B. Wissenschaftliche Mitteilungen

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Multidimensional valuation of life insurance policies and fair value

1 Introduction

The aim of this paper is to set out a formal framework of modern valuation techniques for both life insurance portfolios and contracts. The main idea behind it is to represent a life insurance contract as a linear combination of basic financial instruments. We combine these financial instruments to create a multidimensional valuation portfolio. As there is no liquid market for insurance contracts, we cannot directly value these contracts. However, for many financial instruments a liquid market exists. When speaking about the fair value of liquidly traded financial instruments, then usually refers to market values, i.e. market prices at a specific date.

In order to price financial instruments which are not liquidly traded, we try to replicate cash flows from this financial instrument by a combination of liquidly traded financial instruments (financial engineering, option pricing) and then take the value (or price) of this replication portfolio as the value (or price) of this financial instrument. It is normally not possible to replicate pure insurance risks with financial instruments. But we can replicate *expected* more generally their certainty equivalent cash flows arising from an insurance contract, provided we assume the existence of risk-free instruments for different maturities. In order to stress the fact that not cash flows themselves but their expected value are replicated, we are going to use the term $VaPo$ instead of replication portfolio.

Firstly, the actuarial analysis is performed on this multidimensional valuation portfolio. Secondly, a single number is attached to the valuation portfolio, such as the traditional mathematical reserve or also any value within the family of fair value concepts. The procedure follows [De Felice and Moriconi (2002)].

Specifically, a traditional non-participating life insurance policy can be represented as a linear combination of zero coupon bonds, which are then valued in a second step. If the $VaPo$ consists only of zero coupon bonds the calculation of the fair value and the mathematical reserve is almost the same, only the discount factors

(= value of the zero coupon bond) are different. In more general $VaPo's$, values of other financial instruments (e.g. European options) may also be included.

Initially, we will introduce some notation to represent a life insurance contract and to have a formal framework for our valuation. Thereafter, a general procedure will be outlined which forms a part of our valuation process. Finally, we will also consider the technical gains and losses caused by the observed mortality.

We will illustrate the concept based on the following two examples:

I Endowment policy with annual premium payment,

II Annuity in payment with single premium payment.

2 Notation

In order to have a formal framework for developing the concept, we first need some notation. We need to define the vector space of life insurance policies $\mathcal{G}$ and also the vector space $\mathcal{F}$ spanned by financial instruments e_i (i.e. $\mathcal{F} = \langle e_1, \ldots, e_m \rangle$). In the following we will use a rather general representation of life insurance policies by means of a Markov model. For the standard actuarial notation that is used in this paper we refer to [Gerber (1997)].

The Markov model is described by the contractual functions $a_i^{\text{Pre}}, a_{ij}^{\text{Post}}$ and the transition probabilities p_{ij}:

$a_i^{\text{Pre}}(t)$: Payment at time t, if the person is in state i at time t, e.g. premium payment;

$a_{ij}^{\text{Post}}(t)$: Payment at time $t+1$, if the person is in state i at time t and in state j at time $t+1$, e.g. death benefit;

$p_{ij}(t)$: Probability of switching from state i to state j in the time interval $[t, t+1)$.

We consider a life insurance contract g_x of a person aged x. g_x can be represented by the contractual functions:

$$g_x := g_x(a_i^{\text{Pre}}, a_{ij}^{\text{Post}};\ i, j \in S),$$

where x denotes the age at entry of the insured person and S the set of states. We write $\mathcal{G}$ for the set of all possible life insurance contracts.

The set (vector space) of financial instruments needed for multidimensional valuation is denoted by $\mathcal{F}$. It is convenient to choose financial instruments e_i which form a basis $\mathcal{B} = \{e_1, \ldots, e_m\}$ of $\mathcal{F}$, i.e.

$$\mathcal{F} = \langle e_1, \ldots, e_m \rangle .$$

We will also call the financial instruments e_i *units*, because they span our financial instruments universe. It is important to distinguish the units from their monetary value. With the above notation we can represent each financial instrument using the basis $\mathcal{B}$. Therefore we only need information about the units e_i and not about any other financial instrument.

At this point it is also important to mention that there might be different choices for the vector space $\mathcal{F}$. For a traditional non-participating insurance policy $\mathcal{F}$ will normally consist of zero coupon bonds with different maturities. However, for more sophisticated policy constructions it could also include other instruments such as equities, (European) options or also instruments which represent the difference between the expected cash flows and the random cash flows induced by the effects of mortality.

In the following we will denote by $Z^{(t+k)}$ a zero coupon bond. A zero coupon bond is a security paying the amount of one monetary unit at time $t + k$. With $P(t, t+k)$ we denote the price of this zero coupon bond at time t (see Appendix A). The redemption yield of such a bond can be calculated by the formula

$$P(t, t + k)^{-\frac{1}{k}} - 1 .$$

3 Procedure

The valuation of an insurance contract can be done using the following procedure:

- First, we define a linear mapping φ, the *valuation*, from the set of insurance contracts $\mathcal{G}$ into the vector space $\mathcal{F}$ of valuation portfolios spanned by the units e_i, $i = 1, \ldots, m$:

$$\begin{aligned} \varphi : \mathcal{G} &\rightarrow \mathcal{F} \\ g &\mapsto \sum_{i=1}^{m} \lambda_i(g) e_i , \end{aligned} \tag{1}$$

  where $\lambda_i(g)$ denotes the number of units e_i needed to represent the insurance contract g.

Example of a pure endowment policy of a 30 year old person with maturity 5 years, maturity benefit of one monetary unit and annual premium payment Π. Then

$$\begin{aligned} \lambda_1(g) &= -\Pi & \qquad e_1 &= Z^{(30)} \\ \lambda_2(g) &= -\Pi \cdot {}_1p_{30} & e_2 &= Z^{(30+1)} \\ \lambda_3(g) &= -\Pi \cdot {}_2p_{30} & \text{and} \qquad e_3 &= Z^{(30+2)} \\ \lambda_4(g) &= -\Pi \cdot {}_3p_{30} & e_4 &= Z^{(30+3)} \\ \lambda_5(g) &= -\Pi \cdot {}_4p_{30} & e_5 &= Z^{(30+4)} \\ \lambda_6(g) &= + {}_5p_{30} & e_6 &= Z^{(30+5)} \end{aligned}$$

- In a second step we price the valuation portfolio in a monetary amount by applying the *accounting principle* ψ:

$$\begin{aligned} \psi : \quad & \mathcal{F} \quad \to \mathbb{R} \\ & \sum_{i=1}^{m} \lambda_i(g) e_i \mapsto \psi\Big(\sum_{i=1}^{m} \lambda_i(g) e_i\Big), \end{aligned} \tag{2}$$

defined as a linear mapping, i.e.

$$\psi\Big(\sum_{i=1}^{m} \lambda_i(g) e_i\Big) = \sum_{i=1}^{m} \lambda_i(g) \psi(e_i).$$

Consequence: Again we only need to know $\psi(e_i)$, $i = 1, \ldots, m$, in order to determine the monetary value of an insurance contract. However, this can be complicated as the prices of certain units might be difficult to calculate (in particular if no liquid market exists for them).

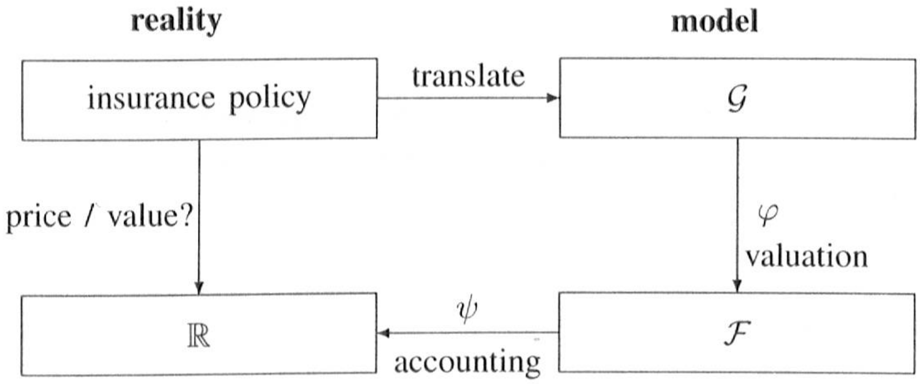
Continuation of the above example: We choose the calculation of the mathematical reserve as the accounting principle. In this example $\psi(e_k)$ is the normal discount factor calculated by the technical interest rate i, i.e.

$$\psi\left(Z^{(30+k)}\right) = \left(\frac{1}{1+i}\right)^k = v^k$$

and

$$\begin{aligned} \psi\Big(\sum_{k=1}^{6} \lambda_k(g) Z^{(30+k-1)}\Big) = & -\Pi - \Pi \cdot {}_1p_{30} \cdot v - \Pi \cdot {}_2p_{30} \cdot v^2 \\ & - \Pi \cdot {}_3p_{30} \cdot v^3 - \Pi \cdot {}_4p_{30} \cdot v^4 + {}_5p_{30} \cdot v^5 . \end{aligned}$$

Schematic representation:



At this point it is useful to remark that the mapping ψ defines the accounting principle applied to the valuation portfolio. There are different possibilities to define ψ, as we will see in a moment. The mapping ψ can be defined such that it represents the valuations in terms of classical mathematical reserves. But it is also possible to define ψ in a way to get a variety of valuations which can be summarized under the concept of fair values.

Remark: As we consider only finite dimensional vector spaces $\mathcal{F}$ in this paper, we do not need to worry about continuity of the mappings φ and ψ as both of them are linear.

4 Valuation process

We divide the valuation process into three steps:

Step 1: We define appropriate *units* $e_i, i = 1, \ldots, m$, where m denotes the number of different units used. Each financial instrument represents a possible cash flow from the payer or to the beneficiary of the policy. The units should form a basis of all valuations for that policy and span the vector space $\mathcal{F} = \langle e_1, \ldots, e_m \rangle$.

Step 2: We perform a multidimensional valuation in units with a *valuation portfolio* ($VaPo$) by use of the mapping φ defined in (1).

Step 3: We transform the *valuation in units* into a *monetary account* by use of the mapping ψ defined in (2).

5 Examples

5.1 Introduction

In Section 4 we described the three steps of the valuation process. We will illustrate this process for an endowment policy and an annuity step by step in the following subsections.

5.2 Endowment policy

5.2.1 Situation

We consider an endowment policy with premium payments for a 50 year old person:

death and maturity benefit	$C =$ CHF 50'000
age at entry	$x =$ 50 years
contract term	$n =$ 5 years

Remarks:

1. The time scale is in years. For the ease of notation, the time is equal to the age of the insured person.
2. Benefits are paid *at the end* of the year when death occurs or at maturity of the contract.
3. Premiums Π are due *at the beginning* of each year.
4. There are no administration charges.

Convention: Payments *receivable* by the policy holder have a *positive* sign whereas payments to be *made* by the policy holder have a *negative* sign.

Description of the insurance contract:
State space $S = \{*, \dagger\}$, where $*$ symbolizes *alive* and $\dagger$ represents *dead*.
Contractual functions:

$$a_*^{\text{Pre}}(t) = \begin{cases} -\Pi\,, & t = 50, \ldots, 54 \\ 0\,, & \text{else} \end{cases}$$

$$a_{*\dagger}^{\text{Post}}(t) = \begin{cases} C\,, & t = 50, \ldots, 54 \\ 0\,, & \text{else} \end{cases} \qquad a_{**}^{\text{Post}}(t) = \begin{cases} C\,, & t = 54 \\ 0\,, & \text{else} \end{cases}$$

The transition probabilities in the period $[t, t+1)$ are as follows:

$$\begin{aligned} p_{**}(t) &= p_t \\ p_{*\dagger}(t) &= q_t \\ p_{\dagger *}(t) &= 0 \\ p_{\dagger\dagger}(t) &= 1 \end{aligned}$$

Insurance contract: $g_{50} = g_{50}(a_*^{\text{Pre}}, a_{*\dagger}^{\text{Post}}, a_{**}^{\text{Post}})$.

Note that the contractual functions and the transition probabilities are defined for $t = 0, 1, 2, \ldots, \omega$ where ω denotes the ultimate age in the life table. This definition is helpful to generalize the procedure (see page 37).

Our task: What is the value of this insurance contract

$$\left.\begin{array}{l} \text{at age } 50 \\ \text{at age } 51 \\ \text{at age } 52 \\ \text{at age } 53 \\ \text{at age } 54 \end{array}\right\} \textit{before} \text{ the premium payment } \Pi?$$

Remark: There are current discussions on fair value and the most important value is the "value at issue", i.e. the value of the contract at age 50. To see the evolution of the contract's value it is also interesting to calculate the values after issue.

In the following we will perform the different steps of the valuation process:

Step 1: Definition of the vector space $\mathcal{F}$

Premium: The premiums Π are paid at the beginning of the period (BoP) $[t, t+1)$, $t = 50, \ldots, 54$. The unit e_t is a zero coupon bond $Z^{(t)}$ with duration $t - 50$ paying CHF 1 at the beginning of age t. The premium payment Π is therefore represented in units of such zero coupon bonds.

Death benefits: The death benefits payable at the end of the period (EoP) $[t, t+1)$ are also considered as zero coupon bonds $Z^{(t+1)}$, $t = 50, \ldots, 54$.

Remark: As the death benefits are paid at the end of the year t they are represented by the units $Z^{(t+1)}$ paying CHF 1 at age $t+1$.

Maturity benefit: The maturity benefit at age 55 is represented by the unit $Z^{(55)}$.

As a consequence we need the following six units:

$$Z^{(t)'}, \ t = 50, 51, 52, 53, 54, 55 .$$

They form the basis

$$\mathcal{B} = \{Z^{50}, Z^{51}, Z^{52}, Z^{53}, Z^{54}, Z^{55}\}$$

for our valuation.

5.2.2 Valuation in units at age 50

Step 2: Valuation in units

We consider two valuation schemes:

- *Scheme A* describes
  - *when* the premiums are received and the benefits are paid.
  - the *amount* of these payments.
- *Scheme B* shows
  - the *sort of units* enclosed in the portfolio: $e_1, \ldots, e_m$.
  - the *number* of the different units: $\lambda_1(g_{50}), \ldots, \lambda_m(g_{50})$.

We get all the information needed from the contractual functions and the transition probabilities.
Observe that Scheme A describes the possible cash flows of premiums and benefits, whereas Scheme B expresses the cash flows in units of financial instruments.
Compiling these schemes for an endowment insurance, we will first carry out the valuation for l_{50} people using the life table and afterwards we will show an abstract valuation for one person. This valuation is general and is valid for all insurance contracts.

Valuation at age 50 for l_{50} people:

Valuation scheme A:

payments in interval $[t, t+1)$	unit e_t	number of units for l_{50} people: premium	death benefit	maturity benefit
BoP 50	$Z^{(50)}$	$-l_{50} \cdot \Pi$		
EoP 50	$Z^{(51)}$		$d_{50} \cdot C$	
BoP 51	$Z^{(51)}$	$-l_{51} \cdot \Pi$		
EoP 51	$Z^{(52)}$		$d_{51} \cdot C$	
BoP 52	$Z^{(52)}$	$-l_{52} \cdot \Pi$		
EoP 52	$Z^{(53)}$		$d_{52} \cdot C$	
BoP 53	$Z^{(53)}$	$-l_{53} \cdot \Pi$		
EoP 53	$Z^{(54)}$		$d_{53} \cdot C$	
BoP 54	$Z^{(54)}$	$-l_{54} \cdot \Pi$		
EoP 54	$Z^{(55)}$		$d_{54} \cdot C$	$l_{55} \cdot C$

Note that the premium payments correspond to short position in the valuation portfolio.

Valuation scheme B:

unit e_t	number of units for l_{50} people: premium	benefits		total
$Z^{(50)}$	$-l_{50} \cdot \Pi$			$-l_{50} \cdot \Pi$
$Z^{(51)}$	$-l_{51} \cdot \Pi$	$d_{50} \cdot C$		$-l_{51} \cdot \Pi + d_{50} \cdot C$
$Z^{(52)}$	$-l_{52} \cdot \Pi$	$d_{51} \cdot C$		$-l_{52} \cdot \Pi + d_{51} \cdot C$
$Z^{(53)}$	$-l_{53} \cdot \Pi$	$d_{52} \cdot C$		$-l_{53} \cdot \Pi + d_{52} \cdot C$
$Z^{(54)}$	$-l_{54} \cdot \Pi$	$d_{53} \cdot C$		$-l_{54} \cdot \Pi + d_{53} \cdot C$
$Z^{(55)}$		$d_{54} \cdot C$	$l_{55} \cdot C$	$l_{55} \cdot C + d_{54} \cdot C$

Valuation at age 50 for one person:

We obtain the valuation schemes A and B for one person by simply dividing the payments by l_{50}. The relations

$$\frac{l_t}{l_x} := \begin{cases} {}_{t-x}p_x = \prod_{k=x}^{t-1} p_k = \prod_{k=x}^{t-1} p_{**}(k) =: p_{**}(x,t)\,, & t > x\,, \\ 1\,, & t = x\,, \\ 0\,, & \text{else}\,, \end{cases}$$

$$\frac{d_t}{l_x} = \begin{cases} \frac{d_t}{l_t}\frac{l_t}{l_x} = q_t \cdot {}_{t-x}\,p_x = q_t \prod_{k=x}^{t-1} p_k =: p_{*\dagger}(t) \cdot p_{**}(x,t)\,, & t > x \\ 0\,, & \text{else,} \end{cases}$$

yield to the following payments in the interval $[t, t+1)$:

premium: $p_{**}(50,t) \cdot a_{*}^{\text{Pre}}(t) \cdot Z^{(t)}$

death benefit: $p_{**}(50,t) \cdot p_{*\dagger}(t) \cdot a_{*\dagger}^{\text{Post}}(t) \cdot Z^{(t+1)}$

maturity benefit: $p_{**}(50,t+1) \cdot a_{**}^{\text{Post}}(t) \cdot Z^{(t+1)}$

Valuation scheme A:

payments in interval $[t, t+1)$	unit e_t	number of units for one person: premium	death benefit	maturity benefit
BoP t	$Z^{(t)}$	$p_{**}(50,t) \cdot a_{*}^{\text{Pre}}(t)$		
EoP t	$Z^{(t+1)}$		$p_{**}(50,t) \cdot p_{*\dagger}(t) \cdot a_{*\dagger}^{\text{Post}}(t)$	$p_{**}(50,t+1) \cdot a_{**}^{\text{Post}}(t)$

Valuation scheme B:

unit e_t	number of units for one person λ_t
$Z^{(t)}$	$p_{**}(50,t) \cdot a_{*}^{\text{Pre}}(t) + p_{**}(50,t-1) \cdot p_{*\dagger}(t-1) \cdot a_{*\dagger}^{\text{Post}}(t-1)$ $+ p_{**}(50,t) \cdot a_{**}^{\text{Post}}(t-1)$

With the mapping φ given in (1) we obtain:

$$\begin{aligned}
\varphi_{50}(g_{50}) &= \sum_{t=0}^{\omega}\Big[p_{**}(50,t)\cdot a_{*}^{\text{Pre}}(t) \\
&\qquad + p_{**}(50,t-1)\cdot p_{*\dagger}(t-1)\cdot a_{*\dagger}^{\text{Post}}(t-1) \\
&\qquad + p_{**}(50,t)\cdot a_{**}^{\text{Post}}(t-1)\Big] Z^{(t)} \\
&= \sum_{t=0}^{\omega} p_{**}(50,t-1)\Big[p_{**}(t)\cdot a_{*}^{\text{Pre}}(t) \\
&\qquad + p_{*\dagger}(t-1)\cdot a_{*\dagger}^{\text{Post}}(t-1) \\
&\qquad + p_{**}(t)\cdot a_{**}^{\text{Post}}(t-1)\Big] Z^{(t)} && (3) \\
&= \sum_{t=50}^{55} p_{**}(50,t-1)\Big[p_{**}(t)\cdot a_{*}^{\text{Pre}}(t) \\
&\qquad + p_{*\dagger}(t-1)\cdot a_{*\dagger}^{\text{Post}}(t-1)t \\
&\qquad + p_{**}(t)\cdot a_{**}^{\text{Post}}(t-1)\Big] Z^{(t)} && (4)
\end{aligned}$$

The last equation follows since $a_{*}^{\text{Pre}}(t) = a_{*\dagger}^{\text{Post}}(t) = a_{**}^{\text{Post}}(t) = 0$ for $t < 50$ and $t > 54$.

The above described algorithms are true for an arbitrary Markov model. The corresponding formulae have to be generalized slightly.

5.2.3 Valuation in units at age 51

Step 2: Valuation in units

Again we consider the valuation for l_{51} people and also for one person.

Valuation at age 51 for l_{51} people:

Our valuation scheme A at age 51 corresponds to the valuation scheme A at age 50 leaving out the premium payment and benefits at age 50.

Valuation scheme A:

payments in interval $[t, t+1)$	unit e_t	number of units for l_{51} people premium	death benefit	maturity benefit
BoP 51	$Z^{(51)}$	$-l_{51} \cdot \Pi$		
EoP 51	$Z^{(52)}$		$d_{51} \cdot C$	
BoP 52	$Z^{(52)}$	$-l_{52} \cdot \Pi$		
EoP 52	$Z^{(53)}$		$d_{52} \cdot C$	
BoP 53	$Z^{(53)}$	$-l_{53} \cdot \Pi$		
EoP 53	$Z^{(54)}$		$d_{53} \cdot C$	
BoP 54	$Z^{(54)}$	$-l_{54} \cdot \Pi$		
EoP 54	$Z^{(55)}$		$d_{54} \cdot C$	$l_{55} \cdot C$

Valuation scheme B:

unit e_t	number of units for l_{51} people premium	benefits		total
$Z^{(51)}$	$-l_{51} \cdot \Pi$			$-l_{51} \cdot \Pi$
$Z^{(52)}$	$-l_{52} \cdot \Pi$	$d_{51} \cdot C$		$-l_{52} \cdot \Pi + d_{51} \cdot C$
$Z^{(53)}$	$-l_{53} \cdot \Pi$	$d_{52} \cdot C$		$-l_{53} \cdot \Pi + d_{52} \cdot C$
$Z^{(54)}$	$-l_{54} \cdot \Pi$	$d_{53} \cdot C$		$-l_{54} \cdot \Pi + d_{53} \cdot C$
$Z^{(55)}$		$d_{54} \cdot C$	$l_{55} \cdot C$	$l_{55} \cdot C + d_{54} \cdot C$

Valuation at age 51 for one person:

We divide the rows of the valuation scheme by l_{51} and obtain the following payments in the interval $[t, t+1)$:

premium: $p_{**}(51, t) \cdot a_*^{\text{Pre}}(t) \cdot Z^{(t)}$

death benefit: $p_{**}(51, t) \cdot p_{*\dagger}(t) \cdot a_{*\dagger}^{\text{Post}}(t) \cdot Z^{(t+1)}$

maturity benefit: $p_{**}(51, t) \cdot a_{**}^{\text{Post}}(t) \cdot Z^{(t+1)}$

Valuation scheme A:

payments in interval $[t, t+1)$	unit e_t	number of units for one person premium	death benefit	maturity benefit
t	$Z^{(t)}$	$p_{**}(51,t) \cdot a_*^{\text{Pre}}(t)$		
	$Z^{(t+1)}$		$p_{**}(51,t) \cdot p_{*\dagger}(t) \cdot a_{*\dagger}^{\text{Post}}(t)$	$p_{**}(51,t+1) \cdot a_{**}^{\text{Post}}(t)$

Valuation scheme B:

unit e_t	number of units λ_t
$Z^{(t)}$	$p_{**}(51,t) \cdot a_*^{\text{Pre}}(t) + p_{**}(51,t-1) \cdot p_{*\dagger}(t-1) \cdot a_{*\dagger}^{\text{Post}}(t-1) + p_{**}(51,t) \cdot a_{**}^{\text{Post}}(t-1)$

With the mapping φ defined in (1) we obtain:

$$
\begin{aligned}
\varphi_{51}(g_{50}) &= \sum_{t=1}^{\omega} \Big[p_{**}(51,t) \cdot a_*^{\text{Pre}}(t) \\
&\qquad + p_{**}(51,t-1) \cdot p_{*\dagger}(t-1) \cdot a_{*\dagger}^{\text{Post}}(t-1) \\
&\qquad + p_{**}(51,t) \cdot a_{**}^{\text{Post}}(t-1) \Big] Z^{(t)} \\
&= \sum_{t=1}^{\omega} p_{**}(51,t-1) \Big[p_{**}(t) \cdot a_*^{\text{Pre}}(t) + p_{*\dagger}(t-1) \cdot a_{*\dagger}^{\text{Post}}(t-1) \\
&\qquad + p_{**}(t) \cdot a_{**}^{\text{Post}}(t-1) \Big] Z^{(t)}
\end{aligned}
$$

5.2.4 Valuation in units at ages 52, 53 and 54

Valuation at age t for l_t people, $t = 52, 53, 54$:

The procedure is the same as at age 51 and the premium payments and benefits at previous ages are left out in the valuation schemes.

Valuation in units at age t for one person, $t = 52, 53, 54$:

The valuation schemes are the same as at age 51 for one person. We simply have to replace 51 by 52, 53 or 54, respectively.

5.3 *Annuity*

5.3.1 Situation

We consider an annuity in payment of a 65 year old person:

annuity payment	$C =$ CHF 15'000 per annum
age at entry	$x = 65$ years

Description of the insurance contract:

State space $S = \{*, \dagger\}$
Contractual functions:

$$a_*^{\text{Pre}}(t) = \begin{cases} C\,, & t = 65, \ldots, \omega \\ 0\,, & \text{else} \end{cases}$$

The transition probabilities in period $[t, t+1)$ are as follows:

$$\begin{aligned} p_{**}(t) &= p_t \\ p_{*\dagger}(t) &= q_t \\ p_{\dagger *}(t) &= 0 \\ p_{\dagger\dagger}(t) &= 1 \end{aligned}$$

Insurance contract: $g_{65} = g_{65}(a_*^{\text{Pre}})$.

Our task: Which is the value of this insurance contract

$$\left.\begin{array}{l} \text{at age } 65 \\ \text{at age } 66 \\ \text{at age } 67 \\ \vdots \\ \text{at age } \omega - 1 \\ \text{at age } \omega \end{array}\right\} \text{always } \textit{before} \text{ the annuity payment } C?$$

Step 1: Definition of the units

The annuity payments C payable at age t are represented by the units $Z^{(t)}$, $t = 65, \ldots, \omega$.

Therefore the basis consists of

$$\mathcal{B} = \{Z^{(t)};\, t = 65, 66, 67, \ldots, \omega\}.$$

5.3.2 Valuation in units at age 65

Step 2: Valuation in units

First, we look at the valuation for l_{65} and then for one person.

Valuation at age 65 for l_{65} people:

Valuation scheme A:

payments in interval $[t, t+1)$	unit e_t	number of units for l_{65} people
BoP 65	$Z^{(65)}$	$l_{65} \cdot C$
BoP 66	$Z^{(66)}$	$l_{66} \cdot C$
BoP 67	$Z^{(67)}$	$l_{67} \cdot C$
BoP 68	$Z^{(68)}$	$l_{68} \cdot C$
$\vdots$	$\vdots$	$\vdots$
BoP $\omega - 1$	$Z^{(\omega-1)}$	$l_{\omega-1} \cdot C$
BoP ω	$Z^{(\omega)}$	$l_{\omega} \cdot C$

Valuation scheme B:

unit e_t	number of units λ_t
$Z^{(65)}$	$l_{65} \cdot C$
$Z^{(66)}$	$l_{66} \cdot C$
$Z^{(67)}$	$l_{67} \cdot C$
$Z^{(68)}$	$l_{68} \cdot C$
$Z^{(69)}$	$l_{69} \cdot C$
$\vdots$	$\vdots$
$Z^{(\omega-1)}$	$l_{\omega-1} \cdot C$
$Z^{(\omega)}$	$l_{\omega} \cdot C$

Valuation at age 65 for one person:

We look at the payments in the interval $[t, t+1)$ and divide them by l_{65}. By analogy with Section (5.2.2), we obtain

Valuation scheme A:

payments in interval$[t, t+1)$	unit e_t	number of units for one person λ_t
$BoP\ t$	$Z^{(t)}$	$p_{**}(65, t) \cdot a_*^{\text{Pre}}(t)$

Valuation scheme B:

unit e_t	number of units λ_t
$Z^{(t)}$	$p_{**}(65, t) \cdot a_*^{\text{Pre}}(t)$

With φ as given in (1) we write in short

$$\varphi_{65}(g_x) = \sum_{t=1}^{\omega} \left[p_{**}(65, t) \cdot a_*^{Pre}(t)\right] Z^{(t)}$$

The valuations at age 66,...,ω are calculated analogously as in Sections 5.2.3 and 5.2.4. At this point it is important to remark that formula (3) remains true also in this case.

6 Monetary Valuation of the VaPo

6.1 Basic concept

As indicated in the introduction, the valuation portfolio is evaluated by the function ψ.

Step 3: Monetary valuation

In (2) we formally defined the map ψ which transforms a valuation in units (read a valuation portfolio) into money. The function ψ can be regarded as an *accounting principle* (or as a pricing system).

$$\psi : \text{Portfolio in units} \mapsto \text{Monetary amount.}$$

Question: How does ψ look like?

There are several possibilities. We consider two of them immediately and show some possible extensions later.

$$\left.\begin{array}{l}\psi_1\\ \psi_2\end{array}\right\}\text{assigns the}\left\{\begin{array}{l}\text{statutory book value}\\ \text{market value}\end{array}\right\}\text{to the units.}$$

First, we look again at the endowment insurance of the 50 year old person and then at the annuity of the person aged 65.
At this point it is worthwhile to mention that all the calculations performed in the classical life insurance mathematics such as the fact that the mathematical reserve equals the present value of the outflows (benefits) minus the present value of the inflows (premiums) are valid also in this context, as only linearity of the different maps is required.

6.1.1 Endowment policy

Statutory book value (Traditional mathematical reserves)

The book value of the $VaPo$ gives us the mathematical reserves (MR), i.e. the value of the zero coupon bonds is calculated with the technical interest rate.

Example: The book value of the zero coupon bond $Z^{(t)}$ with duration to maturity $t-50$ has the value

$$\psi_1(Z^{(t)}) = v^{t-50},$$

where $(1+i)^{-1}$. Here i denotes the technical interest rate.

Market value

The market value of the $VaPo$ gives us the *fair value*. Each unit has its market value.

Example: $\psi_2(Z^{(51)}) := P(50,51)$ corresponds to the value or price of a zero coupon bond at age 50 with maturity one year (see Appendix A.2).

Notation: We denote the value of the $VaPo$ at age t under the map ψ_1 by MR_t and under ψ_2 by $V_{t|t}$.

The valuation scheme B gives us the value of the $VaPo$: We multiply the number of every unit $Z^{(t)}$ by the value of $Z^{(t)}$ and add them up.

Valuation at age 50 for one person

In order to calculate the numerical values, we first calculate the premium Π with the use of the equivalence principle, which says that the present value of the premiums equals the present value of the benefits, i.e.

$$\Pi \cdot \ddot{a}_{50:\overline{5}|} = A_{50:\overline{5}|} \cdot C$$

By using commutation functions of the life table EKM95 with the technical interest rate 2.5% we receive:

$$\text{premium } \Pi = \frac{M_{50} - M_{55} + D_{55}}{N_{50} - N_{55}} \cdot C$$

$$= \frac{13'245 - 12'613 + 23'237}{562'903 - 435'605} \cdot 50'000 = 9'375.21$$

Valuation scheme B:

unit e_t	number of units for one person premium	benefits		total
$Z^{(50)}$	−9'375			−9'375
$Z^{(51)}$	−9'336	208		−9'128
$Z^{(52)}$	−9'293	229		−9'064
$Z^{(53)}$	−9'246	251		−8'995
$Z^{(54)}$	−9'194	275		−8'919
$Z^{(55)}$		302	48'734	49'036

Using the technical interest rate i=2.5% and the two yield curves in Appendix A.4, relating to the years 2000 or 2002, respectively, we obtain the following mathematical reserves and the fair values:

payments at age	$V_{50\|50}$, term structure 2000	$V_{50\|50}$, term structure 2002	MR_{50}
50	−9'375.21	−9'375.21	−9'375.21
51	−8'784.52	−9'043.42	−8'905.20
52	−8'365.82	−8'824.58	−8'627.53
53	−7'946.17	−8'552.66	−8'352.58
54	−7'543.31	−8'252.87	−8'080.14
55	39'689.57	44'037.07	43'340.67
total	−2'325.45	−11.67	0.00

Valuation at age 51 for one person:

Valuation scheme B:

unit	number of units for one person			
e_t	premium	benefits		total
$Z^{(51)}$	−9′375			−9′375
$Z^{(52)}$	−9′332	230		−9′102
$Z^{(53)}$	−9′285	252		−9′033
$Z^{(54)}$	−9′233	277		−8′956
$Z^{(55)}$		303	48′938	49′241

We do not know the value of the zero coupon bonds at age 51 yet. But if we "lock in" the term structure at the beginning of the policy then the value $P(50, 51, 52)$ of $Z^{(52)}$ at age 51 can be calculated as follows (see Appendix A.2):

$$P(50, 51, 52) = \frac{P(50, 52)}{P(50, 51)}\,.$$

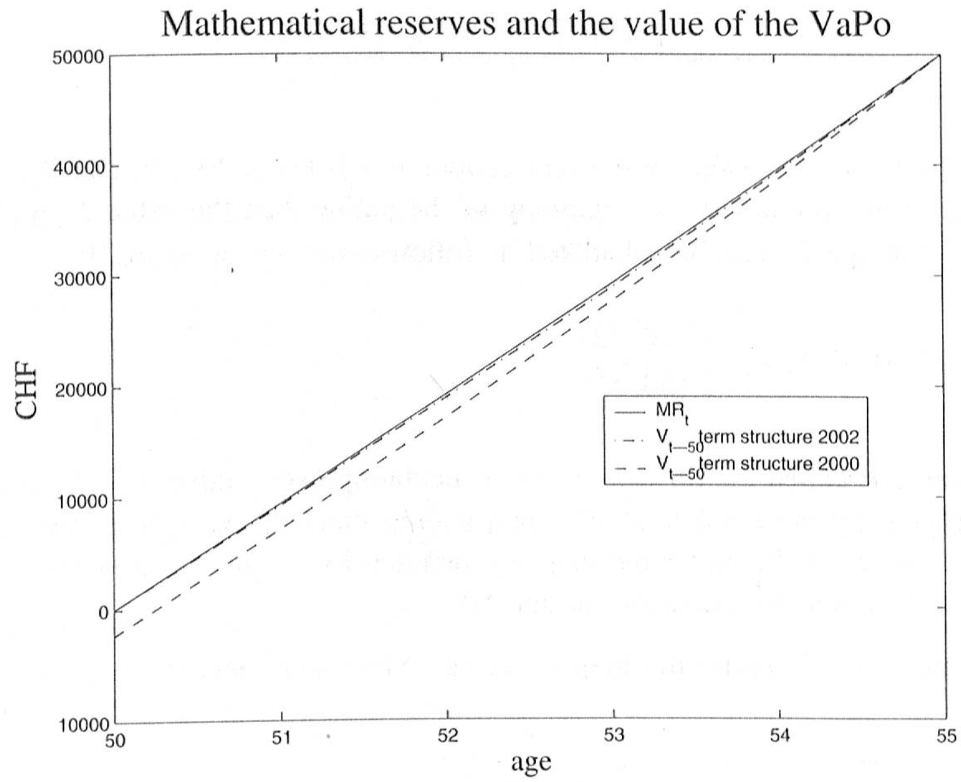
This value is called the *forward price*. Calculating every value of $Z^{(t)}$ at age 51, multiplying by the number of $Z^{(t)}$ and adding them up we receive the value of the $VaPo$ at age 51 under the map ψ_2, denoted by $V_{51|50}$. $V_{51|50}$ is the value of the $VaPo$ at age 51 calculated at age 50.

The value of $Z^{(t)}$ under the map ψ_1 at age 51 is much easier to calculate: it is simply v^{t-51}.

payments at age	$V_{50\|50}$, term structure 2000	$V_{50\|50}$, term structure 2002	MR_{50}
51	−9′375.21	−9′375.21	−9′375.21
52	−8′729.14	−8′944.22	−8′880.22
53	−8′291.26	−8′668.61	−8′597.22
54	−7′870.91	−8′364.76	−8′316.80
55	41′413.25	44′634.08	44′610.04
total	7′146.74	9′281.30	9′440.61

The next table summarizes the different values of the $VaPo$:

values at age t	$V_{t\|50}$, term structure 2000	$V_{t\|50}$, term structure 2002	MR_t
50	$-2'325.45$	-11.67	0.00
51	$7'146.74$	$9'281.30$	$9'440.61$
52	$17'076.73$	$18'842.83$	$19'144.35$
53	$27'521.46$	$28'784.46$	$29'126.71$
54	$38'475.03$	$39'151.72$	$39'405.28$
55	$50'000.00$	$50'000.00$	$50'000.00$



6.1.2 Annuity

Again, we consider the annuity described in Section 5.3 and put the figures of the table ERM 2000 into the valuation scheme B. Note that the terminal age ω of ERM equals to $\omega = 119$.

Valuation scheme B:

units e_t	number of units
$Z^{(65)}$	15'000
$Z^{(66)}$	14'890
$Z^{(67)}$	14'773
$Z^{(68)}$	14'646
$Z^{(69)}$	14'510
$\vdots$	$\vdots$
$Z^{(\omega-1)}$	0
$Z^{(\omega)}$	0

In order to value this scheme we have to know the prices of the zero coupon bonds with duration to maturity up to 119-65=54 years. For the term structure given in the Appendix A.4, zero coupon prices are available only for time to maturities equal to 10 years (term structure 2000) and 30 years (term structure 2002). We consider two possibilities to fill in these gaps. For this purpose we denote by τ the maximal time for which the price of the zero coupon bond is known (e.g. $\tau = 10, \tau = 30$):

- We set the internal interest rate of zero coupon bonds with duration to maturity less than τ years equal to the internal interest rate of the zero coupon bonds $Z^{(65+\tau)}$, i.e.

$$\psi_3(Z^{(65+k)}) = \begin{cases} P(65, 65+k)\,, & k \leq \tau \\ P(65, 65+\tau)^{\frac{k}{\tau}}\,, & k > \tau \end{cases}$$

- We include all the cash flows with time to maturity greater than τ to one cash flow at year τ, i.e.

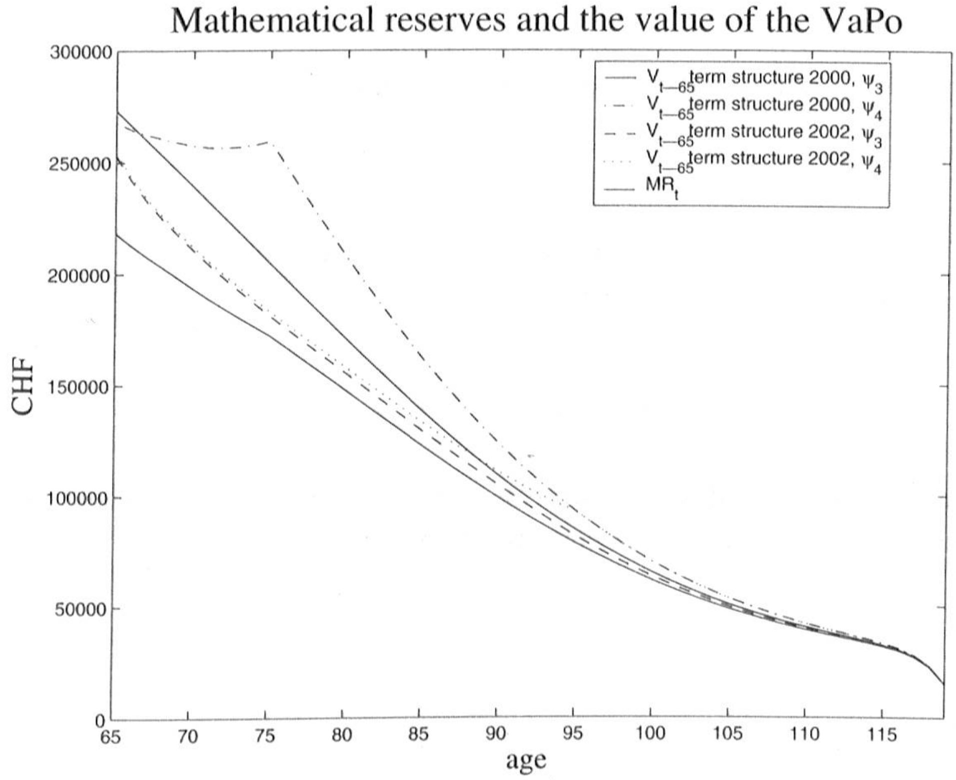
$$\psi_4(Z^{(65+k)}) = \begin{cases} P(65, 65+k)\,, & k \leq \tau \\ P(65, 65+\tau)\,, & k > \tau \end{cases}$$

Using these two versions of ψ and the technical interest rate 2.5% we calculate the fair value and the mathematical reserves at age 65, assuming τ=10 years (term structure 2000) or 30 years (term structure 2002).

payments at age	$V_{65\|65}$, term structure 2000 ψ_3	ψ_4	$V_{65\|65}$, term structure 2002 ψ_3	ψ_4	MR_{65}
65	15'000.00	15'000.00	15'000.00	15'000.00	15'000.00
66	14'330.24	14'330.24	14'752.59	14'752.59	14'527.11
67	13'634.16	13'634.16	14'381.83	14'381.83	14'060.69
68	12'938.64	12'938.64	13'926.19	13'926.19	13'600.41
69	12'272.33	12'272.33	13'426.74	13'426.74	13'145.72
⋮	⋮	⋮	⋮	⋮	⋮
74	9'126.99	9'126.99	10'684.29	10'684.29	10'938.93
75	8'545.98	8'545.98	10'129.13	10'129.13	10'507.72
76	8'029.49	8'402.08	9'571.74	9'571.74	10'078.81
⋮	⋮	⋮	⋮	⋮	⋮
94	1'482.52	3'509.72	2'026.84	2'026.84	2'699.39
95	1'270.70	3'147.86	1'763.04	1'763.04	2'362.02
96	1'077.77	2'793.80	1'511.76	1'564.74	2'045.22
⋮	⋮	⋮	⋮	⋮	⋮
total	218'141.69	267'779.49	252'873.64	254'219.93	272'965.81

As before, we look at the different values for future ages of the contract:

values at age x	$V_{x\|65}$, term structure 2000 ψ_3	ψ_4	$V_{x\|65}$, term structure 2002 ψ_3	ψ_4	MR_x
65	218'141.69	267'779.49	252'873.64	254'219.93	272'965.81
66	212'636.09	264'593.85	241'863.00	243'231.87	266'363.13
67	207'726.18	262'336.58	232'711.39	234'115.55	259'701.41
68	203'086.14	260'632.12	224'834.60	226'284.70	252'982.89
69	198'298.03	258'968.39	217'640.02	219'144.06	246'214.29
⋮	⋮	⋮	⋮	⋮	⋮



6.2 Recursive calculation of the monetary value

6.2.1 Thiele's difference equation

The mathematical reserves can be calculated recursively with Thiele's difference equation:

$$\begin{aligned} MR^i_{t+s} &= a_i^{\text{Pre}}(t+s) \\ &\quad + \sum_{j\in S} v_{t+s}\, p_{ij}(t+s)\left\{a_{ij}^{\text{Post}}(t+s) + MR^j_{t+s+1}\right\}. \end{aligned} \tag{5}$$

We imagine being at age t and identify this t to coincide with the year 2000 or 2002. In the above equation v_{t+s} denotes the annual discount rate during the time interval $[t+s, t+s+1)$ and MR^i_{t+s} the mathematical reserves at age $t+s$ if the policy holder is in state i.
The same method is also applicable to the valuation of the $VaPo$:

Question: How does v_{t+s} look with respect to the calculation of MR_{t+s} and $V_{r+s|t}$?

Answer:

- MR_{t+s}: We use the normal discount factor

$$v_{t+s} = v\,.$$

- $V_{r+s|t}$: The forward short rate $f(t, r+s)$ (see Appendix A.3) gives us the right answer. In (5), replace v_{r+s} by

$$v_{r+s} = \frac{1}{1 + f(t, r+s)} = \frac{P(t, r+s+1)}{P(t, r+s)}\,.$$

As a consequence of this fact, we can calculate fair values by means of Thiele's difference equation, using the appropriate parametrization.

6.2.2 Examples

1. We look again at the endowment policy of a 50 year old person. In the event of surviving s years, the recursion (5) for the value of the $VaPo$ is

$$V_{50+s|50} = a_*^{\text{Pre}}(50+s) + v_{50+s}\, p_{50+s}\, \left\{a_{**}^{\text{Post}}(50+s) + V_{50+s+1|50}\right\} + v_{50+s}\, q_{50+s}\, a_{*\dagger}^{\text{Post}}(50+s)\,,$$

   with the initial condition $V_{56} = 0$, $v_{50+s} = \dfrac{P(50, 51+s)}{P(50, 50+s)}$, $s = 0, \ldots, 5$.

2. Annuity described in Section 6.1.2

$$V_{65+s|65} = a_*^{\text{Pre}}(65+s) + v_{65+s} \cdot p_{65+s} \cdot V_{65+s+1|65}\,,$$

   with the initial condition $V_\omega = 0$, $s = 0, \ldots, \omega$.

6.3 *Some other possibilities for* ψ

In this section we will show another possibility for ψ relating to the use of first and second order tables within the life insurance industry. Under second order tables we understand the best estimate values for q_x etc. Under first order tables we understand tables with a PAD (provision for adverse deviation), which are normally used for pricing and reserving.

In the following we consider an annuity in payment with the following basis elements for $\mathcal{F}$:

$Z^{(i)}$ expected payment of one unit at time i based on second order basis.

$W^{(i)}$ Difference between expected payments of first and second order tables at time i with an amount of 1. We consider $W^{(i)}$ for the modelling of security loadings.

As financial instruments $Z^{(i)}$ represent zero coupon bonds and $W^{(i)}$ swaps expressed in zero coupon bonds reflecting the uncertainty of the insurance contract at time i. For an annuity of amount C at age $x = 65$ we have the following valuation scheme

age	unit $Z^{(i)}$	unit $W^{(i)}$
65	C	0
66	${}_1p_x^{II} \cdot C$	$({}_1p_x^I - {}_1p_x^{II}) \cdot C$
$\vdots$	$\vdots$	$\vdots$
$x+\tau$	${}_\tau p_x^{II} \cdot C$	$({}_\tau p_x^I - {}_\tau p_x^{II}) \cdot C$
$\vdots$	$\vdots$	$\vdots$

Now we can define the following accounting principles:

map	description	definition
ψ_5	book value first order	$\psi_5\left(\sum \lambda_i Z^{(i)} + \sum \mu_i W^{(i)}\right) = \sum (\lambda_i + \mu_i)\, v^{-i}$
ψ_6	book value second order	$\psi_6\left(\sum \lambda_i Z^{(i)} + \sum \mu_i W^{(i)}\right) = \sum \lambda_i v^{-i}$
ψ_7	fair value with PAD	$\psi_7\left(\sum \lambda_i Z^{(i)} + \sum \mu_i W^{(i)}\right) = \sum (\lambda_i + \mu_i)\, P(Z^i)$
ψ_8	fair value w/o PAD	$\psi_8\left(\sum \lambda_i Z^{(i)} + \sum \mu_i W^{(i)}\right) = \sum \lambda_i P(Z^i)$

Remarks:

- $\psi_5(f)$ for $f \in \mathcal{F}$ is usually called the *statutory technical reserve* and $\psi_6(f)$ the *second order mathematical reserve*.
- The value

$$\psi_7(f)/\psi_8(f) - 1 = \left(\sum \mu_i P(x,i)\right) \Big/ \left(\sum \lambda_i P(x,i)\right), x \in \{50, 65\}$$

can be interpreted as the demographic loading for the table.

In a second step we now want to look at the corresponding numerical values. Therefore we consider a 65 year old person together with the generation table ERM2000 for the first order basis and for the second order basis. In order to compare the different types of reserves we assume a technical interest of 2.5% and ignore the single premium payment, i.e.

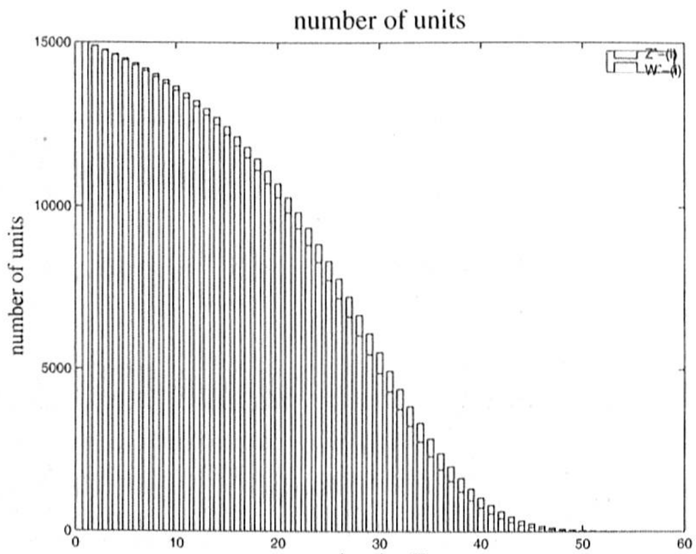
$$S = \{*, \dagger\},$$

$$a_*^{\text{Pre}}(t) = \begin{cases} C, & t = 65, \ldots, \omega \\ 0, & \text{else} \end{cases}$$

all other contractual functions being 0.

In this case we get the following valuation scheme

age	unit $Z^{(i)}$	unit $W^{(i)}$
65	15'000.00	0.00
66	14'881.52	8.77
67	14'754.20	18.32
68	14'617.26	28.89
69	14'469.78	40.64
70	14'310.84	53.67
⋮	⋮	⋮



And further the valuation

map		term structure	extrapolation term structure	value of the unit $Z^{(i)}$	value of the unit $W^{(i)}$	value
ψ_5	book value first order			264′958.07	8′007.74	272′965.81
ψ_6	book value second order			264′958.07		264′958.07
ψ_7	fair value with PAD	2000	ψ_3	213′246.13	4′895.57	218′141.69
		2002	ψ_3	246′491.96	6′381.68	252′873.64
		2000	ψ_4	258′105.70	9′673.79	267′779.49
		2002	ψ_4	247′483.25	6′736.69	254′219.93
ψ_8	fair value w/o PAD	2000	ψ_3	213′246.13		213′246.13
		2002	ψ_3	246′491.96		246′491.96
		2000	ψ_4	258′105.70		258′105.70
		2002	ψ_4	247′483.25		247′483.25

Before ending this section we also want to mention that there are various other possibilities for $\mathcal{F}$ and ψ which can be used to solve the respective concrete problems. An example for a valuation basis $\mathcal{F}$ could consist of $\{Z^{(i)}; i \in \mathbb{N}, e\}$ where $Z^{(i)}$ represent zero coupon bonds with different maturities, reflecting the expected payments and where e denotes a financial instrument representing the difference between the expected payments and the random variable. Therefore e reflects the pure insurance risk.

7 Technical gains / losses

7.1 Introduction

The valuation schemes used yield a valuation for a deterministic model. The question now arises what happens if the actual number of deaths deviates from the expected one.

Let us assume that all transition probabilities remain unchanged except p_x and q_x, which are replaced by their observed values $\tilde{p}_x$ and $\tilde{q}_x$

$$\begin{aligned} p_x &\mapsto \tilde{p}_x \\ q_x &\mapsto \tilde{q}_x \,. \end{aligned}$$

Again the endowment policy and the annuity will be considered. First, we calculate the value of the $VaPo$ with the observed values. Second, we look at the difference between the two values.

7.2 *Examples*

7.2.1 Endowment policy

We consider again the endowment insurance with a cover period of 5 years for a 50 year old person.

Valuation scheme B at age 50:

unit	number of units for one person		
e_t	premium	death benefits	maturity benefits
$Z^{(50)}$	$-\Pi$		
$Z^{(51)}$	$-\tilde{p}_{50} \cdot \Pi$	$\tilde{q}_{50} \cdot C$	
$Z^{(52)}$	$-\tilde{p}_{50} \cdot p_{**}(51,52) \cdot \Pi$	$\tilde{p}_{50} \cdot p_{*\dagger}(51) \cdot C$	
$Z^{(53)}$	$-\tilde{p}_{50} \cdot p_{**}(51,53) \cdot \Pi$	$\tilde{p}_{50} \cdot p_{**}(51,52) \cdot p_{*\dagger}(52) \cdot C$	
$Z^{(54)}$	$-\tilde{p}_{50} \cdot p_{**}(51,54) \cdot \Pi$	$\tilde{p}_{50} \cdot p_{**}(51,53) \cdot p_{*\dagger}(53) \cdot C$	
$Z^{(55)}$		$\tilde{p}_{50} \cdot p_{**}(51,54) \cdot p_{*\dagger}(54) \cdot C$	$\tilde{p}_{50} \cdot p_{**}(51,55) \cdot C$

This valuation scheme is not the same as on page 35. We analyze the difference between the two schemes, which is also an element of $\mathcal{F}$, as $\mathcal{F}$ is a linear vector space. We subtract the number of units of the scheme from the number of units of the valuation scheme B on page 35.

Notation: $\Delta p_x := p_x - \tilde{p}_x, \qquad \Delta q_x := q_x - \tilde{q}_x.$

unit e_t	number of units for one person: premium	death benefits	maturity benefits
$Z^{(50)}$	0		
$Z^{(51)}$	$-\Delta p_{50} \cdot \Pi$	$\Delta q_{50} \cdot C$	
$Z^{(52)}$	$-\dfrac{\Delta p_{50}}{p_{50}} \cdot p_{**}(50,52) \cdot \Pi$	$\dfrac{\Delta p_{50}}{p_{50}} \cdot p_{**}(50,51) \cdot p_{*\dagger}(51) \cdot C$	
$Z^{(53)}$	$-\dfrac{\Delta p_{50}}{p_{50}} \cdot p_{**}(50,53) \cdot \Pi$	$\dfrac{\Delta p_{50}}{p_{50}} \cdot p_{**}(50,52) \cdot p_{*\dagger}(52) \cdot C$	
$Z^{(54)}$	$-\dfrac{\Delta p_{50}}{p_{50}} \cdot p_{**}(50,54) \cdot \Pi$	$\dfrac{\Delta p_{50}}{p_{50}} \cdot p_{**}(50,53) \cdot p_{*\dagger}(53) \cdot C$	
$Z^{(55)}$		$\dfrac{\Delta p_{50}}{p_{50}} \cdot p_{**}(50,54) \cdot p_{*\dagger}(54) \cdot C$	$\dfrac{\Delta p_{50}}{p_{50}} \cdot p_{**}(51,55) \cdot C$

Consequence: It is important to understand gains and losses expressed in *units*, i.e. the annual gains and losses represent also a *portfolio* (difference of two different $VaPo$'s).

Generalization: We use the same notation as on page 35 to look at the difference between the two schemes.

unit e_t	number of units for one person λ_t
$Z^{(x)}$	0
$Z^{(x+1)}$	$-\Delta p_x\, \Pi + \Delta q_x\, C$
$Z^{(t)},\ t > x+1$	$\dfrac{\Delta p_x}{p_x}\Big[- p_{**}(x,t)\, a_*^{\text{Pre}}(t) + p_{**}(x,t-1)\, p_{*\dagger}(t-1)\, a_{*\dagger}^{\text{Post}}(t-1) + p_{**}(x,t)\, a_{**}^{\text{Post}}(t-1)\Big]$

Remark: For $t > x+1$ the technical gain/loss is a multiple of the number of units in the deterministic model. The factor is $\Delta p_x / p_x$.

Example: We take $\tilde{q}_{50} = 1.5 \cdot q_{50}$ and receive the following values:

	unit e_t	number of units for one person: premium	death benefits	maturity benefits	total
gains/losses in units	$Z^{(50)}$	0.00			0.00
	$Z^{(51)}$	−19.53	−104.16		−123.69
	$Z^{(52)}$	−19.44	0.48		−18.96
	$Z^{(53)}$	−19.34	0.53		−18.82
	$Z^{(54)}$	−19.23	0.58		−18.66
	$Z^{(55)}$		0.63	101.52	102.15

We would like to draw the attention to the sign of the number of units. The premium income of the new $VaPo$ is *smaller* and the death benefit at the end of age 50 is much *higher* compared to the old $VaPo$, although we have two negative signs.
The value of the above $VaPo$ is:

	payments at age	$\Delta V_{50\|50}$, term structure 2000	$\Delta V_{50\|50}$, term structure 2002	ΔMR_{50}
monetary gains / losses	50	0.00	0.00	0.00
	51	−119.04	−122.54	−120.67
	52	−17.50	−18.46	−18.05
	53	−16.62	−17.89	−17.47
	54	−15.78	−17.26	−16.90
	55	82.68	91.74	90.29
	total	−86.26	−84.42	−82.81

Consequence: The value of the new $VaPo$ is higher than the old one due to the negative sign. Moreover, the value of the new $VaPo$ with structure 2002 has a positive sign (see the corresponding value on page 44):

$$-11.67 - (-84.42) = 72.75\,.$$

7.2.2 Annuity

We consider the annuity described in Section 5.3 and work out the valuation scheme B with the changed mortality $\tilde{q}_{65}$.

Valuation scheme B at age 65:

unit e_t	number of units for one person λ_t
$Z^{(65)}$	C
$Z^{(66)}$	$\tilde{p}_{65} \cdot C$
$Z^{(67)}$	$\tilde{p}_{65} \cdot p_{**}(66, 67) \cdot C$
$Z^{(68)}$	$\tilde{p}_{65} \cdot p_{**}(66, 68) \cdot C$
$\vdots$	$\vdots$
$Z^{(\omega)}$	$\tilde{p}_{65} \cdot p_{**}(66, \omega) \cdot C$

We look again at the difference of the schemes before and after changing the mortality rates q_{65}:

unit e_t	number of units for one person λ_t
$Z^{(65)}$	0
$Z^{(66)}$	$\Delta p_{65} \cdot C$
$Z^{(67)}$	$\frac{\Delta p_{65}}{p_{65}} \cdot p_{**}(65, 67) \cdot C$
$Z^{(68)}$	$\frac{\Delta p_{65}}{p_{65}} \cdot p_{**}(65, 68) \cdot C$
$\vdots$	$\vdots$
$Z^{(\omega)}$	$\frac{\Delta p_{65}}{p_{65}} \cdot p_{**}(65, \omega) \cdot C$

Generalization: We use the same notation as on page 35 and look at the difference between this scheme and the scheme on page 42.

unit e_t	number of units for one person λ_t
$Z^{(x)}$	0
$Z^{(t)},\ t > x$	$\frac{\Delta p_x}{p_x} \cdot p_{**}(x,t) \cdot a_*^{\text{Pre}}(t)$

Remark: Again, for $t > x$ the technical loss/gain is a multiple of the number of units in the deterministic model with the factor $\Delta p_x / p_x$. If $\tilde{p}_x < p_x$ then the value of the new $VaPo$ is still smaller than the value of the old one because $\Delta p_x = p_x - \tilde{p}_x > 0$.

In numbers:

Valuation scheme B at age 65:

	unit e_t	number of units for one person λ_t
gains/losses in units	$Z^{(65)}$	0.00
	$Z^{(66)}$	54.86
	$Z^{(67)}$	54.42
	$Z^{(68)}$	53.96
	$Z^{(69)}$	53.46
	⋮	⋮

	payments at age	$\Delta V_{65\|65}$ term structure 2000 ψ_3	$\Delta V_{65\|65}$ term structure 2000 ψ_4	$\Delta V_{65\|65}$ term structure 2002 ψ_3	$\Delta V_{65\|65}$ term structure 2002 ψ_4	ΔMR_{50}
monetary gains/losses	65	0.00	0.00	0.00	0.00	0.00
	66	52.79	52.79	54.35	54.35	53.52
	67	50.23	50.23	52.98	52.98	51.80
	68	47.67	47.67	51.30	51.30	50.10
	69	45.21	45.21	49.46	49.46	48.43
	70	42.83	42.83	47.52	47.52	46.77
	⋮	⋮	⋮	⋮	⋮	⋮
	total	748.36	931.23	876.31	881.27	950.33

Consequence: As mentioned before the value of the new $VaPo$ is smaller than the value of the old one. Because the actual number of deaths is higher than the expected one, the annuity payments are reduced and therefore the $VaPo$ has a lower value.

A Time value of money

A.1 *Introduction*

The market value over time of investment bonds depends on the term structure of interest rates. This dependency can be expressed by interest rate curves as:

- the time curve of zero coupon prices;
- yield curves;
- the curve of forward short rates.

A.2 *Price structure of zero coupon bonds*

The market trend of financial instruments is random, therefore we clearly have to say *when* the financial instruments are considered. We assume to be at time t:
A *zero coupon bond* $Z^{(t+T)}$ is a financial instrument paying one cash unit at time $t+T$ with absolute certainty:

$$Z^{t+T} = (\underbrace{0}_{t+1}, 0, \dots, 0, \underbrace{1}_{t+T}, 0, \dots).$$

We will use the notation $Z^{(T)}$ instead of $Z^{(t+T)}$.

Question: How much does $Z^{(T)}$ cost at time $t+\tau$?

Definitions: We call this price $P(t, t+\tau, t+T)$. The price is fixed at time t and will be paid at time $t+\tau$.
If $\tau = 0$, $P(t, t+T) := P(t, t, t+T)$ is named the *spot price*.
If $\tau > 0$, $P(t, t+\tau, t+T)$ is called the *forward price*.
The curve $\gamma_t : s \mapsto P(t, t+s)$ is referred to as the *time-t curve of zero coupon prices*.

Remark: The shape of $\gamma_t(s)$ depends on t. For every t, we get a new curve.

Proposition: Knowing all spot prices, the forward prices can be calculated by the formula

$$P(t, t+\tau, t+T) = \frac{P(t, t+T)}{P(t, t+\tau)}.$$

A.3 Interest rates

Definitions:

$$Y(t, t+\tau, t+T) := P(t, t+\tau, t+T)^{-\frac{1}{T-\tau}} - 1.$$

If $\tau = 0$, $Y(t, t+T) := Y(t, t, t+T)$ is named the *spot rate.*
If $\tau > 0$, $Y(t, t+\tau, t+T)$ is called *forward rate.*

The *yield curve* $\delta_t : s \mapsto Y(t, t+s)$ shows the connection between the interest rates and the time to maturity of zero coupon bonds. It is a snapshot of the term structure at time t.
If $T = \tau + 1$, $f(t, t+\tau) := Y(t, t+\tau, t+\tau+1)$ is called the *forward short rate*
The curve $\alpha_t : s \mapsto f(t, t+s)$ is referred to as the *curve of forward short rates.*

Proposition: The fundamental relations between zero coupon prices and interest rates are the following:

$$P(t, t+\tau, t+T) = \frac{1}{[1 + Y(t, t+\tau, t+T)]^{T-\tau}}$$

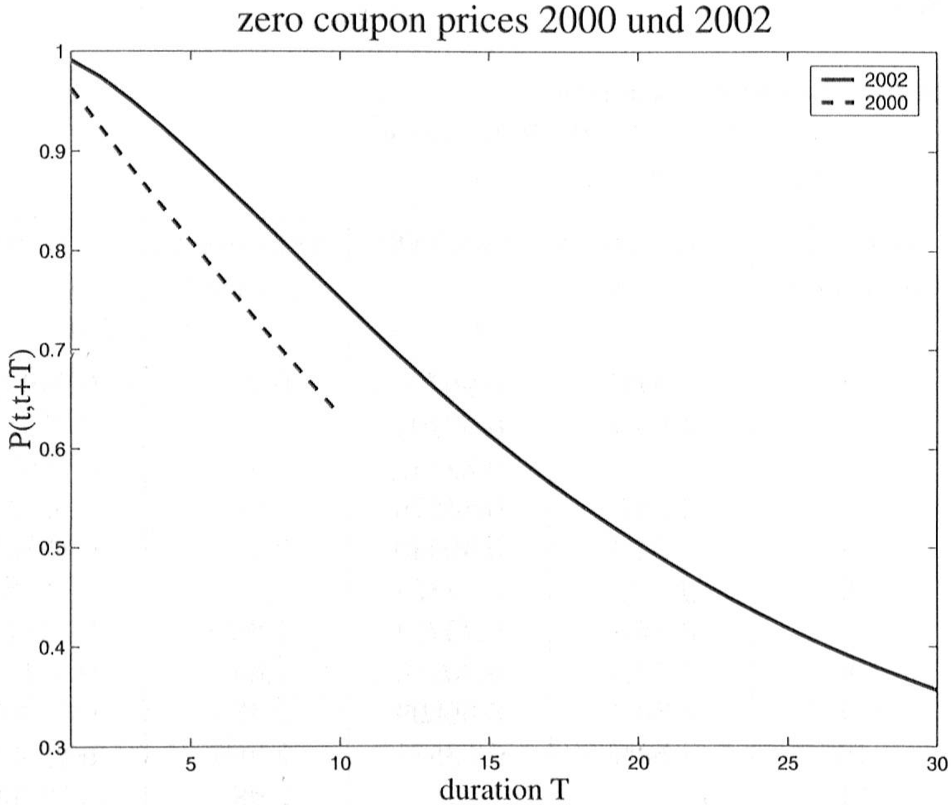
$$[1 + Y(t, t+T)]^T = [1 + Y(t, t+\tau)]^\tau [1 + Y(t, t+\tau, t+T)]^{T-\tau}$$

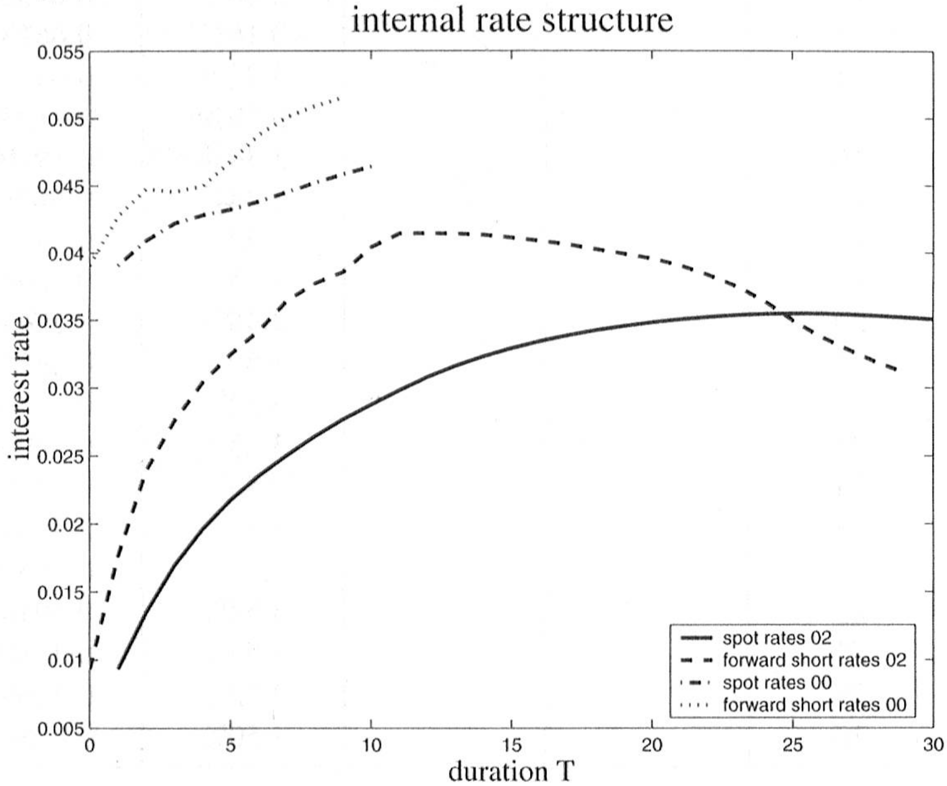
$$P(t, t+T) = \frac{1}{[1 + f(t,t)][1 + f(t, t+1)] \ldots [1 + f(t, t+T-1)]}.$$

A.4 *Term structures*

Term structure 10.5.2000: Swap rates,
term structure 26.11.2002: CHF LIBOR Terminraten,
both converted into zero coupon prices.

duration to maturity s	term structure 10.5.2000 spot rates $Y(t,t+s)$	term structure 10.5.2000 value $P(t,t+s)$	term structure 26.11.2002 spot rates $Y(t,t+s)$	term structure 26.11.2002 value $P(t,t+s)$
1	3.91%	0.96239	0.93%	0.99075
2	4.09%	0.92294	1.35%	0.97355
3	4.22%	0.88342	1.69%	0.95084
4	4.28%	0.84576	1.96%	0.92532
5	4.32%	0.80940	2.17%	0.89806
6	4.38%	0.77322	2.35%	0.86984
7	4.45%	0.73727	2.50%	0.84112
8	4.52%	0.70210	2.64%	0.81157
9	4.58%	0.66809	2.77%	0.78209
10	4.64%	0.63535	2.88%	0.75305
11			2.98%	0.72380
12			3.08%	0.69501
13			3.16%	0.66736
14			3.23%	0.64082
15			3.29%	0.61539
16			3.34%	0.59109
17			3.38%	0.56787
18			3.42%	0.54571
19			3.45%	0.52460
20			3.48%	0.50446
21			3.50%	0.48527
22			3.52%	0.46704
23			3.53%	0.44980
24			3.54%	0.43355
25			3.55%	0.41833
26			3.55%	0.40420
27			3.54%	0.39101
28			3.53%	0.37860
29			3.52%	0.36691
30			3.50%	0.35585


zero coupon prices 2000 und 2002
P(t,t+T)
duration T
2002
2000
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
5
10
15
20
25
30


internal rate structure
interest rate
duration T
spot rates 02
forward short rates 02
spot rates 00
forward short rates 00
0.055
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
5
10
15
20
25
30

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Abstract

This paper illustrates one possible way to value a life insurance contract. The difficulty in giving a value to an insurance contract is the fact that there exists no liquid market for these contracts. However there is a liquid market for many financial instruments. Therefore we represent a life insurance contract as combination of financial instruments with payoffs which match the *expected* liabilities. Instead of valuing the insurance contract, we value the portfolio consisting of the financial instruments. Applying different accounting principles on the same portfolio we attach the mathematical reserve or the fair value to the portfolio.

Zusammenfassung

Dieser Artikel zeigt eine Möglichkeit, wie Lebensversicherungsverträge bewertet werden können. Weil kein liquider Markt für Versicherungsverträge besteht, ist es schwierig, ihnen einen Wert zuzuweisen. Wir umgehen dieses Problem, indem ein Versicherungsvertrag als Kombination von Finanzinstrumenten dargestellt wird mit Payoffs, die den erwarteten Verpflichtungen entsprechen, denn für die meisten Finanzinstrumente existiert ein liquider Markt. Auf diese Weise kann anstelle des Versicherungsvertrages das Portfolio bestehend aus Finanzinstrumenten bewertet werden. Durch Anwendung verschiedener Bewertungsprinzipien auf das Portfolio wird das Deckungskapital oder der Fair Value bestimmt.

Résumé

Cet article décrit une méthode d'évaluation pour les contrats d'assurance vie. La difficulté d'une telle évaluation réside dans l'absence de marché liquide. Nous contournons ce problème en représentant un contrat d'assurance vie comme combinaison d'instruments financiers dont les flux sont identiques à l'espérance mathématique des engagements. Comme il existe un marché liquide pour la plupart des instruments financiers, nous pouvons alors évaluer le portefeuille de ces instruments financiers au lieu du contrat d'assurance. En appliquant différents principes comptables au même portefeuille, on peut calculer la provision mathématique ou la valeur juste.