R. SCHNIEPER, Zürich

Capital and Asset Allocation[1]

1   Introduction

The actuarial approach to capital allocation is usually based on prudential considerations only. The amount of capital required to run an insurance company is determined in such a way that a failure of the company becomes very unlikely. It is purely based on the viewpoint of policyholders or regulators acting on behalf of policyholders. The corporate finance approach to capital allocation is more sophisticated. It is based on the viewpoint of the owners of the firm. It attempts to find the particular combination of debt and equity which maximizes the value of the firm. According to Modigliani and Miller's (1958) famous "Proposition I" the capital structure of a firm, its mix of equity and debt, does not matter in perfect capital markets. If a firm increases its level of debt it increases both its expected return on equity and the volatility of those returns. Shareholders are indifferent to the leverage of the company. They simply discount the higher expected returns associated with a higher leverage with a higher discount rate, thus leaving the value of the equity unchanged.

The fact that capital structure matters in real markets is usually attributed to taxes and to the costs of financial distress. Because interest payments on debt are taxed more favorably than dividends, firms have an incentive to finance themselves through debt rather than equity. Since a high level of debt financing increases the probability of default and the expected costs associated with financial distress, firms have an incentive to keep the debt to equity ratio at a reasonable level. These two conflicting objectives lead to an optimal ratio of debt to equity. For an in depth discussion of a firm's capital structure see e.g. Brealey and Myers (1988).

We adopt an approach which is similar to the corporate finance approach. We consider the conflicting objectives of the owners of the firm who want to maximize the expected return and minimize the risk of the firm. The conflict is resolved by applying Markovitz' mean variance methodology.

We also take into account the viewpoint of the policyholders and of the supervisory authorities. This leads to a restriction of the ratio of debt to equity of

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the firm. We simultaneously optimize the capital structure and the asset allocation of the firm. Our model focuses on economic values of both assets and liabilities.

2 The Model

We consider an insurance company characterized by its portfolio of in force risks which generates a premium income P and by its corresponding portfolio of outstanding liabilities D. The technical reserves D are valued on a discounted basis. It is assumed that the amounts and time of future payments in respect of outstanding liabilities are deterministic and known to the company. This stream of future cash flows is discounted based on the yield curve at the beginning of the period. In each (annual) period the technical reserves thus generate a random return

$$-\widetilde{\mu}_0 D$$

which corresponds to the unwinding of the discount and to the reevaluation of future payments based on the yield curve at the end of the period.
We denote the random return generated by the portfolio of in force risks by

$$\widetilde{\mu}_P P \, .$$

It is the difference of the premium income, paid claims and increase in loss reserves including IBNRs. All amounts are discounted with interest rates corresponding to the yield curve at the end of the period. The assets of the company are invested and generate a random return

$$\widetilde{\mu}_A A \, .$$

The above quantity denotes the total return on assets, i.e. investment income and change in valuation during the period. Let E denote the equity of the company. The total return of the company is

$$\widetilde{\mu}_E E = \widetilde{\mu}_A A + \widetilde{\mu}_P P - \widetilde{\mu}_0 D \, .$$

Obviously we have

$$A = D + E \, .$$

The first issue that we address in this paper is the issue of capital allocation, i.e. we ask ourselves what the optimal amount of equity (E) should be in relation to the debt (D) or equivalently in relation to the premium volume (P).

We make the assumption that the company can invest into two types of assets characterized by different expected returns and by different risks, e.g. a portfolio of bonds and a portfolio of equities. The second issue that we address is the issue of asset allocation, i.e. we determine the optimal mix between the different types of assets.

We use the following notation

$$\mu_x = E(\widetilde{\mu}_x) \qquad \sigma_x^2 = \mathrm{Var}(\widetilde{\mu}_x)$$

where $\widetilde{\mu}_x$ is any of the random rate of return appearing in the article. σ_x is usually referred to as the risk or the volatility pertaining to the corresponding quantity.

We assume that the characteristics of the insurance portfolio (μ_P, σ_P) are given and we vary the leverage and the share of risky asset in the portfolio of the company. The rationale for this approach is that the asset allocation of the firm and to a lesser extent its leverage can be easily adjusted. (The leverage can be increased through a generous dividend policy or a share buy back programme. A decrease of the leverage is only possible if the firm has access to additional capital.) An adjustment of the insurance portfolio however is cumbersome and costly. For a more detailed discussion of the model see R. Schnieper (2000). The latter article also contains a discussion of the optimization of μ_P and σ_P through portfolio management and reinsurance.

3   Capital Allocation

When defining the amount of capital necessary to support the business one has primarily to take into account the interest of the owners and of the clients of the firm. Supervisory authorities and rating agencies act on behalf of the insureds. For stock companies financial analysts look at the firm from the viewpoint of the shareholders.

The owners of the firm have two conflicting objectives. They want to

–   maximize the expected rate of return of the company μ_E and

–   minimize the risk of the firm as measured by σ_E^2.

The reason why the owners care about the risk even if they can diversify their holdings is because of the costs associated with financial distress. If the company has to cease trading because it has lost too much of its capital it has to either recapitalize or dispose of its in force portfolio in a distressed sale. Both transactions are unfavorable to the existing owners of the firm.

According to their preferences the owners put weights on these conflicting objectives and maximize the following objective function

$$2\tau\mu_E - \sigma_E^2 \, .$$

The parameter τ is called the risk tolerance of the company. The approach is the same as Markowitz' mean variance methodology. This methodology is consistent with utility maximization in the following two cases

- quadratic utility functions and

- normal distribution of return.

For a more detailed discussion see H. Panjer et al. (1998). The random return $\tilde{\mu}_P P$ is net of reinsurance and it is not unreasonable to assume that $\tilde{\mu}_P$ is normal. The asset returns $\tilde{\mu}_0$ and $\tilde{\mu}_A$ are usually assumed to be normal. Hence one can assume that $\tilde{\mu}_E$ is distributed according to a normal distribution.

When looking at capital allocation supervisory authorities usually want to limit the risk assumed by the company. This can take the form of a limitation imposed on the leverage of the firm, i.e. on either $\frac{P}{E}$ or $\frac{D}{E}$ or both. More sophisticated jurisdictions impose a minimal capitalization in terms of multiple of the total volatility of the firm's result $(\sigma_E E)$ which is tantamount to imposing an upper bound on σ_E.

To determine the amount of required capital we therefore maximize the objective function

$$2\tau\mu_E - \sigma_E^2$$

and consider the impact of an upper bound on either leverage or volatility (σ_E). In order to focus on insurance risk we assume that the company invests an amount D in the bond portfolio, which replicates the maturities of the liabilities of the company and an amount E in the risk free asset. The risk free rate of return is denoted by r_0. The return on assets of the company is

$$\tilde{\mu}_A A = \tilde{\mu}_0 D + r_0 E$$

and the total return is

$$\tilde{\mu}_E E = r_0 E + \tilde{\mu}_P P \, .$$

Let

$$\lambda = \frac{D}{E}$$

denote the leverage of the company which is to be optimized and let

$$l = \frac{P}{D}$$

denote the ratio of premium to technical reserves which is defined by the portfolio of the company and by the yield curve. The total return of the company becomes

$$\tilde{\mu}_E = r_0 + \tilde{\mu}_P \lambda l \,.$$

Hence the objective function which has to be maximized becomes

$$2\tau\mu_E - \sigma_E^2 = 2\tau(r_0 + \mu_P \lambda l) - \sigma_P^2 \lambda^2 l^2 = \max_{\lambda}!$$

Equating the derivative of the above expression with respect to λ to zero one obtains the optimal leverage

$$\lambda = \tau \frac{\mu_P}{\sigma_P^2} l^{-1}$$

or equivalently the optimal premium to equity ratio

$$\frac{P}{E} = \lambda l = \tau \frac{\mu_P}{\sigma_P^2} \,.$$

Numerical Example

Assuming $\tau = 0.1$, $\mu_P = 10\%$ (loss reserves are discounted) and $\sigma_P = 5\%$, leads to a leverage of eight and to an optimal premium to equity ratio of four which is not unreasonable given the very favorable Sharpe ratio of the portfolio

$$\frac{\mu_P}{\sigma_P} = 2 \,.$$

Remark

Note that the objective function above

$$2\tau\mu_E - \sigma_E^2$$

can be modified by replacing μ_E by a different measure of performance and σ_E^2 by a different measure of risk.

4   Capital and Asset Allocation

We now make the assumption that the company can invest into two types of assets, i.e. a portfolio of bonds with the same maturities as the liabilities of the company and a more risky asset with a higher expected return e.g. a portfolio of equities. The return rate of the bond portfolio is thus $\widetilde{\mu}_0$ and we denote the return rate of the more risky asset by $\widetilde{\mu}_1$. The share of assets invested in the risky asset is denoted by s. The return on assets of the company is

$$\widetilde{\mu}_A A = \widetilde{\mu}_1 s A + \widetilde{\mu}_0 (1 - s) A.$$

The total return of the company becomes

$$\widetilde{\mu}_E E = \widetilde{\mu}_0 ((1 - s) A - D) + \widetilde{\mu}_1 s A + \widetilde{\mu}_P P$$

$$\widetilde{\mu}_E = \widetilde{\mu}_0 \left((1 - s) \frac{A}{E} - \frac{D}{E} \right) + \widetilde{\mu}_1 s \frac{A}{E} + \widetilde{\mu}_P \frac{P}{E}$$

$$\widetilde{\mu}_E = \widetilde{\mu}_0 (1 - s - \lambda s) + \widetilde{\mu}_1 (s + \lambda s) + \widetilde{\mu}_P \lambda l.$$

Let ϱ denote the correlation between $\widetilde{\mu}_0$ and $\widetilde{\mu}_1$. We assume that asset risks and insurance risks are uncorrelated

$$\mathrm{Cov}(\widetilde{\mu}_0, \widetilde{\mu}_P) = \mathrm{Cov}(\widetilde{\mu}_1, \widetilde{\mu}_P) = 0.$$

The expected rate of return and variance of the rate of return become

$$\mu_E(\lambda, s) = \mu_0 (1 - s - \lambda s) + \mu_1 (s + \lambda s) + \mu_P \lambda l$$

$$\sigma_E^2(\lambda, s) = \sigma_0^2 (1 - s - \lambda s)^2 + \sigma_1^2 (s + \lambda s)^2$$
$$+ 2\varrho \sigma_0 \sigma_1 (1 - s - \lambda s)(s + \lambda s) + \sigma_P^2 \lambda^2 l^2.$$

The objective function

$$2\tau \mu_E(\lambda, s) - \sigma_E^2(\lambda, s) = \max_{\lambda, s}!$$

is obtained by plugging in the above expressions for $\mu_E(\lambda, s)$ and $\sigma_E^2(\lambda, s)$ respectively. Deriving the objective function with respect to s and equating the derivative to zero one obtains the value of s which maximizes the objective function

$$s = \frac{\tau(\mu_1 - \mu_0) + \sigma_0^2 - \varrho \sigma_0 \sigma_1}{\sigma_0^2 - 2\varrho \sigma_0 \sigma_1 + \sigma_1^2} \cdot \frac{1}{1 + \lambda}$$

or equivalently

$$s(1 + \lambda) = \frac{s \cdot A}{E} = c$$

with

$$c = \frac{\tau(\mu_1 - \mu_0) + \sigma_0^2 - \varrho\sigma_0\sigma_1}{\sigma_0^2 - 2\varrho\sigma_0\sigma_1 + \sigma_1^2}$$

i.e. the optimal amount of risky assets $(s \cdot A)$ expressed as a percentage of equity (E) is a constant which only depends on the characteristics of asset risks $(\mu_0, \mu_1, \sigma_0, \sigma_1, \varrho)$ and on the risk tolerance of the company (τ). This constant is sometimes referred to as the gearing of the company.

Numerical Example

Let

$$\tau = 0.10 \qquad \mu_P = 0.10 \qquad \sigma_P = 0.05$$

which is as as in section 3. Let

$$l = \frac{P}{D} = 0.5$$

and make the following assumptions concerning asset risks

$$\mu_0 = 0.05, \quad \sigma_0 = 0.025, \quad \mu_1 = 0.125, \quad \sigma_1 = 0.20, \quad \varrho = 0.5 \,.$$

We obtain

$$c = \frac{\tau(\mu_1 - \mu_0) + \sigma_0^2 - \varrho\sigma_0\sigma_1}{\sigma_0^2 - 2\varrho\sigma_0\sigma_1 + \sigma_1^2} = 0.158$$

i.e. the optimal investment policy consists in investing 15.8 % of the equity into the risky asset. It is seen that the optimal gearing of the company is substantially below the values observed in practice, which are often 100 % higher.
Deriving the objective function with respect to λ and equating the derivative to zero, one obtains

$$\lambda = \frac{\tau(s(\mu_1 - \mu_0) + \mu_P l) + \sigma_0^2(s - s^2) - \sigma_1^2 s^2 - \varrho\sigma_0\sigma_1(s - 2s^2)}{\sigma_0^2 s^2 + \sigma_1^2 s^2 - 2\varrho\sigma_0\sigma_1 s^2 + \sigma_P^2 l^2}$$

hence

$$\lambda + 1 = \frac{\tau(s(\mu_1 - \mu_0) + \mu_P l) + \sigma_0^2 s - \varrho \sigma_0 \sigma_1 s + \sigma_P^2 l^2}{\sigma_0^2 s^2 + \sigma_1^2 s^2 - 2\varrho \sigma_0 \sigma_1 s^2 + \sigma_P^2 l^2}$$

inserting

$$s = \frac{c}{1 + \lambda},$$

replacing

$$\kappa = 1 + \lambda$$

and rearranging terms one obtains

$$\kappa^2 \sigma_P^2 l^2 - \kappa(\sigma_P^2 l^2 + \tau \mu_P l) + \sigma_0^2 c^2 - 2\varrho \sigma_0 \sigma_1 c^2 + \sigma_1^2 c^2 - \sigma_0^2 c$$
$$+ \varrho \sigma_0 \sigma_1 c - \tau(\mu_1 - \mu_0)c = 0$$

or

$$\kappa^2 - B\kappa + C = O$$

with

$$B = 1 + \tau \frac{\mu_P}{\sigma_P^2} l^{-1}$$

$$C = \frac{1}{\sigma_P^2 l^2} (\sigma_0^2 c^2 - 2\varrho \sigma_0 \sigma_1 c^2 + \sigma_1^2 c^2 - \sigma_0^2 c + \varrho \sigma_0 \sigma_1 c - \tau(\mu_1 - \mu_0)c)$$

and

$$\kappa = \frac{1}{2} B \pm \frac{1}{2} \sqrt{B^2 - 4C}.$$

In practical examples we have

$$B \gg |C|$$

hence

$$\kappa \simeq B$$
$$1 + \lambda \simeq 1 + \tau \frac{\mu_P}{\sigma_P^2} l^{-1}$$

hence

$$\lambda \simeq \tau \frac{\mu_P}{\sigma_P^2} \, l^{-1}$$

which is the optimal leverage of the simplified model of section 3.

The above expression only depends on the characteristics of the insurance risk (μ_P, σ_P^2, l) and on the risk tolerance of the company (τ).

It is seen that the risk tolerance of the company τ determines both the capital allocation and the asset allocation of the company. The two policies are therefore aligned. The optimum is available in analytical form and is obtained by computing first

$$\lambda = \kappa - 1 \quad \text{with} \quad \kappa = \frac{1}{2}\left(B + \sqrt{B^2 - 4C}\right)$$

and by plugging λ into

$$s = \frac{\tau(\mu_1 - \mu_0) + \sigma_0^2 - \varrho\sigma_0\sigma_1}{\sigma_0^2 - 2\varrho\sigma_0\sigma_1 + \sigma_1^2} \cdot \frac{1}{1+\lambda} = \frac{c}{1+\lambda} .$$

In the case of the above parameters, the optimum is $\lambda = 7.9999$ and $s = 0.01755$. We now analyze the consequences of regulatory constraints. If the company must operate with a leverage below the optimal λ given above, it can improve the value of its objective function by increasing the share s of its risky assets according to the above formula.

Let $\lambda^* \leq \lambda$ be the maximum leverage acceptable to the supervisory authorities. The company can improve the value of its objective function by investing a share $s^* = \frac{C}{1+\lambda^*} \geq s = \frac{C}{1+\lambda}$ in the risky asset.

A contour plot of the objective function

$$2\tau\mu_E(\lambda, s) - \sigma_E^2(\lambda, s)$$

is given in the appendix. It illustrates the above statement. It also shows that the value of the objective function changes relatively little, if the actual leverage is somewhat smaller or somewhat larger than the optimal leverage. However if the share of risky assets is larger than the optimal share, the value of the objective function decreases dramatically. This effect is compounded if the leverage is larger than the optimal leverage. We illustrate this by a numerical example

(λ, s)	μ_E	σ_E	$2\tau\mu_E - \sigma_E^2$
(7.9999, 0.01755)	0.462	0.205	0.050
(4, 0.0316)	0.262	0.110	0.040
(8, 0.20)	0.585	0.403	−0.046
(12, 0.20)	0.845	0.584	−0.172

The first case corresponds to the optimum. The second case corresponds to the situation where the company is restricted to a leverage ratio below four. It consequently increases its share of risky assets to 3.16% of total assets. The overall decrease of the objective function is thus limited (from 0.050 to 0.040). In the third case the firm has a very risky asset allocation policy with $s = 20\%$. This leads to a much lower value of the objective function (-0.046) and to highly volatile returns with $\sigma_E = 40.3\%$. If in addition the leverage is increased from eight to twelve, the situation gets much worse. Note that at the end of the 1990s some insurance companies did invest 20% or more of their assets into equities. Certain of these same companies did also increase their leverage through a very generous dividend policy or through an aggressive expansion policy.

Let us now assume that the supervisory authorities put an upper bound on σ_E rather than on λ

$$\sigma_E(\lambda, s) \leq m.$$

The company will then operate with $\sigma_E(\lambda, s) = m$. Replacing s by $\frac{c}{1+\lambda}$ in the above expression for $\sigma_E^2(\lambda, s)$ we obtain

$$m^2 = \sigma_0^2 \left(1 - \frac{c}{1+\lambda} - \lambda \frac{c}{1+\lambda} \right)^2 + \sigma_1^2 \left(\frac{c}{1+\lambda} + \lambda \frac{c}{1+\lambda} \right)^2$$
$$+ 2\varrho\sigma_0\sigma_1 \left(1 - \frac{c}{1+\lambda} - \lambda \frac{c}{1+\lambda} \right) \left(\frac{c}{1+\lambda} + \lambda \frac{c}{1+\lambda} \right) + \sigma_p^2 l^2 \lambda^2$$

hence

$$\lambda = \frac{1}{\sigma_p l} \sqrt{m^2 - \sigma_0^2 (1-c)^2 + 2\varrho\sigma_0\sigma_1(1-c)c + \sigma_1^2 c^2}$$

which is the optimal leverage corresponding to the above constraint on $\sigma_E(\lambda, s)$. If m is large enough, the solution is real. We illustrate this with a numerical example

| (λ, s)	μ_E	σ_E	$2\tau\mu_E - \sigma_E^2$
$(7.9999, 0.01755)$	0.462	0.205	0.050
$(3.644, 0.0340)$	0.244	0.102	0.038

The first case corresponds to the unconstrained optimum. The second case assumes that regulatory constraints put an upper bound of 0.102 (50% of the optimum) on σ_E. The corresponding optimal leverage and share of risky assets are 3.644 and 3.4% respectively. The reduction of the objective function remains limited (0.038 instead of 0.050).

Literature

R.A. Brealey, S.C. Myers (1988), Principles of Corporate Finance, Mc Graw-Hill.

F. Modigliani, M.H. Miller (1958), The Cost of Capital, Corporation Finance and the Theory of Investment, The American Economic Review, Vol XLVIII, 1958:3.

H. Panjer et al. (1998), Financial Economics, The Actuarial Foundation.

R. Schnieper (2000), Portfolio Optimization, ASTIN Bulletin, Vol. 30, No. 1

René Schnieper
Converium Ltd
General Guisan-Quai 26
Postfach
CH-8022 Zürich

Appendix
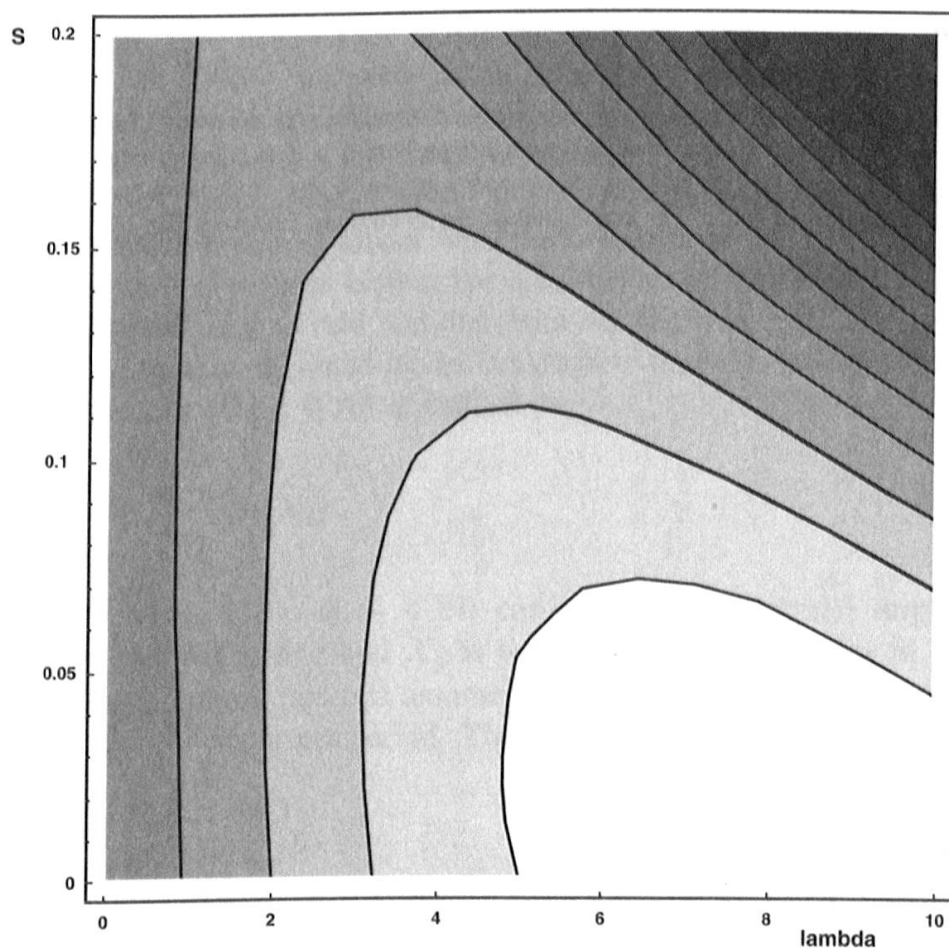
Abstract

The optimal capital allocation of an insurance company is derived in such a way as to take into account the conflicting objectives of the owners of the company for a maximum expected return and a minimum risk. The optimal allocation between two asset classes is determined simultaneously. The implications of regulatory constraints are analyzed.

Zusammenfassung

Die optimale Kapitalzuordnung einer Versicherungsgesellschaft wird hergeleitet. Dabei werden die widersprüchlichen Ziele der Besitzer der Gesellschaft nach maximaler Rendite und nach minimalem Risiko berücksichtigt. Die optimale Zuordnung der Anlagen zwischen zwei Anlagekategorien wird simultan hergeleitet. Die Auswirkungen aufsichtsrechtlicher Auflagen werden analysiert.

Résumé

L'allocation optimale du capital d'une compagnie d'assurance est dérivée. Les objectifs contra-dictoires des propriétaires de la compagnie qui recherchent à la fois le rendement maximum et le risque minimum sont pris en compte. Simultanément l'on dérive l'allocation optimale des actifs entre deux catégories d'actifs. Les conséquences des contraintes imposées par l'autorité de tutelle sont analysées.