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Common mixture in the individual risk model

1 Introduction

We consider the classical individual risk model with a portfolio of n individual risks. In the actuarial literature, attention is devoted to the measurement of the global risk of an insurance portfolio which is done via the aggregate claim amount for a given period. Let S denote the aggregate claim amount for a portfolio of n risks

$$S = X_1 + \cdots + X_n,$$

where X_i $(i = 1, \ldots, n)$ is the total amount of claims for the ith policyholder of the portfolio. We define the random variable (r.v.) X_i by

$$X_i = \begin{cases} B_i, I_i = 1 \\ 0, I_i = 0 \end{cases},$$

where I_i is a Bernoulli r.v. with mean q_i and B_i is a positive r.v. with cumulative distribution function (c.d.f.) F_{B_i}. This construction can be found e.g. in Bowers *et al.* (1997), Klugman *et al.* (1998) and Rolski *et al.* (1999).

The riskiness of an insurance portfolio has usually been studied under the assumption that the I_i's as well as the B_i's $(i = 1, \ldots, n)$ are independent. Recently, dependence between the risks of an insurance portfolio has been examined in various papers. Among them, Albers (1999) and Cossette *et al.* (2002) propose different structures of dependence and evaluate numerically the impact of such dependence between the risks. Wang (1998) (see also the discussion by Meyers (1998)) suggests a set of tools for modeling and combining correlated risks. Also, Wang and Dhaene (1998) and Dhaene and Goovaerts (1997) find the riskiest stop-loss premium for portfolios of dependent risks. Denuit *et al.* (1999) find bounds on the c.d.f. of S when the distribution of the X_i's is known and when no hypotheses are made on the type of correlation structure of the risks. Embrechts *et al.* (1999, 2000, 2001a, 2001b) study the impact and the nature of the correlation between risks in the context of risk management.

In this paper, we propose a structure of dependence on the occurrence random variables that can be used in a context where a certain factor, such as the weather or economic conditions, can have an impact on the whole portfolio. This type

of construction has first been studied by Marshall and Olkin (1988) and in an insurance setting by Wang and Dhaene (1998), Wang (1998) and, Bäuerle and Müller (1998). The structure discussed in this paper can be applied, for example, in crop insurance where the weather greatly affects the harvest of the farmers and in a financial context in the analysis of a portfolio of credit risks. The common mixture model proposed here includes a parameter or a vector of parameters which allows us to quantify the impact on the claim occurrence and on the default of credit risks over a given period.

2 Proposed Structure

We propose here a structure allowing dependence among risks of an insurance portfolio. In many situations, those risks are correlated. They are influenced by common economic situations, geographic locations and other external factors. One way to introduce this kind of dependence is through an external mechanism which shall influence the entire insurance portfolio. The external mechanism can be viewed as a realization of a positive r.v. Θ with c.d.f. denoted by G_Θ. A realization θ of the r.v. Θ represents the influence from external factors and gives the risk level of a given year for a portfolio. In other words, the individual risk model assumes that the occurrence of claims for each policy is function of a r.v. Θ. Both a discrete and continuous distribution can be considered for Θ. We define

$$\Pr(I_i = 1 \mid \Theta = \theta) = 1 - r_i^\theta \quad \text{and} \quad \Pr(I_i = 0 \mid \Theta = \theta) = r_i^\theta, \tag{1}$$

where $(I_i \mid \Theta = \theta)$ $(i = 1, \dots, n)$ are Bernoulli r.v. and r_i is the basic parameter of the conditional distribution of $(I_i \mid \Theta = \theta)$. Each r.v. I_i $(i = 1, \dots, n)$ is influenced by the possible values of the r.v. Θ. For a fixed r_i, the conditional probability of no occurrence given $\Theta = \theta$ is a decreasing function of θ. We also assume that $(I_1 \mid \Theta = \theta), \dots, (I_n \mid \Theta = \theta)$ are independent r.v. For a given distribution G_Θ of the r.v. Θ and fixed marginals for $I_1, \dots, I_n$, we have

$$r_i = e^{M_\Theta^{-1}(p_i)},$$

since

$$\Pr(I_i = 0) = 1 - q_i = p_i$$

$$= \int_0^\infty r_i^\theta \, dG_\Theta(\theta)$$

$$= P_\Theta(r_i)$$

$$= M_\Theta(\ln(r_i)),$$

where $P_\Theta(t), M_\Theta(t)$ are respectively the p.g.f. and the m.g.f. of Θ. We assume that $M_\Theta(t)$ exists for some $t \neq 0$. With the proposed structure, a dependence relation is introduced between the components $\underline{I} = (I_1, \ldots, I_n)$ and the joint c.d.f. of $\underline{I}$ is

$$F_{I_1, \ldots, I_n}(i_1, \ldots, i_n) = \int \prod_{j=1}^n (F_{I_j}^0(i_j))^\theta \, dG_\Theta(\theta), \tag{2}$$

where $(F_{I_j}^0(i_j))^\theta$ is the conditional c.d.f. of I_j given that $\Theta = \theta$.

The random vector $\underline{X} = (X_1, \ldots, X_n)$ has dependent components due to the dependence of the I_i's. In order to obtain the m.g.f. of S, we examine the p.g.f. of $\underline{I} = (I_1, \ldots, I_n)$ since it can be easily (given the assumptions) written in the following form

$$P_{\underline{I}}(\underline{t}) = \int_0^\infty \prod_{i=1}^n (r_i^\theta + (1 - r_i^\theta)t_i) \, dG_\Theta(\theta).$$

Following Cossette *et al.* (2002), we obtain

$$M_{\underline{X}}(\underline{t}) = \int_0^\infty \prod_{i=1}^n (r_i^\theta + (1 - r_i^\theta)M_{B_i}(t_i)) \, dG_\Theta(\theta),$$

which can be written as

$$M_{\underline{X}}(\underline{t}) = \int_0^\infty M_{\underline{X}|\Theta=\theta}(\underline{t}) \, dG_\Theta(\theta). \tag{3}$$

It follows from (3) that the m.g.f. of the aggregate claim amount S is

$$M_S(t) = M_{\underline{X}}(t, \dots, t)$$

$$= \int_0^\infty M_{\underline{X}|\Theta=\theta}(t, \dots, t) \, dG_\Theta(\theta) \tag{4}$$

$$= \int_0^\infty M_{S|\Theta=\theta}(t) \, dG_\Theta(\theta),$$

where $(S \mid \Theta = \theta)$ is the sum of the independent random variables $(X_i \mid \Theta = \theta)$ $(i = 1, \dots, n)$. It is clear that

$$F_S(s) = \int_0^\infty F_{S|\Theta=\theta}(s) \, dG_\Theta(\theta). \tag{5}$$

In the case of a continuous distribution for Θ, we discretize G_Θ and use the usual numerical methods such as Depril's algorithm and the compound Poisson approximation to compute $F_{S|\Theta=\theta}(s)$. The application of the compound Poisson approximation will be discussed in the following section.

A measure of the global risk of an insurance portfolio which is often used is the stop-loss premium defined by

$$\pi_S(d) = E\left[(S - d)_+\right],$$

where d is the retention level. In our context, the stop-loss premium for a fixed retention level d is the mixture

$$\pi_S(d) = \int_0^\infty \pi_{S|\Theta=\theta}(d) \, dG_\Theta(\theta).$$

Now let us look at the expectation and the variance of S in the context just presented. The expectation of S is given by

$$E[S] = \sum_{i=1}^n E[B_i] q_i$$

$$= \sum_{i=1}^n E[B_i](1 - P_\Theta(r_i)).$$

As for the variance of S, we have

$$
\begin{aligned}
\mathrm{Var}[S] &= \sum_{i=1}^{n} \mathrm{Var}[X_i] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{Cov}[X_i, X_j] \\
&= \sum_{i=1}^{n} \left(E[B_i^2] q_i - E^2[B_i] q_i^2 \right) \\
&\quad + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[B_i] E[B_j] \left(\int_0^{\infty} (1 - (r_i)^\theta)(1 - (r_j)^\theta) \, dG_\Theta(\theta) - q_i q_j \right) \\
&= \sum_{i=1}^{n} \left(E[B_i^2] q_i - E^2[B_i] q_i^2 \right) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[B_i] E[B_j] (P_\Theta(r_i r_j) - p_i p_j) \,.
\end{aligned}
$$

Certain choices of distribution for Θ give interesting results. A discrete distribution for Θ leads to the following c.d.f. for S

$$
F_S(s) = \sum_{k=1}^{m} \Pr(\Theta = \theta_k) F_{S|\Theta=\theta_k}(s) \,,
$$

where m is a strictly positive integer. When $m = 1$, we have a degenerate distribution for Θ which corresponds to the case of independent risks. Also, if we make the assumption that both the r.v. I_i $(i = 1, \ldots, n)$ and the r.v. B_i $(i = 1, \ldots, n)$ are identically distributed then an explicit expression can be found for $F_S(s)$. Let us suppose that I_i $(i = 1, \ldots, n)$ have a common c.d.f. F_I which is a Bernoulli distribution with parameter p. The conditional r.v. $I \mid \Theta = \theta$ is also a Bernoulli r.v. but with parameter r^θ.

Also, suppose that B_i $(i = 1, \ldots, n)$ have a common c.d.f. F_B. Under these assumptions, the explicit form of the c.d.f. of the r.v. S is as follows

$$F_S(s) = \int_0^\infty F_{S|\Theta=\theta}(s)\, dG_\Theta(\theta)$$

$$= \int_0^\infty \left\{ (r^\theta)^n + \sum_{k=1}^n \binom{n}{k}(1 - r^\theta)^k (r^\theta)^{n-k} F_B^{*k}(s) \right\} dG_\Theta(\theta)$$

$$= \int_0^\infty \left\{ (r^\theta)^n + \sum_{k=1}^n \binom{n}{k}\left[\sum_{j=0}^k \binom{k}{j}(-1)^j (r^\theta)^j\right](r^\theta)^{n-k} F_B^{*k}(s) \right\} dG_\Theta(\theta)$$

$$= \int_0^\infty \left\{ (r^\theta)^n + \sum_{k=1}^n \binom{n}{k} \sum_{j=0}^k \binom{k}{j}(-1)^j (r^{n-k+j})^\theta F_B^{*k}(s) \right\} dG_\Theta(\theta)$$

$$= M_\Theta(n \ln(r)) + \sum_{k=1}^n \sum_{j=0}^k \binom{n}{k}\binom{k}{j}(-1)^j M_\Theta((n - k + j)\ln(r)) F_B^{*k}(s),$$

where F_B^{*k} is the k-fold convolution of F_B.

As an illustration, we examine the graphs of the quantile functions and the stop-loss premiums $\pi_S(d)$ for 3 portfolios of 10 identically distributed risks. The interest in the quantile function lies, notably, in the evaluation of the Value-at-risk in a risk management context (see e.g. Embrechts *et al.* 1999, 2000, 2001a, 2001b).

In a first example, the following properties hold for the three portfolios:

1. the claim amount random variables $B_1, \ldots, B_{10}$ are exponentially distributed with mean 1;

2. the occurrence r.v. I_i have a Bernoulli distribution with parameter $q_i = 0.1$ for $i = 1, \ldots, 10$;

3. the r.v. Θ has a logarithmic distribution with mass probability function and probability generating function

$$\Pr(\Theta = k) = \frac{\gamma^k}{-k \ln(1 - \gamma)}, \qquad k = 1, 2, \ldots$$

and

$$P_\Theta(t) = \frac{\ln(1 - \gamma t)}{\ln(1 - \gamma)}, \qquad |t| < 1.$$

For each portfolio we have chosen the values 0.5, 0.95, or 0.999999 for the parameter γ.

In a second example, the portfolios have the same properties except for the r.v. Θ which in this case has an extended truncated negative binomial distribution with p.g.f.

$$P_\Theta(t) = \frac{(1 - \beta t)^{-r} - 1}{(1 - \beta)^{-r} - 1}, \qquad 0 < \beta < 1; \ -1 < r < 0.$$

We choose $r = -0.5$ for the three portfolios and let the parameter β take the values 0.5, 0.95, and 0.9999. Note that the p.g.f. of the logarithmic distribution can be obtained from the p.g.f. of the extended truncated negative binomial distribution by taking the limit as $r \to 0$ (see e.g. Panjer and Willmot (1992) or Klugman *et al.* (1998)).

It is clear from the figures below that the dangerousness of the r.v. Θ has a significant impact on both the quantile function and $\pi_S(d)$. Observe that the stop-loss premium is monotone increasing with the parameters γ and β of respectively the logarithmic and the extended truncated negative binomial distributions. This observation has yet to be proven formally. This is not observed for the quantile function. However, it seems that for values greater than a given point, the quantile function increases as the degree of dependence increases. Let us also mention that the values used in the figures are exact and that no simulation methods have been used.
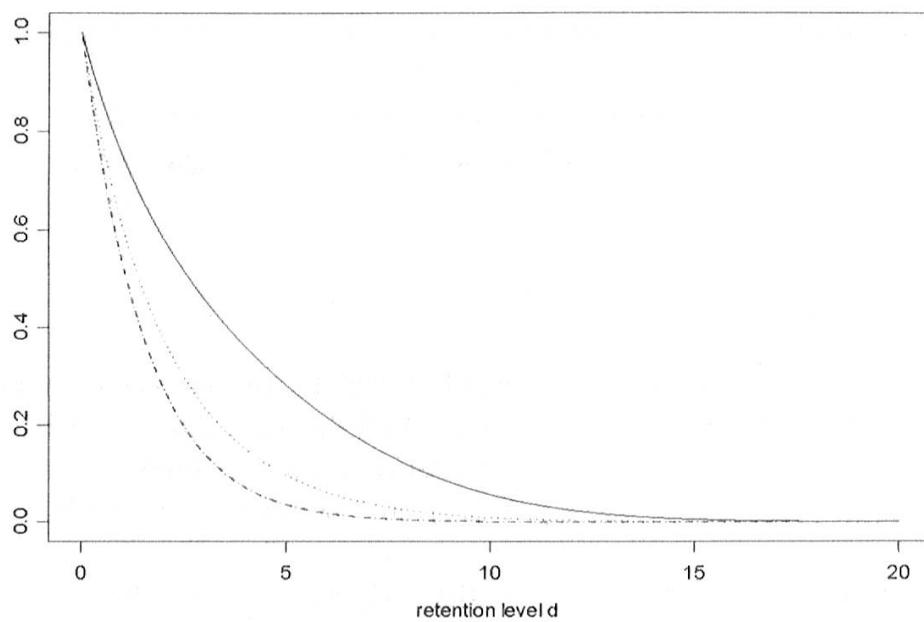
Figure 1. Stop-loss premiums for the logarithmic model with γ equal to 0.9999 (continuous line), 0.95 (dotted line) and 0.5 (dashed line).
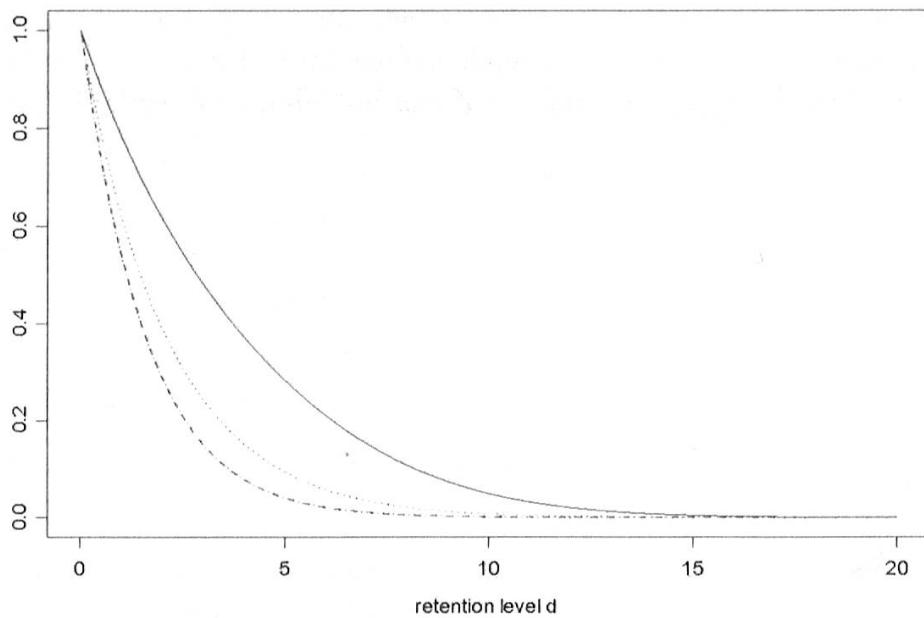
Figure 2. Stop-loss premiums for the extended negative binomial model with γ equal to 0.9999 (continuous line), 0.95 (dotted line) and 0.5 (dashed line).
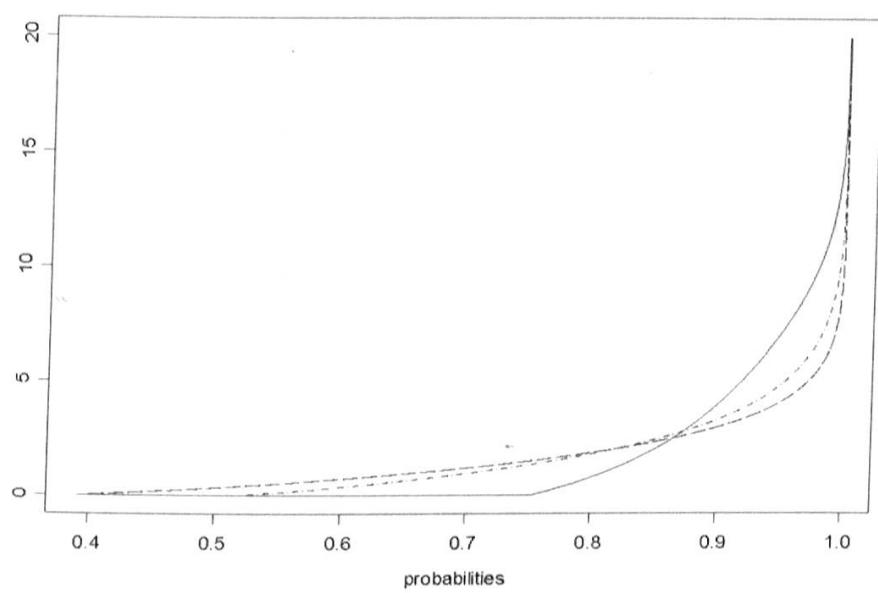
Figure 3. Quantiles for the logarithmic model with γ equal to 0.9999 (continuous line), 0.95 (dotted line) and 0.5 (dashed line).
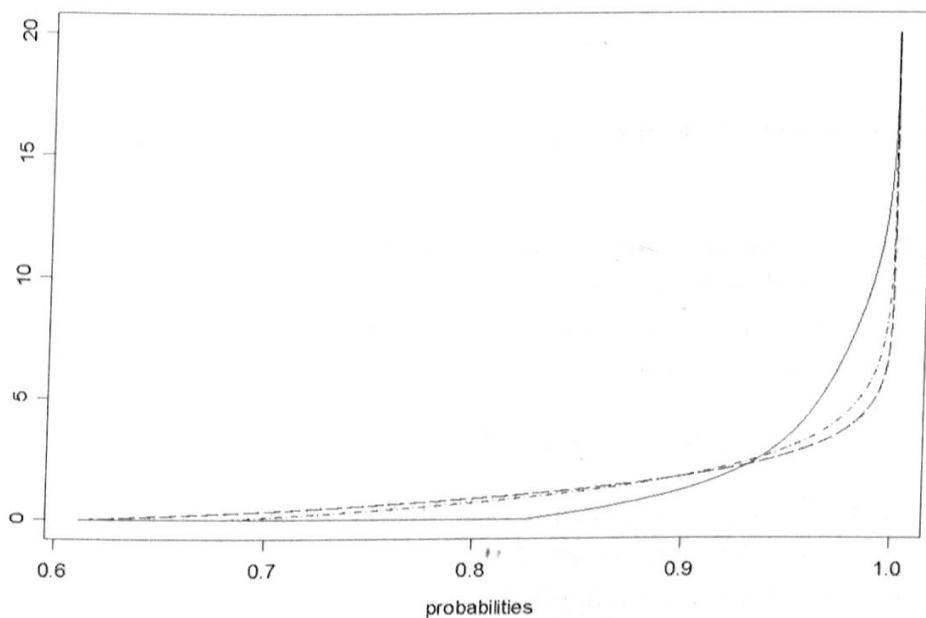

Figure 4. Quantiles for the extended negative binomial model with γ equal to 0.9999 (continuous line), 0.95 (dotted line) and 0.5 (dashed line).
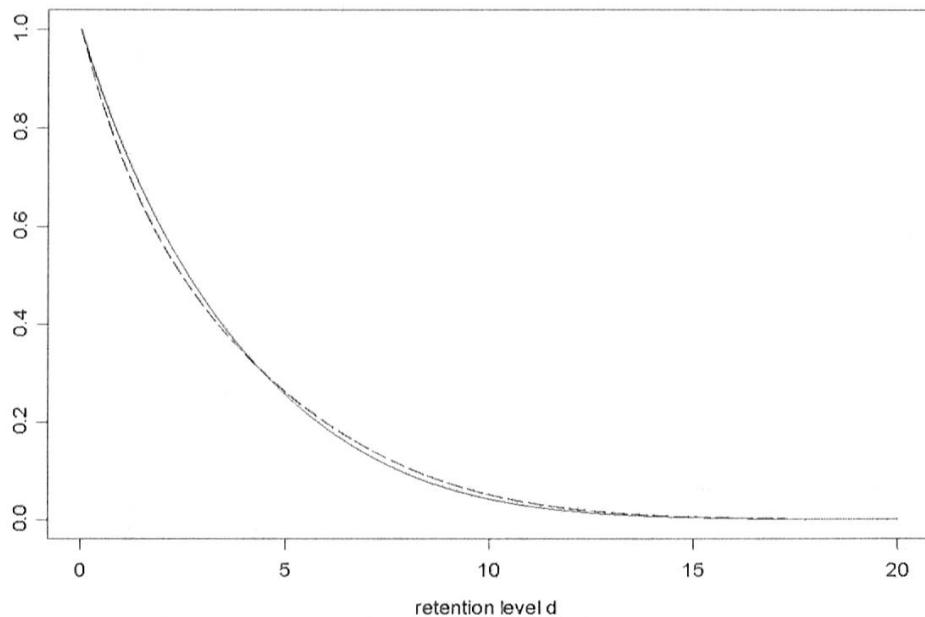
Figure 5. Stop-loss premiums for the logarithmic model (continuous line) and the extended negative binomial model (dashed line).

3   The Compound Poisson Approximation

The computation of the distribution of the total amount of claims in the individual model with independent risks is generally difficult in the case of a large portfolio. It has been shown that one way to overcome this problem is to use the compound Poisson approximation. The main idea is to approximate each r.v. X_i in $S = \sum_{i=1}^{n} X_i$ by a compound Poisson r.v. that we denote by Y_i. This allows the use of Panjer's algorithm (see Rolski *et al.* (1999), Klugman *et al.* (1998)) to compute the approximated distribution of S. The compound Poisson approximation has usually been applied in the past in cases where the B_i's and the I_i's are independent, see Rolski *et al.* (1999), Klugman *et al.* (1998) or Bowers *et al.* (1997) for actuarial applications and Barbour *et al.* (1999) for a general survey in applied probability.

In this section, we use the compound Poisson approximation to evaluate F_S but, contrary to the case of independent risks, we do not apply it directly on the dependent r.v. X_i $(i = 1, \dots, n)$. We first assume as being known the correlation structure previously presented and then apply theapproximation to the

independent conditional random variables representing the total amount of claims given the risk parameter θ. This approach is similar to the one used in Genest *et al.* (2000). The compound Poisson approximation with correlated risks has also been studied, in an actuarial context, by Goovaerts and Dhaene (1996). Their results are based on Chen (1975), and Arratia *et al.* (1990) which have studied the Poisson approximation in a general setting (see also Barbour (1992)). Their approach differs in the fact that they approximate a sum of dependent Bernoulli r.v. with a Poisson distribution. In our approach, we approximate the distribution of S by a mixture of compound Poisson distributions due to the dependence structure proposed in this paper.

Suppose first that Θ is discrete i.e. $\Theta \in \{\theta_1, \ldots, \theta_m\}$ with

$$\Pr(\Theta = \theta_k) = \alpha_k, \qquad k = 1, \ldots, m.$$

In the case of a continuous distribution for Θ, one can discretize the distribution and pursue as follows. Discretization methods are discussed in Panjer and Willmot (1992) or Klugman *et al.* (1998).

Let F_T denote the c.d.f. of T which approximates F_S and let the conditional r.v.

$$(T \mid \Theta = \theta_k) = \sum_{i=1}^{n} (Y_i \mid \Theta = \theta_k)$$ be a compound Poisson r.v. with parameters λ_k and F_{Z_k}, where $(Y_i \mid \Theta = \theta_k) = \sum_{j=1}^{N_i \mid \Theta = \theta_k} B_{i,j}$ is a compound Poisson r.v.

with parameters $\lambda_{i,k}$ and F_{B_i}. The random variables $B_{i,j}$ ($i = 1, \ldots, n; j = 1, \ldots, N_i \mid \Theta = \theta_k$) are independent and identically distributed random variables with common c.d.f. F_{B_i}. The r.v. $N_i \mid \Theta = \theta_k$ is a Poisson distributed r.v. such that

$$E[N_i \mid \Theta = \theta_k] = \lambda_{i,k} = 1 - (r_i)^{\theta_k},$$

in order for the mean number of claims to be identical in the individual model and its compound Poisson approximation. We will also present below another compound Poisson approximation in which the probability of no claim is identical in the individual model and its approximation. It is easy to show that the parameters of the conditional compound Poisson r.v. $(T \mid \Theta = \theta_k)$ are respectively

$$\lambda_k = \sum_{i=1}^{n} \lambda_{i,k}$$
$$= \sum_{i=1}^{n} (1 - (r_i)^{\theta_k})$$

and

$$F_{Z_k}(x) = \sum_{i=1}^{n} \frac{(1 - (r_i)^{\theta_k})}{\lambda_k} F_{B_i}(x).$$

Finally, the c.d.f. of the (unconditional) aggregate claim amount T for the whole portfolio is clearly the following mixture

$$F_T(s) = \sum_{k=1}^{m} \alpha_k F_{T|\Theta=\theta_k}(s), \qquad s \geq 0, \tag{6}$$

where

$$F_{T|\Theta=\theta_k}(s) = \Pr(M_k = 0 \mid \Theta = \theta_k) + \sum_{j=1}^{\infty} \Pr(M_k = j \mid \Theta = \theta_k) F_{Z_k}^{*j}(s),$$

and $(M_k \mid \Theta = \theta_k) \sim \text{Poisson}(\lambda_k)$ and $F_{Z_k}^{*j}$ is the jth convolution of Z_k.

In most cases, there is no explicit form for F_T. We can either apply Panjer's algorithm or the Fast Fourier transform (see Rolski *et al.* (1999) or Klugman *et al.* (1998) for details on these two approaches) to compute $F_{T|\Theta=\theta_k}$ for each $k = 1, 2, \ldots, m$. Then, we apply (6) to obtain F_T.

Since $(T \mid \Theta = \theta_k) \sim \text{compound Poisson}(\lambda_k, F_{Z_k})$, the m.g.f. of $(T \mid \Theta = \theta_k)$ is

$$\begin{aligned} M_{T|\Theta=\theta_k}(t) &= P_{M_k|\Theta=\theta_k}(M_{Z_k}(t)) \\ &= \exp(\lambda_k(M_{Z_k}(t) - 1)), \end{aligned}$$

which leads to the m.g.f. of T

$$M_T(t) = \sum_{k=1}^{m} \alpha_k M_{T|\Theta=\theta_k}(t).$$

The expectation and variance of T are respectively

$$\begin{aligned} E[T] &= \sum_{k=1}^{m} \alpha_k \sum_{i=1}^{n} \lambda_k \frac{(1 - (r_i)^{\theta_k})}{\lambda_k} E[B_i] \\ &= E[S], \end{aligned}$$

and

$$\mathrm{Var}[T] = E[T^2] - E^2[T]$$

$$= \sum_{k=1}^{m} \alpha_k \left\{ \sum_{i=1}^{n} (1 - (r_i)^{\theta_k}) E[B_i^2] \right\}$$

$$+ \sum_{k=1}^{m} \alpha_k \left\{ \left(\sum_{i=1}^{n} (1 - (r_i)^{\theta_k}) E[B_i] \right)^2 - \sum_{i=1}^{n} (1 - (r_i)^{\theta_k}) E[B_i]^2 \right\}$$

$$- \left(\sum_{k=1}^{m} \alpha_k \sum_{i=1}^{n} (1 - (r_i)^{\theta_k}) E[B_i] \right)^2.$$

We are also interested in evaluating the quality of the approximation of the c.d.f. of the total amount of claims S of the individual model by the c.d.f. of the total amount of claims T. We use two measures to evaluate the distance between S and T; they are the total variation distance and the stop-loss distance. In Rolski *et al.* (1999), these two distances are respectively defined as follows:

$$d_{\mathrm{TV}}(S, T) = \sup_{B \in \mathbf{B}(\mathbb{R})} |\mathrm{Pr}(S \in B) - \mathrm{Pr}(T \in B)|,$$

and

$$d_{\mathrm{SL}}(S, T) = \sup_{d \geq 0} |E[(S - d)_+] - E[(T - d)_+]|.$$

Proposition 1 *With the structure proposed in section 2, one obtains*

$$d_{\mathrm{TV}}(S, T) \leq \sum_{k=1}^{m} \alpha_k \sum_{i=1}^{n} (1 - (r_i)^{\theta_k})^2,$$

and

$$d_{\mathrm{SL}}(S, T) \leq \sum_{k=1}^{m} \alpha_k \sum_{i=1}^{n} E[B_i](1 - (r_i)^{\theta_k})^2.$$

Proof We first compare the distribution of X_i and Y_i where Y_i is as previously defined. For each $B \in \mathbf{B}(\mathbb{R})$,

$$
\Pr(X_i \in B) - \Pr(Y_i \in B) = \sum_{k=1}^{m} \alpha_k \left\{ (r_i)^{\theta_k} 1_{[0,\infty)}(B) + (1 - (r_i)^{\theta_k}) F_{B_i}(B) \right\}
$$
$$
- \sum_{k=1}^{m} \alpha_k \left\{ e^{-\lambda_{i,k}} 1_{[0,\infty)}(B) + \lambda_{i,k} e^{-\lambda_{i,k}} F_{B_i}(B) \right.
$$
$$
+ \sum_{j=2}^{\infty} \frac{\lambda_{i,k}^j}{j!} e^{-\lambda_{i,k}} F_{B_i}^{*j}(B) \bigg\}
$$
$$
\leq \sum_{k=1}^{m} \alpha_k \left\{ ((r_i)^{\theta_k} - e^{-\lambda_{i,k}}) 1_{[0,\infty)}(B) \right.
$$
$$
+ (1 - (r_i)^{\theta_k} - \lambda_{i,k} e^{-\lambda_{i,k}}) F_{B_i}(B) \bigg\}.
$$

Since $(r_i)^{\theta_k} \leq e^{-\lambda_{i,k}}$, we obtain

$$
\Pr(X_i \in B) - \Pr(Y_i \in B) \leq \sum_{k=1}^{m} \alpha_k (1 - (r_i)^{\theta_k})^2.
$$

The second inequality is obtained in a similar way

$$
E[(Y_i - d)_+] - E[(X_i - d)_+] = \sum_{k=1}^{m} \alpha_k \sum_{j=1}^{\infty} \frac{\lambda_{i,k}^j}{j!} e^{-\lambda_{i,k}} \int_d^{\infty} (x - d) \, dF_{B_i}^{*j}(x)
$$
$$
- \sum_{k=1}^{m} \alpha_k (1 - (r_i)^{\theta_k}) \int_d^{\infty} (x - d) \, dF_{B_i}(x)
$$
$$
\leq \sum_{k=1}^{m} \alpha_k \sum_{j=2}^{\infty} \frac{\lambda_{i,k}^j}{j!} e^{-\lambda_{i,k}} \int_d^{\infty} (x - d) \, dF_{B_i}^{*j}(x)
$$
$$
\leq \sum_{k=1}^{m} \alpha_k \sum_{j=2}^{\infty} \frac{\lambda_{i,k}^j}{j!} e^{-\lambda_{i,k}} j E[B_i]
$$
$$
\leq \sum_{k=1}^{m} \alpha_k E[B_i] \lambda_{i,k}^2.
$$

It is shown in Rolski *et al.* (1999) that $d_{\mathrm{TV}}(T, S) \leq \sum_{i=1}^{n} d_{\mathrm{TV}}(Y_i, X_i)$ and $d_{\mathrm{SL}}(T, S) \leq \sum_{i=1}^{n} d_{\mathrm{SL}}(Y_i, X_i)$. Therefore,

$$d_{\mathrm{TV}}(T, S) \leq \sum_{k=1}^{m} \alpha_k \sum_{i=1}^{n} (1 - (r_i)^{\theta_k})^2,$$

and

$$d_{\mathrm{SL}}(T, S) \leq \sum_{k=1}^{m} \alpha_k \sum_{i=1}^{n} E[B_i](1 - (r_i)^{\theta_k})^2. \qquad \blacksquare$$

Another way to approximate the individual risk model by the collective risk model is to use the compound Poisson approximation such that the probability of no claim is identical. For this second approximation, we denote the random variables and the Poisson parameters with a "$'$". In this approximation, the r.v. $(N_i' \mid \Theta = \theta_k)$ is a Poisson distributed r.v. such that

$$\Pr(N_i' = 0 \mid \Theta = \theta_k) = e^{-\lambda_{i,k}'}$$
$$= \Pr(I_i = 0 \mid \Theta = \theta_k) = (r_i)^{\theta},$$

for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. In the proposed structure, we must let the parameter λ_k' of the Poisson distribution of $(M_k' \mid \Theta = \theta_k)$ be equal to $\lambda_k' = \sum_{i=1}^{n} \lambda_{i,k}' = \sum_{i=1}^{n} -\ln((r_i)^{\theta_k})$ instead of $\lambda_k' = \sum_{i=1}^{n} (1 - (r_i)^{\theta_k})$ of the first compound Poisson approximation given. The following proposition gives the bounds on the two distances between S and T'.

Proposition 2 *For the second compound Poisson approximation, one has*

$$d_{\mathrm{TV}}(T', S) \leq \sum_{k=1}^{m} \alpha_k \sum_{i=1}^{n} (1 - (r_i)^{\theta_k})^2,$$

and

$$d_{\mathrm{SL}}(T', S) \leq \sum_{k=1}^{m} \alpha_k \sum_{i=1}^{n} E[B_i](1 - (r_i)^{\theta_k})^2.$$

Proof We first compare the distribution of X_i and Y_i. For each $B \in \mathbf{B}(\mathbb{R})$

$$
\begin{aligned}
\Pr(X_i \in B) - \Pr(Y_i' \in B) = & \sum_{k=1}^{m} \alpha_k \left\{ (r_i)^{\theta_k} 1_{[0,\infty)}(B) + (1 - (r_i)^{\theta_k}) F_{B_i}(B) \right\} \\
& - \sum_{k=1}^{m} \alpha_k \left\{ e^{-\lambda_{i,k}'} 1_{[0,\infty)}(B) + \lambda_{i,k}' e^{-\lambda_{i,k}} F_{B_i}(B) \right. \\
& \left. + \sum_{j=2}^{\infty} \left(\frac{\lambda_{i,k}'}{j!} \right)^{j} e^{-\lambda_{i,k}'} F_{B_i}^{*j}(B) \right\} \\
\leq & \sum_{k=1}^{m} \alpha_k (1 - (r_i)^{\theta_k} - \lambda_{i,k}' e^{-\lambda_{i,k}'}) F_{B_i}(B)
\end{aligned}
$$

Since $1 - (r_i)^{\theta_k} \leq \lambda_{i,k}'$, we obtain

$$
\Pr(X_i \in B) - \Pr(Y_i' \in B) \leq \sum_{k=1}^{m} \alpha_k (1 - (r_i)^{\theta_k})^2
$$

The result follows from the inequalities given above from Rolski *et al.* (1999). The second inequality is obtained as in proposition 1. ∎

The two approximation methods produce very similar results, particularly when q_i is small.

If Θ is a degenerated r.v., then the I_i's are independent and the bound becomes

$$
\begin{aligned}
d_{\mathrm{TV}}(T, S) \leq & \sum_{k=1}^{m} \alpha_k \sum_{i=1}^{n} (1 - (r_i)^{\theta_k})^2 \\
= & \sum_{i=1}^{n} (1 - 2p_i + p_i^2) \\
= & \sum_{i=1}^{n} q_i^2,
\end{aligned}
$$

which is the bound obtained by Gerber (1984).

If the B_i's are identically distributed, then

$$
d_{\mathrm{TV}}(T, S) \leq \sum_{k=1}^{m} \alpha_k \frac{1 - e^{\lambda_k}}{\lambda_k} \sum_{i=1}^{n} (1 - (r_i)^{\theta_k})^2.
$$

An application of the compound Poisson approximation and its quality is examined in the next section.

4   Numerical Illustration

We consider the individual risk model in a credit risk context where a financial institution holds a portfolio of credit risks X_{ij}^k ($k = 1, \dots, n_{ij}$; $i = 1, \dots, a$; $j = 1, \dots, b$) classified according to b ($j = 1, \dots, b$) different risk classes and with possible loss amount at default B_i ($i = 1, \dots, a$) in units, say, of $100\,000\$$. We denote by q_j ($j = 1, \dots, b$) the probability of default of a credit risk in class j and by n_{ij} the number of credit risks in the jth class with loss unit B_i in case of default. The total number of credit risks in the portfolio is $m = \sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}$ and the maximum number of loss units for the portfolio is $\sum_{i=1}^{a}\sum_{j=1}^{b} n_{ij}B_i$. The characteristics of such a portfolio can be displayed as in the Table 1.

Loss unit/Risk class	q_1	$\dots$	q_j	$\dots$	q_b
B_1	n_{11}	$\dots$	n_{1j}	$\dots$	n_{1b}
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
B_i	n_{i1}	$\dots$	n_{ij}	$\dots$	n_{ib}
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
B_a	n_{a1}	$\dots$	n_{aj}	$\dots$	n_{ab}

Table 1. Characteristics of the portfolio

In the present section, we are interested in the comparison of the riskiness of different credit portfolios with the characteristics described above. For that purpose, we use the stop-loss premium as a risk measure. It has been discussed in different papers (e.g. Bäuerle and Müller (1998), Dhaene *et al.* (2000)) that in the class of all multivariate risks $(X_{11}^1, \dots, X_{ab}^{n_{ab}})$ with given marginals, the maximal stop-loss premium of $S = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} X_{ij}^k$ is obtained when the risks X_{ij}^k are mutually comonotonic.

Definition 3 *The risks $Y_1, Y_2, \ldots, Y_n$ are said to be mutually comonotonic if any of the following equivalent conditions hold*

1. *The c.d.f. $F_{Y_1,\ldots,Y_n}$ of $(Y_1, \ldots, Y_n)$ satisfies*

$$F_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \min(F_{Y_1}(y_1), \ldots, F_{Y_n}(y_n))$$

for all $x_1, \ldots, x_n \geq 0$.

2. *There exists a random variable Z and non-decreasing functions $g_1, \ldots, g_n$ on $\mathbf{R}$ such that $(Y_1, \ldots, Y_n)$ is equal in distribution to $(g_1(Z), \ldots, g_n(Z))$*

3. *For any uniformly distributed random variable U on $[0,1]$, we have that $(Y_1, \ldots, Y_n)$ is equal in distribution to $(F_{Y_1}^{-1}(U), \ldots, F_{Y_n}^{-1}(U))$.*

Based on Dhaene *et al.* (2000), the maximal stop-loss premium for the portfolio just described is

$$E[(S-d)_+] = \begin{cases} \displaystyle\sum_{j=1}^{b} q_j \sum_{i=1}^{a} n_{ij} B_i - dq_1, & \text{if } 0 \leq d \leq \sum_{i=1}^{a} n_{i1} B_i \\[2em] \displaystyle\sum_{j=t+1}^{b} q_j \sum_{i=1}^{a} n_{ij} B_i - \Big(d - \sum_{j=1}^{t}\sum_{i=1}^{a} n_{ij} B_i\Big) q_{t+1}, & \\ \quad \text{if } \displaystyle\sum_{j=1}^{t}\sum_{i=1}^{a} n_{ij} B_i \leq d \leq \sum_{j=1}^{t+1}\sum_{i=1}^{a} n_{ij} B_i; & \\ \quad 1 \leq t \leq \displaystyle\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} X_{ij}^{k} & \\[1em] 0, & \text{if } d \geq \displaystyle\sum_{j=1}^{b}\sum_{i=1}^{a} n_{ij} B_i. \end{cases}$$

Let us now numerically compare the behavior of the stop-loss premium for a credit portfolio with a common mixture correlation structure as described in section 2. In the example that follows, we suppose

$$\Pr(I_i = 1 \mid \Theta = \theta) = 1 - (r_i)^\theta \quad \text{and} \quad \Pr(I_i = 0 \mid \Theta = \theta) = (r_i)^\theta,$$

where the risk parameter Θ follows a logarithmic distribution with parameter $\gamma \in (0,1)$ and $\Theta \in \{1, 2, \ldots\}$. Given that

$$\Pr(I_i = 1) = 1 - P_\Theta(r_i),$$

we have

$$q_i = 1 - \frac{\ln(1 - \gamma r_i)}{\ln(1 - \gamma)},$$

and

$$r_i = \frac{1 - (1 - \gamma)^{1-q_i}}{\gamma}.$$

The credit portfolio has 4 risk classes with probability of default $q_1 = 2.5\%$, $q_2 = 5\%$, $q_3 = 7.5\%$, and $q_4 = 10\%$. Each risk class contains $n_{ij} = 25$ credit risks with 10 possible loss unit in case of default $B_1 = 1$, $B_2 = 2, \ldots, B_{10} = 10$. The maximal total amount of loss S equals 5500 and $E[S] = 343.75$. In Table 2, we compare the stop-loss premium for a retention level of 0 to 4000 for independent risks, dependent risks with parameter $\gamma = 0.5, 0.9$ and comonotonic risks. We also compare each one of these cases with its compound Poisson approximation, where the Poisson parameter is chosen such that $\lambda_i = q_i$ $(i = 1, 2, \ldots, a)$.

The above results show that, for an heterogeneous portfolio, the stop-loss premium increases uniformly with the degree of dependence between the credit risks' default occurrences and that for credit portfolios with correlated default occurrences, the Poisson approximation is good for every retention level as in the independence case.

d	Indep. exact	Indep. CP ap.	$\gamma = 0.5$ Exact	$\gamma = 0.5$ CP ap.	$\gamma = 0.9$ Exact	$\gamma = 0.9$ CP ap.	Comon —
0	343.750	343.750	343.750	343.750	343.750	343.750	343.750
200	143.755	143.758	145.504	145.656	187.914	187.975	323.750
400	2.943	3.369	56.189	56.266	114.729	114.818	303.750
600	0.000	0.000	22.644	22.713	72.461	72.567	283.750
800	0.000	0.000	9.266	9.336	46.259	46.371	263.750
1000	0.000	0.000	3.775	3.827	29.535	29.644	243.750
1200	0.000	0.000	1.507	1.540	18.734	18.835	223.750
1400	0.000	0.000	0.583	0.602	11.746	11.835	204.375
1600	0.000	0.000	0.217	0.227	7.249	7.325	189.375
1800	0.000	0.000	0.077	0.082	4.385	4.447	174.375
2000	0.000	0.000	0.026	0.028	2.589	2.638	159.375
2200	0.000	0.000	0.008	0.009	1.484	1.523	144.375
2400	0.000	0.000	0.002	0.003	0.822	0.851	129.375
2600	0.000	0.000	0.001	0.001	0.438	0.458	114.375
2800	0.000	0.000	0.000	0.000	0.222	0.235	100.625
3000	0.000	0.000	0.000	0.000	0.106	0.115	90.625
3200	0.000	0.000	0.000	0.000	0.047	0.053	80.625
3400	0.000	0.000	0.000	0.000	0.019	0.023	70.625
3600	0.000	0.000	0.000	0.000	0.007	0.009	60.625
3800	0.000	0.000	0.000	0.000	0.002	0.003	50.625
4000	0.000	0.000	0.000	0.000	0.001	0.001	40.625

Table 2. Stop-loss premiums

5   Dependence structure with Archimedean Copulas

In the present section, we show that the model proposed in section 2 corresponds to the individual risk model with a dependence structure defined via an Archimedean copula or an approximation of an Archimedean copula. We first begin

by giving basic elements on copulas. The reader may refer to Joe (1997), Nelsen (1999) and Genest and Mackay (1986) for a general introduction, and to Wang (1998), Frees and Valdez (1998), Embrechts *et al.* (1999, 2000, 2001a, 2001b), Cossette *et al.* (2002), and Bouye *et al.* (2000) for applications in actuarial science and risk management.

Suppose that $Y_1, \ldots, Y_n$ are r.v. with marginal distributions. Then, for any multivariate distribution $F_{Y_1, \ldots, Y_n}$ the following representation holds for a copula C

$$F_{Y_1, \ldots, Y_n}(y_1, \ldots, y_n) = C(F_{Y_1}(y_1), \ldots, F_{Y_n}(y_n)). \tag{7}$$

For continuous r.v. $Y_1, \ldots, Y_n$, the representation (7) is unique.

A copula C is the distribution function of a random vector with Uniform-$[0, 1]$ marginals. The advantage of (7) is that it separately defines the dependence structure which is made through the copula C and the marginals F_{Y_i} ($i = 1, \ldots, n$). Numerous copulas can be found in the literature (see e.g. Joe (1997), Nelsen (1999), and references therein). The simplest one is the independence copula $C(u_1, \ldots, u_n) = u_1, \ldots, u_n$. An important class of copulas is the Archimedean family of copulas which has been originally considered by Genest and MacKay (1986). An example of a copula from this family is the Cook-Johnson copula which is written as

$$C(u_1, \ldots, u_n) = \left(\left(\sum_{i=1}^{n} u_i^{-\alpha} \right) - (n-1) \right)^{-1/\alpha}, \qquad \alpha > 0.$$

This copula, together with the Frank copula and the Gumbel-Hougaard copula from this family, have been used in actuarial applications (see e.g. Frees and Valdez (1998)). All the copulas from this family can be expressed as

$$C(u_1, \ldots, u_n) = \phi^{-1}(\phi(u_1) + \ldots + \phi(u_n)),$$

where $\phi: (0, 1] \to [0, \infty)$ such that $\phi(1) = 0$ and

$$(-1)^k \frac{d^k}{dx^k} \phi(x) \geq 0, \qquad 1 \leq k \leq n. \tag{8}$$

If (8) is verified for all integers $n \geq 1$, then ϕ is completely monotone and its inverse ϕ^{-1} is the Laplace transform of a distribution K whose support is $[0, \infty)$. Following Marshall and Olkin (1988) (see also Joe (1997)), $F_{Y_1, \ldots, Y_n}$ may be viewed as a mixture of powers i.e. it can be written in the form

$$F_{Y_1, \ldots, Y_n}(y_1, \ldots, y_n) = \int \prod_{j=1}^{n} (F_{Y_j}^0(y_j))^\theta \, dG_\Theta(\theta), \tag{9}$$

where $F_{Y_j}^0(x) = \exp(-\phi(F_{Y_j}(x)))$. If, for instance, G_Θ is a gamma distribution with parameter $\dfrac{1}{\alpha}$ and 1, and $\phi(x) = x^{-\alpha} - 1$ for $\alpha > 0$, then we have a Cook-Johnson copula.

In our case, the relation of dependence is introduced with an Archimedean copula through the random vector $(I_1, \dots, I_n)$ with joint c.d.f.

$$F_{I_1,\dots,I_n}(i_1,\dots,i_n) = C(F_{I_1}(i_1),\dots,F_{I_n}(i_n))$$
$$= \int \prod_{j=1}^{n} \{F_{I_j}^0(i_j)\}^\theta \, dG_\Theta(\theta), \tag{10}$$

where $F_{I_j}^0(x) = \exp(-\phi(F_{I_j}(x)))$. For every Bernoulli r.v. I_j, there exists a r.v. $I_{j,\theta}$ with c.d.f. $(F_{I_j}^0(x))^\theta$. This r.v. is still a Bernoulli r.v. and with expectation $q_{j,\theta} = 1 - (\exp(-\phi(p_j)))^\theta$ for $j = 1,\dots,n$. Under this representation, the Bernoulli r.v. $I_{i,\theta},\dots,I_{n,\theta}$ are mutually independent for a given realization θ of the r.v. Θ.

Note that (10) corresponds to (2) when $\exp(-\phi(p_j)) = r_j$ for $j = 1,\dots,n$. We have thus written the dependence structure proposed in section 2 with an Archimedean copula through a common mixture. When G_Θ is discrete, an example of copula is the Frank copula and when G_Θ is continuous, (10) does not have an explicit form. In that case, one can use discretization methods (see Panjer and Willmot (1992) or Klugman *et al.* (1998)) to make G_Θ a discrete distribution.

Again, we are interested in finding the c.d.f. of the total amount of claims of a portfolio. Given that F_S rarely has an explicit form when the multivariate distribution of $\underline{I} = (I_1,\dots,I_n)$ is defined via copulas, we will, as in section 2, first determine $M_S(t)$ which will allow us to obtain $F_S(s)$ by the Fast Fourier Transform inversion method. The m.g.f. of S in the case of a dependence structure defined with copulas as a common mixture is

$$M_S(t) = E[e^{tS}]$$
$$= E[e^{t(X_1+\dots+X_n)}]$$
$$= E[E[e^{t(I_1 B_1+\dots+I_n B_n)} \mid \Theta]] \tag{11}$$
$$= E[E[e^{tI_1 B_1} \mid \Theta]] \times \dots \times E[E[e^{tI_n B_n} \mid \Theta]]$$
$$= \int M_{S_\theta}(t) \, dG_\Theta,$$

where

$$M_{S_\theta}(t) = \prod_{j=1}^{n} (P_{I_{j,\theta}}(M_{B_j}(t))), \tag{12}$$

and

$$P_{I_{j,\theta}}(t) = (1 - q_{j,\theta}) + tq_{j,\theta}.$$

Given that the dependence structure obtained with an Archimedean copula corresponds to the previous proposed structure, the m.g.f. of S in the case of a continuous distribution G_θ can be approximated as follows by discretization

$$\widetilde{M_S}(t) = \sum_{\theta_k \in A} \prod_{j=1}^{n} P_{I_{j,\theta_k}}(M_{B_j}(t)) \Pr(\widetilde{\Theta} = \theta_k), \tag{13}$$

where $\widetilde{\Theta}$ is the discrete approximation of the r.v. Θ and (12) is obtained by numerical evaluation. To discretize G_θ, we find θ_0^* (> 0) and ν an integer of the set $A = \{\theta_0^*, \theta_0^* + h, \theta_0^* + 2h, \ldots, \theta_0^* + \nu h\}$ such that, for a given ε (ex: $\varepsilon = 10^{-6}$), $G_\Theta(\theta_0^*) = \varepsilon$ and $G_\Theta(\theta_0^* + \nu h) = 1 - \varepsilon$. The parameter h is set according to the number $\nu + 1$ of desired discretization points.

In this section, we have demonstrated that if the dependence structure is defined via an Archimedean copula, then it can be written as the dependence structure presented in section 2. Consequently, another way to determine $\widetilde{F}_S$ is to use, as in section 3, the compound Poisson approximation on F_S.

6   Extension

We have considered in this paper the case where only one risk factor Θ affects the possible occurrence of losses. One may wish to apply the proposed structure in section 2 in a context where different sources of risk affect the occurrence of losses. In crop insurance for instance, not only the weather may be considered in the evaluation of the probability of a bad harvest for a given year.

Let $\underline{\Theta} = (\Theta_1, \Theta_2, \ldots, \Theta_p)$ be a vector of p risk parameters and let $\phi \colon \mathbf{R}^p \to (0, \infty)$ be a function combining the different factors which have an impact on the occurrence of a loss. The conditional probabilities of the occurrence random variables $(I_i \mid \underline{\Theta} = \underline{\theta})$ $(i = 1, \ldots, n)$ defined previously become in such a context

$$\Pr(I_i = 1 \mid \underline{\Theta} = \underline{\theta}) = 1 - r_i^{\phi(\underline{\theta})} \quad \text{and} \quad \Pr(I_i = 0 \mid \underline{\Theta} = \underline{\theta}) = r_i^{\phi(\underline{\theta})},$$

where $(I_i \mid \underline{\Theta} = \underline{\theta})$ $(i = 1, \ldots, n)$ are Bernoulli random variables and r_i is the basic parameter of the conditional distribution of $(I_i \mid \underline{\Theta} = \underline{\theta})$. For a given multivariate distribution $G_{\underline{\Theta}}$ of the random vector $\underline{\Theta}$, we have

$$
\begin{aligned}
\Pr(I_i = 0) &= \int_{\mathbf{R}^p} r_i^{\phi(\underline{\theta})} \, dG_{\underline{\Theta}}(\underline{\Theta}) \\
&= P_{\underline{\Theta}}(r_i) \\
&= M_{\underline{\Theta}}(\ln r_i),
\end{aligned}
$$

where $P_{\underline{\Theta}}(t)$ and $M_{\underline{\Theta}}(t)$ are respectively the multivariate p.g.f. and the m.g.f. of $\underline{\Theta}$. With steps similar to the ones used in section 2, we obtain the following c.d.f. for S and the stop-loss premium $\pi_S(d)$ for the extension just exposed

$$
F_S(s) = \int_{\mathbf{R}^p} F_{S \mid \underline{\Theta} = \underline{\theta}}(s) \, dG_{\underline{\Theta}}(\underline{\theta}),
$$

and

$$
\pi_S(d) = \int_{\mathbf{R}^p} \pi_{S \mid \underline{\Theta} = \underline{\theta}}(d) \, dG_{\underline{\Theta}}(\underline{\theta}).
$$

Note that other types of common mixture models than the one discussed in this paper can be found in Joe (1997).

7   Acknowledgements

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Abstract

We propose a dependence structure based on common mixture models to allow a possible correlation between risks in the individual risk model. This model can be applied in an insurance context or in a credit risk context where one or more factors have an impact on the experience of the whole portfolio. We measure the global risk of a portfolio via the aggregate claim amount distribution. The evaluation of such a quantity can become cumbersome in cases of large portfolios. To overcome this problem, we use the compound Poisson approximation within the proposed common mixture model and then evaluate the quality of this approximation. We also give numerical examples in which we examine the riskiness of portfolios under the correlation structure proposed in this paper and apply the compound Poisson approximation. Finally, we establish the link between the common mixture model proposed and the family of Archimedean copulas.

Zusammenfassung

Wir schlagen eine Abhänigkeitsstruktur vor, die auf Mischungsmodellen basiert, um Korellationen zwischen den Risiken im individuellen Risikomodell zu ermöglichen. Dieses Modell kann im Zusammenhang mit Versicherungs- oder mit Kreditrisiken angewendet werden, wo eine oder mehrere Faktoren den Verlauf des gesamten Portefeuilles beeinflussen. Wir messen das globale Risiko eines Portefeuilles via aggregierte Schadenhöhenverteilung. Die Bestimmung einer solchen Grösse kann im Fall grosser Portefeuilles mühsam werden. Um dieses Problem in den Griff zu bekommen, verwenden wir die Poisson Gesamtschadenapproximation innerhalb des vorgeschlagenen Mischungsmodells. Anschliessend bestimmen wir die Qualität dieser Approximation. Anhand numerischer Beispiele wird das Risiko eines Portefeuilles unter der im Artikel vorgeschlagenen Korrelationsstruktur untersucht und die Poisson Gesamtschadenapproximation angewendet. Schliesslich zeigen wir den Zusammmenhang zwischen dem vorgeschlagenen Mischungsmodell und der Familie der Archimedischen Kopulas auf.

Résumé

Nous proposons une structure de dépendance basée sur des modèles de mélange, afin de permettre des corrélations entre les risques dans le modèle de risque individuel. Ce modèle peut être appliqué dans le contexte des assurance ou dans celui du risque de crédit, où un ou plusieurs facteurs influencent l'expérience de l'ensemble du portefueille. Nous mesurons le risque global d'un portefeuille avec la distribution du montant total des sinistre. L'évaluation d'une telle valeur peut se révéler difficile dans le cas de grands portefeuilles. Pour surmonter ce problème, nous utilisons l'approximation de Poisson du montant total des sinistres à l'aide du modèle de mélange proposé. Nous évaluons ensuite la qualité de cette approximation. Nous proposons également un example numérique dans lequel nous examinons le risque d'un portefeuille sous la structure de corrélation proposée dans cet article et appliquons l'approximation de Poisson du montant total des sinistres. Finalement nous établissons le lien entre le modèle de mélange proposé et la famille des copules archimédiennes.