^2\, \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}$$

and

$$\varrho + \mathbf{q}'\boldsymbol{\nu} = \varrho + q\, \mathbf{1}'\boldsymbol{\nu}$$

In this special case, the same quota is applied to each of the subportfolios and both, the variance of the retention and the expected return, are increasing functions of q.

Since a low variance of the retention as well as a high expected return are desirable, the choice of $q \in [0, 1]$ in the special case considered before, and hence the choice of $\mathbf{q} \in [\mathbf{0}, \mathbf{1}]$ in the general case, should result from a trade–off between the variance of the retention and the expected return.

Letting $\varrho = 0$, we are thus led to the following two optimization problems:

- Maximization Problem: *Choose* $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$ *maximizing the expected return* $\mathbf{q}'\boldsymbol{\nu}$ *over all* $\mathbf{q} \in [\mathbf{0}, \mathbf{1}]$ *for which the variance of the retention* $\mathbf{q}'\boldsymbol{\Sigma}\mathbf{q}$ *is equal to some given* $\sigma^2 \in [0, \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}]$.
- Minimization Problem: *Choose* $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$ *minimizing the variance of the retention* $\mathbf{q}'\boldsymbol{\Sigma}\mathbf{q}$ *over all* $\mathbf{q} \in [\mathbf{0}, \mathbf{1}]$ *for which the expected return* $\mathbf{q}'\boldsymbol{\nu}$ *is equal to some given* $\nu \in [0, \mathbf{1}'\boldsymbol{\nu}]$.

In the present paper we study these optimization problems without assuming that the coordinates of $\mathbf{S}$ are uncorrelated.

This paper is organized as follows: In Section 2 we present a quite general example which generalizes an example considered by Mack [2002] and which demonstrates that in many cases the expected return is indeed (up to an additive constant not affecting the optimization problems) equal to $\mathbf{q}'\boldsymbol{\nu}$ for some $\boldsymbol{\nu} \in \mathbf{R}^n$ with $\mathbf{1}'\boldsymbol{\nu} > 0$. In Section 3 we present explicit solutions of auxiliary optimization problems in which the constraint $\mathbf{q} \in [\mathbf{0}, \mathbf{1}]$ is neglected; the proofs of these results are given in the Appendix. In Section 4 we give conditions on the model parameters $\boldsymbol{\Sigma}$ and $\boldsymbol{\nu}$ which ensure that, for any reasonable choice of the parameters σ^2 or ν, the unique solutions $\mathbf{q}^*$ of the optimization problems studied before satisfy the constraint $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$. In Section 5 we give some results on the optimization problems incorporating the constraint $\mathbf{q} \in [\mathbf{0}, \mathbf{1}]$.

2 An Example

The following example generalizes an example considered by Mack [2002; p. 391] who assumes that the coordinates of the vector of losses $\mathbf{S}$ are independent. Here we do not assume that the coordinates of $\mathbf{S}$ are independent or uncorrelated.

Let $\boldsymbol{\pi} \in \mathbf{R}^n$ be a vector whose coordinates represent the primary insurer's premiums for the different subportfolios and define $\boldsymbol{\mu} := E[\mathbf{S}]$. We assume that $\mathbf{1}'\boldsymbol{\mu} > 0$.

Let us first consider the case of no reinsurance. Then the primary insurer's *return* is given by

$$\Pi(\mathbf{1}) := \mathbf{1}'\boldsymbol{\pi} - \mathbf{1}'\mathbf{S}$$

and the variance of the primary insurer's loss is

$$\mathrm{var}[\mathbf{1}'\mathbf{S}] = \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}$$

Since $\mathbf{S}$ is a random vector, the return $\Pi(\mathbf{1})$ is a random variable.

Let us now consider a quota share reinsurance contract with a vector of quotas $\mathbf{q} \in [\mathbf{0}, \mathbf{1}]$. We assume that the reinsurance premium is

$$\beta\, E[(\mathbf{1}-\mathbf{q})'\mathbf{S}] + \gamma\ \mathrm{var}[(\mathbf{1}-\mathbf{q})'\mathbf{S}]$$

with $\beta \in [1, \infty)$ and $\gamma \in [0, \infty)$ satisfying $\beta > 1$ or $\gamma > 0$, which corresponds to a premium principle containing the *expected value principle* ($\beta > 1, \gamma = 0$) and the *variance principle* ($\beta = 1, \gamma > 0$) as special cases; see e.g. Bühlmann [1970] or Schmidt [2002]. Then the primary insurer's *return* is given by

$$\begin{aligned}\Pi(\mathbf{q}) := & \Big(\mathbf{1}'\boldsymbol{\pi} - \mathbf{1}'\mathbf{S}\Big) \\ & - \Big(\beta\, E[(\mathbf{1}-\mathbf{q})'\mathbf{S}] + \gamma\ \mathrm{var}[(\mathbf{1}-\mathbf{q})'\mathbf{S}] - (\mathbf{1}-\mathbf{q})'\mathbf{S}\Big)\end{aligned}$$

and the expected return is

$$\begin{aligned}E[\Pi(\mathbf{q})] &= \Big(\mathbf{1}'\boldsymbol{\pi} - \mathbf{1}'\boldsymbol{\mu}\Big) \\ &\quad - \Big(\beta(\mathbf{1}-\mathbf{q})'\boldsymbol{\mu} + \gamma(\mathbf{1}-\mathbf{q})'\boldsymbol{\Sigma}(\mathbf{1}-\mathbf{q}) - (\mathbf{1}-\mathbf{q})'\boldsymbol{\mu}\Big) \\ &= \Big(\mathbf{1}'\boldsymbol{\pi} - \beta\mathbf{1}'\boldsymbol{\mu} - \gamma\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}\Big) + \mathbf{q}'\Big((\beta-1)\boldsymbol{\mu} + 2\gamma\boldsymbol{\Sigma}\mathbf{1}\Big) - \gamma\mathbf{q}'\boldsymbol{\Sigma}\mathbf{q}\end{aligned}$$

Furthermore, the variance of the primary insurer's retention $\mathbf{q}'\mathbf{S}$ is

$$\mathrm{var}[\mathbf{q}'\mathbf{S}] = \mathbf{q}'\mathbf{\Sigma q}$$

We now study the problem of maximizing the expected return under the condition that the variance of the retention is fixed in advance.

Consider $\sigma^2 \in \mathbf{R}_+$. For all $\mathbf{q} \in \mathbf{R}^n$ satisfying $\mathbf{q}'\mathbf{\Sigma q} = \sigma^2$, the expected return becomes

$$E[\Pi(\mathbf{q})] = \left(\mathbf{1}'\boldsymbol{\pi} - \beta\mathbf{1}'\boldsymbol{\mu} - \gamma\mathbf{1}'\mathbf{\Sigma 1} - \gamma\sigma^2\right) + \mathbf{q}'\left((\beta-1)\boldsymbol{\mu} + 2\gamma\mathbf{\Sigma 1}\right)$$

Letting $\varrho := \mathbf{1}'\boldsymbol{\pi} - \beta\mathbf{1}'\boldsymbol{\mu} - \gamma\mathbf{1}'\mathbf{\Sigma 1} - \gamma\sigma^2$ and $\boldsymbol{\nu} := (\beta-1)\boldsymbol{\mu} + 2\gamma\mathbf{\Sigma 1}$ we obtain $\mathbf{1}'\boldsymbol{\nu} > 0$ and

$$E[\Pi(\mathbf{q})] = \varrho + \mathbf{q}'\boldsymbol{\nu}$$

Therefore, the maximization problem

Maximize	$E[\Pi(\mathbf{q})]$
over the set	$\{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\mathbf{\Sigma q} = \sigma^2\}$

is equivalent with the maximization problem

Maximize	$\mathbf{q}'\boldsymbol{\nu}$
over the set	$\{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\mathbf{\Sigma q} = \sigma^2\}$

which, together with the dual minimization problem, will be studied in the sequel. In the case of the variance principle ($\beta = 1, \gamma > 0$), it follows from Theorem 4.5 that the maximization problem has a unique solution $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$ whenever the parameter σ^2 fulfills the natural condition $\sigma^2 \in [0, \mathbf{1}'\mathbf{\Sigma 1}]$.

Exactly the same maximization problem would result if fixed costs were incorporated in the reinsurance premium.

3 Auxiliary Optimization Problems

For the remainder of this paper we assume that $\varrho = 0$.

In the present section we give two results

- on maximization of the expected return under a constraint on the variance of the retention and
- on minimization of the variance of the retention under a constraint on the expected return.

The proofs of these results are given in the Appendix.

Let us first consider maximization of the expected return under a constraint on the variance of the retention:

3.1 Theorem (Maximization). *For $\sigma^2 \in \mathbf{R}_+$, the optimization problem*

Maximize the expected return	$\mathbf{q}'\boldsymbol{\nu}$
over the set	$\{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\Sigma\mathbf{q} = \sigma^2\}$

has the unique solution

$$\mathbf{q}^* := \left(\frac{\sigma^2}{\boldsymbol{\nu}'\Sigma^{-1}\boldsymbol{\nu}}\right)^{1/2} \Sigma^{-1}\boldsymbol{\nu}$$

and the maximum is $(\sigma^2 \cdot \boldsymbol{\nu}'\Sigma^{-1}\boldsymbol{\nu})^{1/2}$.

Let us now consider minimization of the variance of the retention under a constraint on the expected return:

3.2 Theorem (Minimization). *For $\nu \in \mathbf{R}_+$, the optimization problem*

Minimize the variance of the retention	$\mathbf{q}'\Sigma\mathbf{q}$
over the set	$\{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\boldsymbol{\nu} = \nu\}$

has the unique solution

$$\mathbf{q}^* := \frac{\nu}{\boldsymbol{\nu}'\Sigma^{-1}\boldsymbol{\nu}} \Sigma^{-1}\boldsymbol{\nu}$$

and the minimum is $\nu^2/\boldsymbol{\nu}'\Sigma^{-1}\boldsymbol{\nu}$.

These two optimization problems are dual to each other in a sense which is made precise in the following corollary:

3.3 Corollary (Duality). *For $\sigma^2 \in \mathbf{R}_+$ and $\nu \in \mathbf{R}_+$ satisfying*

$$\sigma^2 = \frac{\nu^2}{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}}$$

the solutions of the optimization problems

Maximize the expected return	$\mathbf{q}'\boldsymbol{\nu}$
over the set	$\{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\boldsymbol{\Sigma}\mathbf{q} = \sigma^2\}$

and

Minimize the variance of the retention	$\mathbf{q}'\boldsymbol{\Sigma}\mathbf{q}$
over the set	$\{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\boldsymbol{\nu} = \nu\}$

are identical.

It is remarkable that, for each of the optimization problems and for any choice of the constraints, the solution $\mathbf{q}^*$ is a multiple of the vector $\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}$.

4 The Constraints on the Vector of Quotas

In both optimization problems considered in the previous section, the constraint $\mathbf{q} \in [\mathbf{0}, \mathbf{1}]$ has been neglected and it is not even guaranteed that at least the solution $\mathbf{q}^*$ fulfills the constraint $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$. The results are nevertheless useful since the constraint $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$ will turn out to be fulfilled under certain additional assumptions on the model parameters.

In the present section, we give necessary and sufficient conditions on the model parameters $\boldsymbol{\Sigma}$ and $\boldsymbol{\nu}$ under which the solution $\mathbf{q}^*$

- of the maximization problem with respect to some $\sigma^2 \in (0, \infty)$ or
- of the minimization problem with respect to some $\nu \in (0, \infty)$

satisfies $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$. The case where $\sigma^2 = 0$ or $\nu = 0$ will henceforth be neglected since it yields the optimal solution $\mathbf{q}^* = \mathbf{0}$.

Let us first consider the constraint $\mathbf{q}^* \geq \mathbf{0}$.

The following result is evident from Theorems 3.1 and 3.2 :

4.1 Theorem (Positivity). *The following conditions are equivalent*:

(a) *The vector of expected returns satisfies $\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} \geq \mathbf{0}$.*

(b) *For every* $\sigma^2 \in (0, \infty)$, *the solution* $\mathbf{q}^*$ *of the maximization problem with respect to* σ^2 *satisfies* $\mathbf{q}^* \geq \mathbf{0}$.

(c) *For every* $\nu \in (0, \infty)$, *the solution* $\mathbf{q}^*$ *of the minimization problem with respect to* ν *satisfies* $\mathbf{q}^* \geq \mathbf{0}$.

In the case $\boldsymbol{\nu} \geq \mathbf{0}$, the inequality $\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} \geq \mathbf{0}$ is fulfilled whenever any two coordinates of $\mathbf{S}$ are uncorrelated, and this condition in turn is fulfilled whenever the coordinates of $\mathbf{S}$ are independent.

Let us now consider the constraint $\mathbf{q}^* \leq \mathbf{1}$.

Define

$$M(\boldsymbol{\nu}) := \max\Big\{\mathbf{e}_i'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} \,\Big|\, i \in \{1, \ldots, n\}\Big\}$$

where $\mathbf{e}_i$ denotes the i–th unit vector of $\mathbf{R}^n$. In the case $\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} \geq \mathbf{0}$ we have $M(\boldsymbol{\nu}) > 0$.

For the maximization problem, we have the following result:

4.2 Theorem (Maximization). *Assume that* $\sigma^2 > 0$ *and let* $\mathbf{q}^*$ *denote the solution of the maximization problem*

Maximize the expected return	$\mathbf{q}'\boldsymbol{\nu}$
over the set	$\{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\boldsymbol{\Sigma}\mathbf{q} = \sigma^2\}$

Then $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$ *if and only if* $\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} \geq \mathbf{0}$ *and* $\sigma^2 \leq \boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}/(M(\boldsymbol{\nu}))^2$.

A completely analogous result holds for the minimization problem:

4.3 Theorem (Minimization). *Assume that* $\nu > 0$ *and let* $\mathbf{q}^*$ *denote the solution of the minimization problem*

Minimize the variance of the retention	$\mathbf{q}'\boldsymbol{\Sigma}\mathbf{q}$
over the set	$\{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\boldsymbol{\nu} = \nu\}$

Then $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$ *if and only if* $\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} \geq \mathbf{0}$ *and* $\nu \leq \boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}/M(\boldsymbol{\nu})$.

Both results are immediate from Theorems 3.1 and 3.2.

We have thus found upper bounds

– for the constraint on the variance of the retention and

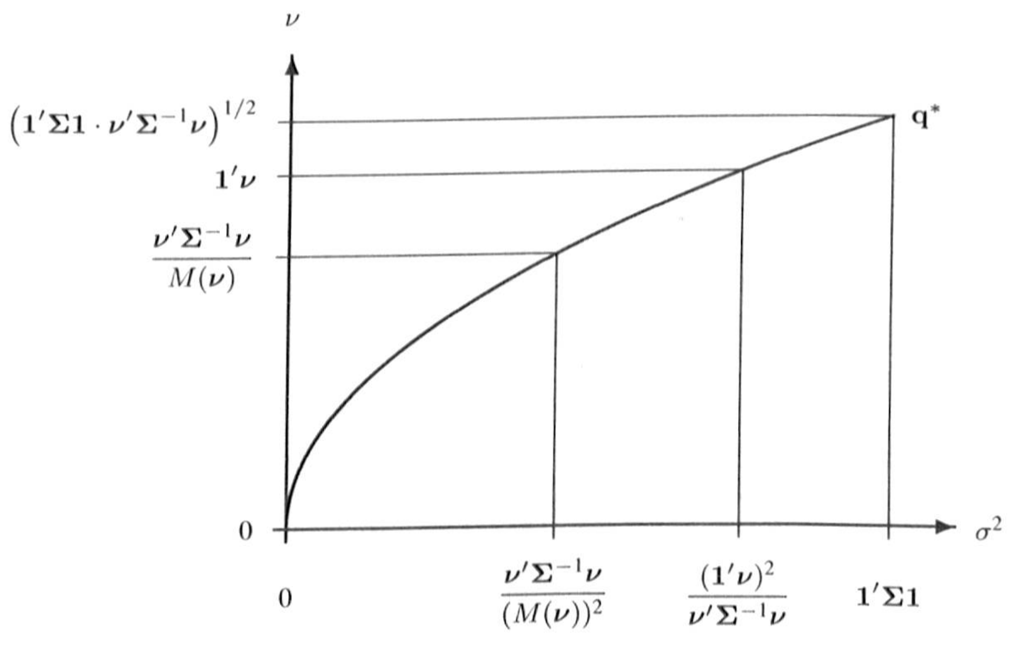
– for the constraint on the expected return

which in the case $\mathbf{\Sigma}^{-1}\boldsymbol{\nu} \geq \mathbf{0}$ guarantee $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$.

Using the relation between the solutions of the maximum problems and the solutions of the minimum problems provided by Corollary 3.3, we obtain the table

$\mathbf{q}^*$	ν	σ^2
$\mathbf{0}$	0	0
$\dfrac{1}{M(\boldsymbol{\nu})}\,\mathbf{\Sigma}^{-1}\boldsymbol{\nu}$	$\dfrac{\boldsymbol{\nu}'\mathbf{\Sigma}^{-1}\boldsymbol{\nu}}{M(\boldsymbol{\nu})}$	$\dfrac{\boldsymbol{\nu}'\mathbf{\Sigma}^{-1}\boldsymbol{\nu}}{(M(\boldsymbol{\nu}))^2}$
$\dfrac{\mathbf{1}'\boldsymbol{\nu}}{\boldsymbol{\nu}'\mathbf{\Sigma}^{-1}\boldsymbol{\nu}}\,\mathbf{\Sigma}^{-1}\boldsymbol{\nu}$	$\mathbf{1}'\boldsymbol{\nu}$	$\dfrac{(\mathbf{1}'\boldsymbol{\nu})^2}{\boldsymbol{\nu}'\mathbf{\Sigma}^{-1}\boldsymbol{\nu}}$
$\left(\dfrac{\mathbf{1}'\mathbf{\Sigma}\mathbf{1}}{\boldsymbol{\nu}'\mathbf{\Sigma}^{-1}\boldsymbol{\nu}}\right)^{1/2}\mathbf{\Sigma}^{-1}\boldsymbol{\nu}$	$\left(\mathbf{1}'\mathbf{\Sigma}\mathbf{1}\cdot\boldsymbol{\nu}'\mathbf{\Sigma}^{-1}\boldsymbol{\nu}\right)^{1/2}$	$\mathbf{1}'\mathbf{\Sigma}\mathbf{1}$

In the case $\boldsymbol{\nu} \geq \mathbf{0}$, the previous table has the following graphical representation:



The table and its graphical representation should be interpreted in the sense that $\mathbf{q}^*$ simultaneously

– maximizes $\mathbf{q}'\boldsymbol{\nu}$ over all $\mathbf{q}$ such that $\mathbf{q}'\mathbf{\Sigma}\mathbf{q} = \sigma^2$ and satisfies $\mathbf{q}^{*\prime}\boldsymbol{\nu} = \nu$ and
– minimizes $\mathbf{q}'\mathbf{\Sigma}\mathbf{q}$ over all $\mathbf{q}$ such that $\mathbf{q}'\boldsymbol{\nu} = \nu$ and satisfies $\mathbf{q}^{*\prime}\mathbf{\Sigma}\mathbf{q}^* = \sigma^2$.

The curve in the (σ^2, ν)–plane is called the *efficient frontier*; see e.g. Schnieper [2000].

In the case $\boldsymbol{\nu} \geq \mathbf{0}$, the following lemma justifies the ordering of the values of $\mathbf{q}^*$ and of ν and σ^2 in the table and its graphical representation:

4.4 Lemma (Inequalities). *Assume that* $\boldsymbol{\nu} \geq \mathbf{0}$. *Then*

$$\begin{aligned}
\frac{1}{M(\boldsymbol{\nu})} &\leq \frac{\mathbf{1}'\boldsymbol{\nu}}{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}} \leq \left(\frac{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}}\right)^{1/2} \\
\frac{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}}{M(\boldsymbol{\nu})} &\leq \mathbf{1}'\boldsymbol{\nu} \qquad \leq \left(\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}\cdot\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}\right)^{1/2} \\
\frac{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}}{(M(\boldsymbol{\nu}))^2} &\leq \frac{(\mathbf{1}'\boldsymbol{\nu})^2}{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}} \leq \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}
\end{aligned}$$

Proof. Since $\boldsymbol{\nu} \geq \mathbf{0}$, we have $\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} \leq M(\boldsymbol{\nu})\cdot\boldsymbol{\nu}'\mathbf{1} = M(\boldsymbol{\nu})\cdot\mathbf{1}'\boldsymbol{\nu}$, and we also have $\mathbf{1}'\boldsymbol{\nu} \leq (\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}\cdot\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu})^{1/2}$; see Harville [1997; Corollary 14.10.3]. This proves the assertion. □

The following result characterizes the case where all inequalities of Lemma 4.4 are equalities. This characterization is valid without the assumption $\boldsymbol{\nu} \geq \mathbf{0}$ made in Lemma 4.4.

4.5 Theorem (Equalities). *The following conditions are equivalent*:

(a) *There exists some* $c \in (0,\infty)$ *such that* $\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} = c\,\mathbf{1}$.

(b) *There exists some* $c \in (0,\infty)$ *such that* $\boldsymbol{\nu} = c\,\boldsymbol{\Sigma}\mathbf{1}$.

(c) *The vector of expected returns satisfies*

$$\boldsymbol{\nu} = \frac{\mathbf{1}'\boldsymbol{\nu}}{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}\,\boldsymbol{\Sigma}\mathbf{1}$$

(d) *The vector of expected returns satisfies*

$$\begin{aligned}
\frac{1}{M(\boldsymbol{\nu})} &= \frac{\mathbf{1}'\boldsymbol{\nu}}{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}} = \left(\frac{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}}\right)^{1/2} \\
\frac{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}}{M(\boldsymbol{\nu})} &= \mathbf{1}'\boldsymbol{\nu} \qquad = \left(\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}\cdot\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}\right)^{1/2} \\
\frac{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}}{(M(\boldsymbol{\nu}))^2} &= \frac{(\mathbf{1}'\boldsymbol{\nu})^2}{\boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}} = \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}
\end{aligned}$$

(e) *The vector of expected returns satisfies* $(\mathbf{1}'\boldsymbol{\nu})^2 = \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1} \cdot \boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu}$.

(f) *For each of the optimization problems and for any choice of the parameters* σ^2 *or* ν, *there exists some* $q \in (0, \infty)$ *such that the solution* $\mathbf{q}^*$ *satisfies* $\mathbf{q}^* = q\mathbf{1}$.

(g) *For any choice of* $\sigma^2 \in \mathbf{R}_+$, *the solution* $\mathbf{q}^*$ *of the maximization problem with respect to* σ^2 *satisfies*

$$\mathbf{q}^* = \left(\frac{\sigma^2}{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}\right)^{1/2} \mathbf{1}$$

(h) *For any choice of* $\nu \in \mathbf{R}_+$, *the solution* $\mathbf{q}^*$ *of the minimization problem with respect to* ν *satisfies*

$$\mathbf{q}^* = \frac{\nu}{\mathbf{1}'\boldsymbol{\nu}} \mathbf{1}$$

Proof. Obviously, (a) and (b) are equivalent. Assume now that (b) holds. Then we have $\mathbf{1}'\boldsymbol{\nu} = c\,\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}$, which yields the value of c. Therefore, (b) implies (c), which means that (b) and (c) are equivalent. Assume again that (b) holds. Then we have

$$\begin{aligned} M(\boldsymbol{\nu}) &= c \\ \mathbf{1}'\boldsymbol{\nu} &= c\,\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1} \\ \boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} &= c^2\,\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1} \end{aligned}$$

Therefore, (b) implies (d). Obviously, (d) implies (e). Assume now that (e) holds. Then we have $\boldsymbol{\nu} = c\,\boldsymbol{\Sigma}\mathbf{1}$ for some $c \in \mathbf{R}$; see Harville [1997; Corollary 14.10.3]. This yields $\mathbf{1}'\boldsymbol{\nu} = c\,\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}$ and hence $c \in (0, \infty)$. Therefore, (e) implies (b). Finally, the equivalence of (b) and each of (f), (g) and (h) is immediate from of Theorems 3.1 and 3.2. □

Theorem 4.5 is important since it provides necessary and sufficient conditions on the vector of expected returns under which

- for any choice of $\sigma^2 \in [0, \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}]$, the solution $\mathbf{q}^*$ of the maximization problem with respect to σ^2 satisfies $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$, and
- for any choice of $\nu \in [0, \mathbf{1}'\boldsymbol{\nu}]$, the solution $\mathbf{q}^*$ of the minimization problem with respect to ν satisfies $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$.

This means that, under the equivalent conditions of Theorem 4.5, the constraint $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$ is fulfilled for any reasonable choice of the parameters σ^2 or ν.

The previous result deserves some comments:

- Condition (b) of Theorem 4.5 is easy to check since it does not require the inversion of the matrix $\boldsymbol{\Sigma}$.
- Condition (b) of Theorem 4.5 is fulfilled if and only if, for each of the optimization problems and any choice of the parameters σ^2 or ν, the same quota is applied to every subportfolio. Therefore, condition (b) of Theorem 4.5 may be regarded as an equilibrium condition for the subportfolios of the heterogeneous portfolio.

Condition (b) of Theorem 4.5 can also be interpreted in terms of the losses S_i of the subportfolios $i \in \{1, \ldots, n\}$ and the total loss $S := \sum_{i=1}^n S_i = \mathbf{1}'\mathbf{S}$:

- Condition (b) of Theorem 4.5 is fulfilled if and only if for all subportfolios the expected return is proportional with the same proportionality factor to the contribution $\operatorname{cov}[S_i, S]$ of the subportfolio to the variance of the total loss. This is a version of the *covariance principle*, applied to the vector of expected returns.
- In the special case where the losses of the subportfolios are uncorrelated or even independent, condition (b) of Theorem 4.5 is fulfilled if and only if the ratio $\nu_i / \operatorname{var}[S_i]$ of the expected return and the variance of the loss is the same for all subportfolios.
- If condition (b) of Theorem 4.5 is fulfilled and if $\boldsymbol{\nu} \geq \mathbf{0}$, then the loss of every subportfolio is positively correlated with the total loss.

We conclude this section with an example.

4.6 Example. Let $n := 2$ and

$$\boldsymbol{\Sigma} := \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Then we have $\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1} = 10$ and

$$\boldsymbol{\Sigma}^{-1} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$$

It follows that $\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} \geq \mathbf{0}$ if and only if the coordinates of the vector of expected returns $\boldsymbol{\nu}$ satisfy $4\nu_2 \leq 2\nu_1 \leq 5\nu_2$. We consider different choices of the vector of expected returns:

(1) Let

$$\boldsymbol{\nu} := \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then we have $\mathbf{1}'\boldsymbol{\nu} = 3$ and

$$\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and hence

$$\begin{aligned} \boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} &= 1 \\ M(\boldsymbol{\nu}) &= 1 \end{aligned}$$

Therefore, the optimal vector of quotas $\mathbf{q}^*$ satisfies $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$ for every choice of $\sigma^2 \in [0, 1]$ and for every choice of $\nu \in [0, 1]$.

(2) Let

$$\boldsymbol{\nu} := \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

Then we have $\mathbf{1}'\boldsymbol{\nu} = 10$ and

$$\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{1}$$

Therefore, the optimal vector of quotas $\mathbf{q}^*$ satisfies $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$ for every choice of $\sigma^2 \in [0, \mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}] = [0, 10]$ and for every choice of $\nu \in [0, \mathbf{1}'\boldsymbol{\nu}] = [0, 10]$, which means that every reasonable constraint in any of the optimization problems yields an optimal vector of quotas $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$.

(3) Let

$$\boldsymbol{\nu} := \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Then we have $\mathbf{1}'\boldsymbol{\nu} = 7$ and

$$\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and hence

$$\begin{aligned} \boldsymbol{\nu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\nu} &= 5 \\ M(\boldsymbol{\nu}) &= 1 \end{aligned}$$

Therefore, the optimal vector of quotas $\mathbf{q}^*$ satisfies $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$ for every choice of $\sigma^2 \in [0, 5]$ and for every choice of $\nu \in [0, 5]$.

5 Interval–Constrained Minimization

We have seen that the problem of maximizing the expected return of the primary insurer under a constraint on the variance of the retention and the problem of minimizing the variance of the retention of the primary insurer under a constraint on the expected return may both result in an optimal vector of quotas $\mathbf{q}^*$ violating the constraint $\mathbf{q}^* \in [\mathbf{0}, \mathbf{1}]$. Therefore, the question arises whether or not the corresponding optimization problems incorporating the interval–constraint $\mathbf{q} \in [\mathbf{0}, \mathbf{1}]$ can be solved as well.

For the minimization problem, we have the following result:

5.1 Theorem (Interval–Constrained Minimization). *Consider $\nu \in [0, \mathbf{1}'\boldsymbol{\nu}]$. Then the optimization problem*

Minimize	$\mathbf{q}'\Sigma\mathbf{q}$
over the set	$D_{[\mathbf{0},\mathbf{1}]}(\boldsymbol{\nu}, \nu) := \{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\boldsymbol{\nu} = \nu, \mathbf{q} \in [\mathbf{0}, \mathbf{1}]\}$

has a unique solution.

Proof. The map $\langle\, .\, ,\, .\, \rangle_\Sigma : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ given by $\langle \mathbf{x}, \mathbf{y} \rangle_\Sigma := \mathbf{x}'\Sigma\mathbf{y}$ is an inner product on the Euclidean space $\mathbf{R}^n$ which turns $\mathbf{R}^n$ into a Hilbert space. Since $\mathbf{R}^n$ has finite dimension, the induced norm $\|\, .\, \|_\Sigma : \mathbf{R}^n \to \mathbf{R}_+$ given by $\|\mathbf{x}\|_\Sigma := (\langle \mathbf{x}, \mathbf{x} \rangle_\Sigma)^{1/2} = (\mathbf{x}'\Sigma\mathbf{x})^{1/2}$ is equivalent with the Euclidean norm.
The set $D_{[\mathbf{0},\mathbf{1}]}(\boldsymbol{\nu}, \nu)$ is nonempty and convex, and it is also closed under the Euclidean norm. Since the norm $\|\, .\, \|_\Sigma$ is equivalent with the Euclidean norm, it follows that the nonempty and convex set $D_{[\mathbf{0},\mathbf{1}]}(\boldsymbol{\nu}, \nu)$ is also closed under the norm $\|\, .\, \|_\Sigma$. Now the assertion follows from the projection theorem in Hilbert spaces; see e.g. Swartz [1997; Theorem 6.13]. □

For those readers which are not familiar with arguments from functional analysis, we present an alternative proof of the theorem which is more lengthy but elementary:

Alternative Proof. The map $f : \mathbf{R}^n \to \mathbf{R}_+$ given by $f(\mathbf{q}) := \mathbf{q}'\Sigma\mathbf{q}$ is continuous and the set $D_{[\mathbf{0},\mathbf{1}]}(\boldsymbol{\nu}, \nu)$ is compact. Therefore, there exist some $\widetilde{\mathbf{q}} \in D_{[\mathbf{0},\mathbf{1}]}(\boldsymbol{\nu}, \nu)$ satisfying

$$\widetilde{\mathbf{q}}'\Sigma\widetilde{\mathbf{q}} \le \mathbf{q}\Sigma\mathbf{q}$$

for all $\mathbf{q} \in D_{[\mathbf{0},\mathbf{1}]}(\boldsymbol{\nu}, \nu)$.

Consider now an arbitrary solution $\widehat{\mathbf{q}} \in D_{[0,1]}(\boldsymbol{\nu}, \nu)$. Then we have

$$\widehat{\mathbf{q}}'\boldsymbol{\Sigma}\widehat{\mathbf{q}} = \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}}$$

Since $0 \le (\widehat{\mathbf{q}} - \widetilde{\mathbf{q}})'\boldsymbol{\Sigma}(\widehat{\mathbf{q}} - \widetilde{\mathbf{q}})$, we obtain

$$\widehat{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}} \le \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}}$$

For $c \in (0, 1)$, define

$$\mathbf{q} := c\,\widetilde{\mathbf{q}} + (1-c)\,\widehat{\mathbf{q}}$$

Since $D_{[0,1]}(\boldsymbol{\nu}, \nu)$ is convex, we have $\mathbf{q} \in D_{[0,1]}(\boldsymbol{\nu}, \nu)$ and hence

$$\widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}} \le \mathbf{q}'\boldsymbol{\Sigma}\mathbf{q}$$

Moreover, straightforward computation yields

$$\begin{aligned}\mathbf{q}'\boldsymbol{\Sigma}\mathbf{q} &= \Big(c\,\widetilde{\mathbf{q}} + (1-c)\,\widehat{\mathbf{q}}\Big)'\boldsymbol{\Sigma}\Big(c\,\widetilde{\mathbf{q}} + (1-c)\,\widehat{\mathbf{q}}\Big) \\ &\le \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}}\end{aligned}$$

Therefore, we have

$$\Big(c\,\widetilde{\mathbf{q}} + (1-c)\,\widehat{\mathbf{q}}\Big)'\boldsymbol{\Sigma}\Big(c\,\widetilde{\mathbf{q}} + (1-c)\,\widehat{\mathbf{q}}\Big) = \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}}$$

Since $\widehat{\mathbf{q}}'\boldsymbol{\Sigma}\widehat{\mathbf{q}} = \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}}$, this implies $\widehat{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}} = \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}}$, hence $(\widehat{\mathbf{q}} - \widetilde{\mathbf{q}})'\boldsymbol{\Sigma}(\widehat{\mathbf{q}} - \widetilde{\mathbf{q}}) = 0$, and thus $\widehat{\mathbf{q}} = \widetilde{\mathbf{q}}$. Therefore, the solution is unique. □

While the previous result is very satisfactory from a theoretical point of view, it does not yield an explicit solution of the optimization problem. We can nevertheless indicate a way to determine the solution of the optimization problem considered in Theorem 5.1:

For the remainder of this section, let $\mathbf{q}^*$ denote the solution of the optimization problem

$$\begin{array}{ll} \textit{Minimize} & \mathbf{q}'\boldsymbol{\Sigma}\mathbf{q} \\ \textit{over the set} & D(\boldsymbol{\nu},\nu) := \{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\boldsymbol{\nu} = \nu\} \end{array}$$

and let $\widetilde{\mathbf{q}}$ denote the solution of the optimization problem

$$\begin{array}{ll} \textit{Minimize} & \mathbf{q}'\boldsymbol{\Sigma}\mathbf{q} \\ \textit{over the set} & D_{[\mathbf{0},\mathbf{1}]}(\boldsymbol{\nu},\nu) := \{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\boldsymbol{\nu} = \nu, \mathbf{q} \in [\mathbf{0},\mathbf{1}]\} \end{array}$$

Since $\mathbf{q}^{*\prime}\boldsymbol{\Sigma}\mathbf{q}^* \leq \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}}$, we have $\mathbf{q}^* \in [\mathbf{0},\mathbf{1}]$ if and only if $\mathbf{q}^* = \widetilde{\mathbf{q}}$. The following result may help to determine $\widetilde{\mathbf{q}}$ in the case $\mathbf{q}^* \notin [\mathbf{0},\mathbf{1}]$:

5.2 Theorem. *Assume that* $\mathbf{q}^* \notin [\mathbf{0},\mathbf{1}]$. *Then at least one of the coordinates of* $\widetilde{\mathbf{q}}$ *is equal to* 0 *or* 1.

Proof. We assume that all coordinates of $\widetilde{\mathbf{q}}$ are distinct from 0 and 1 and we show that this implies $\mathbf{q}^* \in [\mathbf{0},\mathbf{1}]$.
Consider $\mathbf{q} \in D(\boldsymbol{\nu},\nu)$. Since all coordinates of $\widetilde{\mathbf{q}}$ are distinct from 0 and 1, there exists some $c \in (0,1)$ such that all coordinates of the vector $\widetilde{\mathbf{q}} + c\,(\mathbf{q}-\widetilde{\mathbf{q}})$ are contained in the interval $[0,1]$. Then we have $\widetilde{\mathbf{q}} + c\,(\mathbf{q}-\widetilde{\mathbf{q}}) \in D_{[\mathbf{0},\mathbf{1}]}(\boldsymbol{\nu},\nu)$ and hence

$$\begin{aligned} \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}} &\leq \Big(\widetilde{\mathbf{q}} + c\,(\mathbf{q}-\widetilde{\mathbf{q}})\Big)'\boldsymbol{\Sigma}\Big(\widetilde{\mathbf{q}} + c\,(\mathbf{q}-\widetilde{\mathbf{q}})\Big) \\ &= \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}} + 2c\,\widetilde{\mathbf{q}}'\boldsymbol{\Sigma}(\mathbf{q}-\widetilde{\mathbf{q}}) + c^2\,(\mathbf{q}-\widetilde{\mathbf{q}})'\boldsymbol{\Sigma}(\mathbf{q}-\widetilde{\mathbf{q}}) \end{aligned}$$

which yields

$$0 \leq 2\,\widetilde{\mathbf{q}}'\boldsymbol{\Sigma}(\mathbf{q}-\widetilde{\mathbf{q}}) + c\,(\mathbf{q}-\widetilde{\mathbf{q}})'\boldsymbol{\Sigma}(\mathbf{q}-\widetilde{\mathbf{q}})$$

Since $c \in (0,1)$, we obtain

$$0 \leq 2\,\widetilde{\mathbf{q}}'\boldsymbol{\Sigma}(\mathbf{q}-\widetilde{\mathbf{q}}) + (\mathbf{q}-\widetilde{\mathbf{q}})'\boldsymbol{\Sigma}(\mathbf{q}-\widetilde{\mathbf{q}})$$

and hence

$$\begin{aligned} \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}} &\leq \widetilde{\mathbf{q}}'\boldsymbol{\Sigma}\widetilde{\mathbf{q}} + 2\,\widetilde{\mathbf{q}}'\boldsymbol{\Sigma}(\mathbf{q}-\widetilde{\mathbf{q}}) + (\mathbf{q}-\widetilde{\mathbf{q}})'\boldsymbol{\Sigma}(\mathbf{q}-\widetilde{\mathbf{q}}) \\ &= \mathbf{q}'\boldsymbol{\Sigma}\mathbf{q} \end{aligned}$$

Since $\mathbf{q} \in D(\boldsymbol{\nu},\nu)$ was arbitrary, this implies $\mathbf{q}^* = \widetilde{\mathbf{q}}$. □

To illustrate the previous result, we present an example:

5.3 Example. Let $n := 2$ and

$$\boldsymbol{\Sigma} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\nu} := \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

as well as

$$\nu := 9$$

Theorem 3.2 yields

$$\mathbf{q}^* = \begin{pmatrix} 9/34 \\ 36/34 \end{pmatrix}$$

and hence $\mathbf{q}^* \notin [\mathbf{0}, \mathbf{1}]$. By Theorem 5.2, $\widetilde{\mathbf{q}}$ is on the boundary of the interval $[\mathbf{0}, \mathbf{1}]$. It is easily verified that the only vectors in $D_{[\mathbf{0},\mathbf{1}]}(\boldsymbol{\nu}, \nu)$ which are on the boundary of $[\mathbf{0}, \mathbf{1}]$ are the vectors

$$\mathbf{q}_1 \;:=\; \begin{pmatrix} 1 \\ 7/8 \end{pmatrix} \qquad \text{and} \qquad \mathbf{q}_2 \;:=\; \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

and that these vectors satisfy

$$\mathbf{q}_1'\boldsymbol{\Sigma}\mathbf{q}_1 \;=\; 113/64 \qquad \text{and} \qquad \mathbf{q}_2'\boldsymbol{\Sigma}\mathbf{q}_2 \;=\; 80/64$$

Therefore, we have $\widetilde{\mathbf{q}} = \mathbf{q}_2$.

For the case $\mathbf{q}^* \notin [\mathbf{0}, \mathbf{1}]$, it has been suggested in the literature that the coordinates of $\mathbf{q}^*$ which are not contained in the interval $[0, 1]$ should be truncated; see Bühlmann [1970; p. 115] and Mack [2002; p. 391]. This approach is not correct since it usually leads to a violation of the constraint $\mathbf{q}'\boldsymbol{\nu} = \nu$.

5.4 Example. In Example 5.3, truncation of the coordinates of the vector

$$\mathbf{q}^* = \begin{pmatrix} 9/34 \\ 36/34 \end{pmatrix}$$

which are not contained in the interval $[0, 1]$ produces the vector

$$\widehat{\mathbf{q}} = \begin{pmatrix} 9/34 \\ 1 \end{pmatrix}$$

which is not contained in $D(\boldsymbol{\nu}, \nu)$ since $\widehat{\mathbf{q}}'\boldsymbol{\nu} = 143/17 < 9$.

It can be seen from the argument used in the alternative proof of Theorem 5.1 that the optimization problem

$$\begin{array}{ll} \textit{Maximize} & \mathbf{q}'\boldsymbol{\nu} \\ \textit{over the set} & E_{[0,1]}(\Sigma, \sigma^2) := \{\mathbf{q} \in \mathbf{R}^n \mid \mathbf{q}'\Sigma\mathbf{q} = \sigma^2, \mathbf{q} \in [0,1]\} \end{array}$$

with $\sigma^2 \leq \mathbf{1}'\Sigma\mathbf{1}$ has a solution as well. However, is seems to be rather delicate to settle the question of the uniqueness of the solution.

6 Appendix

Throughout this section, let $\mathbf{c} \in \mathbf{R}^n \setminus \{0\}$ and let $\Phi \in \mathbf{R}^{n \times n}$ be a symmetric and regular matrix.

Our first result concerns maximization of $\mathbf{y}'\mathbf{c}$ under a constraint on $\mathbf{y}'\Phi\mathbf{y}$:

6.1 Theorem (Maximization). *Consider $\varphi \in \mathbf{R}_+$. Then the optimization problem*

$$\begin{array}{ll} \textit{Maximize} & \mathbf{y}'\mathbf{c} \\ \textit{over the set} & E(\Phi, \varphi) := \{\mathbf{y} \in \mathbf{R}^n \mid \mathbf{y}'\Phi\mathbf{y} = \varphi^2\} \end{array}$$

has the unique solution

$$\mathbf{y}^* := \left(\frac{\varphi^2}{\mathbf{c}'\Phi^{-1}\mathbf{c}}\right)^{1/2} \Phi^{-1}\mathbf{c}$$

and the maximum is $\left(\varphi^2 \cdot \mathbf{c}'\Phi^{-1}\mathbf{c}\right)^{1/2}$.

Proof. The assertion is obvious in the case $\varphi = 0$. Assume now that $\varphi \in (0, \infty)$. It is clear that $\mathbf{y}^* \in E(\Phi, \varphi)$. For each $\mathbf{y} \in E(\Phi, \varphi)$, we have

$$\mathbf{c}'\mathbf{y} = \left(\frac{\varphi^2}{\mathbf{c}'\Phi^{-1}\mathbf{c}}\right)^{-1/2} \mathbf{y}'\Phi\mathbf{y}^*$$

Since $0 \leq (\mathbf{y} - \mathbf{y}^*)'\Phi(\mathbf{y} - \mathbf{y}^*) = 2\varphi^2 - 2\,\mathbf{y}'\Phi\mathbf{y}^*$, we have

$$\mathbf{y}'\Phi\mathbf{y}^* \leq \mathbf{y}^{*\prime}\Phi\mathbf{y}^*$$

and hence

$$\mathbf{c}'\mathbf{y} \leq \mathbf{c}'\mathbf{y}^*$$

which proves that $\mathbf{y}^*$ is a solution.

Consider now an arbitrary solution $\widehat{\mathbf{y}}$. Then we have

$$\mathbf{c}'\widehat{\mathbf{y}} = \mathbf{c}'\mathbf{y}^*$$

and hence

$$\widehat{\mathbf{y}}'\boldsymbol{\Phi}\mathbf{y}^* = \mathbf{y}^{*\prime}\boldsymbol{\Phi}\mathbf{y}^*$$

Since $\widehat{\mathbf{y}}'\boldsymbol{\Phi}\mathbf{y}^* = \mathbf{y}^{*\prime}\boldsymbol{\Phi}\mathbf{y}^* = \varphi^2 = \widehat{\mathbf{y}}'\boldsymbol{\Phi}\widehat{\mathbf{y}}$, we obtain $(\widehat{\mathbf{y}} - \mathbf{y}^*)'\boldsymbol{\Phi}(\widehat{\mathbf{y}} - \mathbf{y}^*) = 0$ and hence $\widehat{\mathbf{y}} = \mathbf{y}^*$. Therefore, the solution is unique. □

Our second result concerns minimization of $\mathbf{y}'\boldsymbol{\Phi}\mathbf{y}$ under a constraint on $\mathbf{y}'\mathbf{c}$:

6.2 Theorem (Minimization). *Consider $c \in \mathbf{R}$. Then the optimization problem*

Minimize	$\mathbf{y}'\boldsymbol{\Phi}\mathbf{y}$
over the set	$D(\mathbf{c}, c) := \{\mathbf{y} \in \mathbf{R}^n \mid \mathbf{y}'\mathbf{c} = c\}$

has the unique solution

$$\mathbf{y}^* := \frac{c}{\mathbf{c}'\boldsymbol{\Phi}^{-1}\mathbf{c}}\,\boldsymbol{\Phi}^{-1}\mathbf{c}$$

and the minimum is $c^2/\mathbf{c}'\boldsymbol{\Phi}^{-1}\mathbf{c}$.

Proof. It is clear that $\mathbf{y}^* \in D(\mathbf{c}, c)$. For each $\mathbf{y} \in D(\mathbf{c}, c)$, we have

$$\begin{aligned}\mathbf{y}'\boldsymbol{\Phi}\mathbf{y}^* &= \frac{c}{\mathbf{c}'\boldsymbol{\Phi}^{-1}\mathbf{c}}\,\mathbf{y}'\mathbf{c} \\ &= \frac{c^2}{\mathbf{c}'\boldsymbol{\Phi}^{-1}\mathbf{c}}\end{aligned}$$

and hence

$$\mathbf{y}'\boldsymbol{\Phi}\mathbf{y}^* = \mathbf{y}^{*\prime}\boldsymbol{\Phi}\mathbf{y}^*$$

This yields $0 \le (\mathbf{y} - \mathbf{y}^*)'\boldsymbol{\Phi}(\mathbf{y} - \mathbf{y}^*) = \mathbf{y}'\boldsymbol{\Phi}\mathbf{y} - \mathbf{y}^{*\prime}\boldsymbol{\Phi}\mathbf{y}^*$ and hence

$$\mathbf{y}^{*\prime}\boldsymbol{\Phi}\mathbf{y}^* \le \mathbf{y}'\boldsymbol{\Phi}\mathbf{y}$$

which proves that $\mathbf{y}^*$ is a solution.
Consider now an arbitrary solution $\widehat{\mathbf{y}}$. Since $\widehat{\mathbf{y}}'\boldsymbol{\Phi}\mathbf{y}^* = \mathbf{y}^{*\prime}\boldsymbol{\Phi}\mathbf{y}^* = \widehat{\mathbf{y}}'\boldsymbol{\Phi}\widehat{\mathbf{y}}$, we obtain $(\widehat{\mathbf{y}} - \mathbf{y}^*)'\boldsymbol{\Phi}(\widehat{\mathbf{y}} - \mathbf{y}^*) = 0$ and hence $\widehat{\mathbf{y}} = \mathbf{y}^*$. Therefore, the solution is unique. □

The two preceding results are related as follows:

6.3 Corollary (Duality). *Consider $\varphi^2 \in \mathbf{R}_+$ and $c \in \mathbf{R}$ satisfying*

$$\varphi^2 = \frac{c^2}{\mathbf{c}'\mathbf{\Phi}^{-1}\mathbf{c}}$$

Then the solutions of the optimization problems

Maximize	$\mathbf{y}'\mathbf{c}$
over the set	$\{\mathbf{y} \in \mathbf{R}^n \mid \mathbf{y}'\mathbf{\Phi}\mathbf{y} = \varphi^2\}$

and

Minimize	$\mathbf{y}'\mathbf{\Phi}\mathbf{y}$
over the set	$\{\mathbf{y} \in \mathbf{R}^n \mid \mathbf{y}'\mathbf{c} = c\}$

are identical.

This is immediate from Theorems 6.1 and 6.2.

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Abstract

In the present paper we study optimal quota share reinsurance for a heterogeneous portfolio with possibly dependent lines of business. More precisely, we determine quotas which maximize the expected return when the variance of the retention is fixed or minimize the variance of the retention when the expected return is fixed. The results require only that the variance of the vector of losses of the different lines of business is regular. We thus extend results which are known for the case of uncorrelated lines of business.

Zusammenfassung

In der vorliegenden Arbeit untersuchen wir die optimale Form der Quoten–Rückversicherung für einen heterogenen Bestand, bei dem Abhängigkeiten zwischen den einzelnen Geschäftszweigen bestehen können. Wir bestimmen Quoten, die bei gegebener Varianz des Selbstbehaltes den erwarteten Ertrag maximieren oder bei gegebenem Ertrag die Varianz des Selbstbehaltes minimieren. Die Ergebnisse erfordern nur die Regularität der Varianz des Vektors der Verluste der einzelnen Geschäftszweige und verallgemeinern daher bekannte Ergebnisse für den Fall unkorrelierter Geschäftszweige.

Résumé

Dans cette article on étudie la forme optimale de la réassurance quote-part pour un portefeuille hétérogène dont les différents secteurs peuvent être dépendants. Plus précisément, on détermine les quotes-parts qui maximisent le revenu quand la variance du risque retenu est donnée ou minimisent la variance du risque retenu quand le revenu est donné. Les résultats sont valables sous la seule condition que la variance du vecteur des pertes des différents secteurs soit régulière. Ils généralisent donc des résultats qui sont connus dans le cas des secteurs non-corrélés.