Zeitschrift: Mitteilungen / Schweizerische Aktuarvereinigung = Bulletin / Association Suisse des Actuaires = Bulletin / Swiss Association of Actuaries

Herausgeber: Schweizerische Aktuarvereinigung

Band: - (2003)

Heft: 2

Artikel: A note on the expected present value of dividends with a constant barrier in the discrete time model

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DOI: https://doi.org/10.5169/seals-967412

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A note on the expected present value of dividends with a constant barrier in the discrete time model

1 Introduction

The aim of the present study is to formalize the dividend payment policies in the discrete case for a non-life insurance portfolio, and to obtain the expected present value of the dividend payments.
The classical model analyses the solvency of non-life insurance portfolios using the probability of ruin as the criterion. The discrete case has been studied by various authors, for example, Bowers et al. (1987), Gerber (1988), Michel (1989), Shiu (1989), Willmot (1993), Willmot and Cai (2001), De Vylder (1996) or Li and Garrido (2002). Section 2 deals with the alternative approach to be found in the literature proposing the pay-out of part of the reserves in the form of dividends (Bühlmann (1970), Gerber (1972, 1981), Paulsen and Gjessing (1997), Siegl and Tichy (1996, 1999)).
Section 3 deals with the analysis of the dividend payments when the model is modified to have a constant dividend barrier $b(t) = b$, assuming discrete payments, and presents a method for solving such problems. We prove that, as in the continuous case, the probability of ruin is unity. Bühlmann (1970) and Gerber (1972) obtain the expectation of the present value assuming two different particular cases for the distribution of aggregated cost in one period. We present a solution for the general case, i.e. for any discrete aggregated cost distribution. The system of linear equations that allows one to find the expectation of the present value of the dividend payments is obtained, and it is solved using the matrix form of the system. We also include, in Section 4, a recursive solution, alternative to Gerber (1972).

2 Dividend policy in the discrete case

Following Bühlmann (1970) we take a discrete dividend policy to be that which makes the payouts at given times, t_i for $i = 1, 2, 3, \ldots$, as long as the level of reserves at time t_i surpasses the cap represented by the dividend barrier. Consider the equidistant times t_i for $i = 1, 2, 3, \ldots$ with $t_0 = 0$, the time unit being one year.

The level of reserves at t_i before dividend payments, R_i^*, can be defined as $R_i^* = u + c \cdot t_i - SS_i - SD_{i-1}$, where SS_i is the aggregate of claims in the period $[0, t_i]$, u is the initial reserve at t_0, c is the annual premium income, $SD_i = D_1 + D_2 + \cdots + D_s$, $\forall s \leq i$ is the sum of the dividend payments in an interval $[0, t_i]$, where $SD_0 = 0$, and $D_i = \text{Max}\{(R_i^* - b), 0\}$ the dividends paid out at t_i for $i = 1, 2, 3, \ldots$.
Let v be a constant annual discount rate for all the periods and t_k the discrete time of ruin. Then the expected present value of the dividend payments, assuming that there are dividend payments only up to the time of ruin is

$$W(u,b) = E\left[\sum_{i=1}^{k} D_i \cdot v^{t_i}\right] \qquad \text{with} \quad t_k = \text{Min}\{t_i \mid R_i^* < 0\}\,.$$

3 Constant barrier: calculation of $W(u,b)$

We shall now generalize the calculation of the expected present value of the dividends, following the approach of Bühlmann (1970) and Gerber (1972), for the calculation of $W(u,b)$ in a modified model with a constant dividend barrier, b.
We consider $S_i = SS_i - SS_{i-1}$. We assume that S_i are i.i.d. random variables with common probability function $P_s = P[S = s]$ and distribution function $F_S(s) = P[S \leq s]$ for $s = 0, 1, 2, \ldots$. For simplicity, we redefine c as $c \cdot t_1$ so, for a positive security loading, $E[S] < c$.
The solution of the problem involves considering the random variable of the total accumulated claims in a period as a discrete random variable, and the hypothesis that all monetary values $(u, b, c, \ldots)$ are multiples of some given unit. Neither of these conditions implies any major restriction on the validity of the model: in the case of the monetary values, we simply have to change the reference unit, and in the case of the claims, we shall just have to previously discretize the random variable if it is not already discrete.
In the constant dividend barrier case the probability of ruin is 1 in the continuous case (Bühlmann (1970)). We prove that this is also true in the discrete case. In the discrete case ruin probability is $\psi(u,b) = P[t_k < \infty]$.

Theorem 1 *Ruin probability in a model with a constant dividend barrier assuming discrete payments is one,* $\psi(u,b) = 1$

Proof. $\psi(u,b)$ for $u = b$, considering the situation at time t_1, is

$$\psi(b,b) \cdot (1 - F_S(c)) = \sum_{s=c+1}^{b+c} \psi(b+c-s, b) \cdot P_s + 1 - F_S(b+c) \tag{1}$$

We know, for $h \geq 0$, that $\psi(b-h,b) \geq \psi(b,b)$, and rearranging terms, (1) can be written as

$$\psi(b,b)\cdot(1-F_S(b+c)) \geq 1-F_S(b+c) \tag{2}$$

and in view of (2), $\psi(b,b) \geq 1$, then

$$\psi(b-h,b) \geq \psi(b,b) \geq 1$$

which implies $\psi(u,b) = \psi(b,b) = 1$. ■

Bühlmann (1970) proposed a system of finite difference equations to calculate $W(u,b)$, considering the situation at time t_1, and solving it for the particular case in which the variation in the reserves is dichotomous, taking only the values -1 and 1. Since the only random factor considered in the model is the occurrence of claims, the case that Bühlmann calculated implies that the claims in a given period can only take the values $(c+1)$ and $(c-1)$. Gerber (1972) considers the system where the variation in the reserves can take the values $1, 0, -1, -2, \ldots$ and the claims in a given period are multiples of the premium.

To generalize the calculation of $W(u,b)$, we shall analyse the situation of the process at time t_1, so,

$$W(u,b) = v\cdot E[W(R_1^* - D_1, b) + D_1] \tag{3}$$

The dividend payments in t_1 will depend on whether $R_1^* = u+c-s$ is greater or lesser than the level of the barrier b:

- **Case 1:** R_1^* is greater than the level of the barrier b. In this case, the dividend payments in t_1, $D_1 = SD_1$, are positive, with their amount being the difference between R_1^* and the barrier b, i.e. $D_1 = u+c-s-b$. Also, to obtain $W(u,b)$ the calculated future dividends must be discounted to t_1, which are given by $W(b,b)$.

- **Case 2:** R_1^* is less than or equal to the level of the barrier b, independently of what happened in the interval $(0, t_1]$. In this case, for the calculation of $W(u,b)$, we must discount $W(u+c-s,b)$.

To determine the expression for the expected present value of the dividend payments, we shall formalize the two cases described previously, by setting up a system of linear equations.

According to the initial level of reserves u, such that $u \leq b$, one can define $b+1$ equations for the calculation of $W(u,b)$ with $u = 0, \dots, b$.

Theorem 2 *For* $x = 0, 1, \dots, c, c+1, \dots, b$

$$W(b-x,b) = v \cdot \left[W(b,b) \cdot F_S(c-x) + \sum_{s=0}^{c-(x+1)} (c-s-x) \cdot P_s \right.$$
$$\left. + \sum_{s=1}^{b} W(b-s,b) \cdot P_{s+c-x} \right] \tag{4}$$

Proof. $W(u,b)$ is calculated, using the law of total probability, as the discounted sum of D_1 and the expected present value of the future dividends in t_1:

- **If the initial level of reserves coincides with the barrier level,** $u = b$
  First, let us consider the case in which the total of claims s coincides with the premium income c. At t_1 therefore, the level of reserves is $R_1^* = b + c - s = b$, then $D_1 = 0$.
  In those cases when the amount of claims s lies in the interval $[0, c-1]$, there will be dividend payments, since $R_1^* = b + c - s$ is greater than b, with $D_1 = c - s$, so that the level of reserves after dividend payment will be $u = b$.
  Finally, let us consider the cases in which the aggregate claims amount s lies in the interval $[c+1, b+c]$. The level of reserves at t_1, $R_1^* = b+c-s$, is less than b, then $D_1 = 0$.
  Obviously R_1^* and D_1 depend on s, so we can write

s	R_1^*	D_1	$R_1^* - D_1$
$[0, c-1]$	$b+c-s$	$c-s$	b
c	b	0	b
$[c+1, b+c]$	$b+c-s$	0	$b+c-s$

  Then, from (3)

$$W(b,b) = v \cdot \left[W(b,b) \cdot F_S(c) + \sum_{s=0}^{c-1} (c-s) \cdot P_s \right.$$
$$\left. + \sum_{s=1}^{b} W(b-s,b) \cdot P_{s+c} \right] \tag{5}$$

- **If the initial level of reserves is below the barrier by less than c units, $b - c < u < b$.**
The equation for $u = b - x$, when $x = 1, \ldots, c-1$ results from taking into account that the level of reserves at t_1 is

$$R_1^* = b - x + c - s \tag{6}$$

If $s < c - x$ then (6) is greater than b, and therefore leads to dividend payment, where $D_1 = c - s - x$ would have to be paid out, leaving the new level of reserves at b.
If $s > c - x$ then (6) is less than b. In this case, there will be no dividend payment, $D_1 = 0$. Also, so as not to cause ruin, one must have that $b - x + c - s \geq 0 \Rightarrow s \leq b - x + c$. Hence, the amount of s has to lie in the interval $[c - x + 1, c - x + b]$.
Lastly, if $s = c - x$, $R_1^* = b - x + c - s = b$ and $D_1 = 0$.
Then

s	R_1^*	D_1	$R_1^* - D_1$
$[0, c - x - 1]$	$b - x + c - s$	$c - s - x$	b
$c - x$	b	0	b
$[c - x + 1, c - x + b]$	$b - x + c - s$	0	$b - x + c - s$

So, from (3)

$$W(b - x, b) = v \cdot \left[W(b, b) \cdot F_S(c - x) + \sum_{s=0}^{c-(x+1)} (c - s - x) \cdot P_s \right.$$
$$\left. + \sum_{s=1}^{b} W(b - s, b) \cdot P_{s+c-x} \right] \tag{7}$$

- **If the initial level of reserves is below the barrier by at least c units, $0 \leq u \leq b - c < b$.**
Now, for $u = b - x$, when $x = c, c + 1, \ldots, b$, the level of reserves is $R_1^* = b - x + c - s$, which is therefore always less than b given the values of x. There is therefore no dividend payment.
So, from (3)

$$W(b - x, b) = v \cdot \sum_{s=0}^{b+(c-x)} W(b - x + c - s, b) \cdot P_s \tag{8}$$

where the upper-limit of the sum prevents the case in which the level of reserves is negative, $b - x + c - s \geq 0 \Rightarrow s \leq b + (c - x)$.

We can observe that the expressions (5), (7) and (8) are included in the general expression (4). ■

The matrix form of the system defined in expression (4) is obtained in Appendix A.

4 Analysis of the $c = 1$ case

If we assume that the distribution of the total cost in a period is concentrated in multiples of the premium, so $c = 1$, it is possible to calculate $W(u, b)$ as a recursive process. We present an alternative solution to that of Gerber (1972). As $E[S] < c = 1$, then $P_0 > 0$.

Theorem 3 *For* $x = 0, 1, \ldots, b-1$

$$W(b-x, b) = \frac{1}{C_1(x)} \cdot \left(C_3(x) + \sum_{s=x+1}^{b} W(b-s, b) \cdot C_2(s, x) \right) \tag{9}$$

where $C_1(x)$, $C_2(s, x)$ *and* $C_3(x)$ *are calculated in a recursive form,*

$$C_1(x+1) = C_1(x) \cdot (1 - v \cdot P_1) - v \cdot P_0 \cdot C_2(x+1, x)$$

$$C_2(s, x+1) = C_2(s, x) \cdot v \cdot P_0 + C_1(x) \cdot v \cdot P_{s-x}, \quad s = x+2, \ldots, b$$

$$C_3(x+1) = v \cdot P_0 \cdot C_3(x) = v^{x+2} \cdot P_0^{x+2}$$

where $C_1(0) = 1 - v \cdot P_1 - v \cdot P_0$, $C_2(s, 0) = v \cdot P_{s+1}$ *for* $s = 1, \ldots, b$, $C_3(0) = v \cdot P_0$ *and* $W(0, b) = \frac{C_3(b)}{C_1(b)}$.

The proof of Theorem 3 is in Appendix B.

Appendix A

It can be readily verified that the generalization of the system presented in Theorem 2, and defined by equations (5), (7) and (8), can be written in matrix form $v \cdot A \cdot \overline{w} + v \cdot D = \overline{w}$, where A is the matrix of coeffcients made up of different submatrices $A = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$, with M_1 a vector of $(c+1)$ components $(E[S] < c$, then $F_S(c-1) > 0)$, M_2 a matrix of order $(c+1) \times b$, M_3 a null vector of $(b-c)$ components and M_4 a matrix of order $(b-c) \times b$. The matrix A is therefore a square matrix of order $(b+1)$,

$$A = \begin{pmatrix}
F_s(c) & P_{c+1} & P_{c+2} & P_{c+3} & & \cdots & \cdots & P_{c+b} \\
F_s(c-1) & P_c & P_{c+1} & P_{c+2} & & \cdots & \cdots & P_{c+b-1} \\
F_s(c-2) & P_{c-1} & P_c & P_{c+1} & & \cdots & \cdots & P_{c+b-2} \\
\vdots & \vdots & \vdots & \vdots & & \cdots & \cdots & \cdots \\
F_s(0) & P_1 & P_2 & P_3 & & \cdots & \cdots & P_b \\
0 & P_0 & P_1 & P_2 & \cdots & \cdots & \cdots & P_{b-1} \\
0 & 0 & P_0 & P_1 & \cdots & \cdots & \cdots & P_{c+b-1} \\
0 & 0 & 0 & P_0 & \cdots & \cdots & \cdots & P_{c+b-2} \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & P_{c+b-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & P_0 & \cdots & P_c
\end{pmatrix}$$

The vector of independent terms D is of order $(b+1) \times 1$, formed by c first elements different from zero, and the remaining $b+1-c$ elements equal to zero. $\overline{w}$ is the vector of $b+1$ unknowns,

$$D = \begin{pmatrix}
\sum_{s=0}^{c-1} (c-s) \cdot P_s \\
\sum_{s=0}^{c-2} (c-s-1) \cdot P_s \\
\sum_{s=0}^{c-3} (c-s-2) \cdot P_s \\
\vdots \\
P_0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\qquad
\overline{w} = \begin{pmatrix}
W(b,b) \\
W(b-1,b) \\
W(b-2,b) \\
\vdots \\
W(b-c,b) \\
W(b-c-1,b) \\
\vdots \\
W(0,b)
\end{pmatrix}$$

The solution of system is $\overline{w} = [I - v \cdot A]^{-1} \cdot v \cdot D$. As the spectral norm of the matrix $v \cdot A$ is less than one, $[I - v \cdot A]$ is regular.

Appendix B

Proof. From (4) when $c = 1$ and $x = 0$, we obtain

$$W(b,b) = \frac{1}{1 - v \cdot P_1 - v \cdot P_0} \cdot \left(v \cdot P_0 + v \cdot \sum_{s=1}^{b} W(b-s,b) \cdot P_{s+1} \right) \quad \text{(B1)}$$

and for $x = 1$

$$W(b-1,b) = \frac{v}{1 - v \cdot P_1} \cdot \left[W(b,b) \cdot P_0 + \sum_{s=2}^{b} W(b-s,b) \cdot P_s \right] \quad \text{(B2)}$$

If we substitute (B1) in (B2), simplifying terms gives

$$W(b-1,b) = \frac{v^2 \cdot P_0^2 + \sum_{s=2}^{b} W(b-s,b) \cdot (v^2 \cdot P_0 \cdot P_{s+1} + v \cdot P_s \cdot (1 - v \cdot P_1 - v \cdot P_0))}{(1 - v \cdot P_1) \cdot (1 - v \cdot P_1 - v \cdot P_0) - v^2 \cdot P_0 \cdot P_2} \quad \text{(B3)}$$

Putting $C_1(1) = (1 - v \cdot P_1) \cdot (1 - v \cdot P_1 - v \cdot P_0) - v^2 \cdot P_0 \cdot P_2$, $C_2(s,1) = v^2 \cdot P_0 \cdot P_{s+1} + v \cdot P_s \cdot (1 - v \cdot P_1 - v \cdot P_0)$ $\forall s > 1$ and $C_3(1) = v^2 \cdot P_0^2$, (B3) can be re-written as $W(b-1,b) = \frac{1}{C_1(1)} \cdot \left(C_3(1) + \sum_{s=2}^{b} W(b-s,b) \cdot (C_2(s,1)) \right)$, and generalized to (9).

Now we show (9) by induction. We assume (9) is true for x, and we show that it is true for $x+1$. From (4) when $c = 1$ and $u = b - x - 1$, $(0 < x \leq b)$

$$(1 - v \cdot P_1) \cdot W(b-x-1,b) = v \cdot \left[W(b-x,b) \cdot P_0 + \sum_{s=1}^{b-x-1} W(b-x-1-s,b) \cdot P_{s+1} \right] \quad \text{(B4)}$$

We substitute (9) in (B4),

$$W(b-x-1,b) = \frac{v \cdot P_0 \cdot C_3(x) + \sum_{s=x+2}^{b} W(b-s,b) \cdot (v \cdot P_0 \cdot C_2(s,x) + C_1(x) \cdot v \cdot P_{s-x})}{C_1(x) \cdot (1 - v \cdot P_1) - v \cdot P_0 \cdot C_2(x+1,x)} \quad \text{(B5)}$$

which implies

$$W(b-(x+1),b) = \frac{1}{C_1(x+1)} \cdot \left(C_3(x+1) + \sum_{s=x+2}^{b} W(b-s,b) \cdot C_2(s,x+1) \right) \quad \text{(B6)}$$

so (9) is demonstrated.
From (B5) and (B6), we obtain the recursive formula for $C_1(x)$, $C_2(s,x)$ and $C_3(x)$, where the initial values are obtained from (B1) and (9).
To calculate $W(0,b)$, we calculate (4) for $c = 1$ and $x = b$, and (9) for $x = b-1$, and we solve the system. ■

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Abstract

The process of free reserves in a non-life insurance portfolio as defined in the classical model of risk theory is modified by the introduction of dividend policies that set maximum levels for the accumulation of reserves. The work presents a general solution to calculate the expected present value of dividends based on a system of linear equations for discrete dividend payments in the case of a constant dividend barrier.

Résumé

On modifie le processus de réserves libres d'une compagnie d'assurances non-vie, tel qu'il est définit dans le modèle classique de la théorie du risque, en introduisant une politique de distribution de dividendes qui pose un niveau maximum d'accumulation de réserves. Ce travail présente une solution générale du calcul de l'espérance mathématique de la valeur actuelle des dividendes distribués, basée sur un système d'équations linéaires des paiements discrets dans le cas d'une barrière constante.

Zusammenfassung

Der Prozess der freien Reserven in einem Nichtleben-Versicherungsportefeuille, wie er im klassischen Modell der Risikotheorie definiert ist, wird modifiziert durch eine Dividendenpolitik, welche obere Schranken für das Äufnen der Reserven setzt. Für den Fall einer konstanten Dividendenschranke wird eine allgemeine Lösung für die Berechnung des erwarteten Barwerts der Dividenden vorgestellt. Diese Lösung basiert auf einem System linearer Gleichungen für diskrete Dividendenzahlungen.