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Pension and semi-retirement

1 Introduction

The present note is motivated by a new Norwegian act on group pension schemes. If a member of such a scheme continues to work part-time after the retirement age stipulated by the scheme, then the pension payments should be reduced relative to the work-rate of the member, e. g. when the member works 80 % of full position, then he will receive 20 % of full pension. As a part of the pension is not taken out, the premium reserve becomes higher than if the member had fully retired and taken out full pension. This increase in the reserve should at some stage be applied to increase the level of full pension. In the present note we discuss a dynamic approach.

Let u denote the retirement age stipulated by the pension scheme. The premium payment stops at age u. Premiums, benefits, and reserves should be determined such that the equivalence principle is fulfilled, and we apply standard notation for annuities and commutation functions, cf. e. g. Jordan (1967). Let V_x denote the premium reserve at age $x \geq u$.

If the member fully retires at age u, then he will receive a pension of size B per annum for the rest of his life. On the other hand, if he continues to work in a full position till age $x > u$, then no pension will be taken out before that age, so that we will have

$$V_u = \frac{D_x}{D_u} V_x \,, \tag{1}$$

which can be used to determine the pension B_x.

An intermediate case occurs if the member continues to work part-time after age u. Let φ_x be his work-rate at age $x \geq u$, that is, he works $100\varphi_x\,\%$ of a full position. Then he takes out $100(1-\varphi_x)\,\%$ of full pension. The question is now, should the remaining $100\varphi_x\,\%$ be used to increase the amount of full pension with immediate effect, or should the increase take place when he fully retires? In the present note we consider the former case.

We first consider the case with continuous pension and then look at discrete annual payment. In the latter case we consider two sub-cases. In the first one the increase of the pension arising from the pension not taken out, has immediate effect, in the second case its effect starts the following year.

2 The continuous case

We assume that the pension is paid continuously and let B_x denote the rate of full pension at age x. The pension not taken out in the time interval $[x, x+dx)$ is $B_x \varphi_x \, dx$ and should be used as premium for a continuous life annuity with rate $B_{x+dx} - B_x$. Thus we obtain

$$(B_{x+dx} - B_x)\overline{a}_x = B_x \varphi_x \, dx \tag{2}$$

with $B_u = B$, and dividing by dx and letting dx approach zero give the differential equation

$$\left(\frac{d}{dx} B_x\right)\overline{a}_x = \varphi_x B_x$$

with the solution

$$B_x = B e^{\int_u^x \frac{\varphi_r}{\overline{a}_r}\, dr} . \tag{3}$$

In particular, if the work-rate has a constant value φ, we obtain

$$B_x = B\left(e^{\int_u^x \frac{1}{\overline{a}_r}\, dr}\right)^{\varphi} . \tag{4}$$

We have

$$\int_u^x \frac{1}{\overline{a}_r}\, dr = \int_u^x \frac{D_r}{\overline{N}_r}\, dr = -\int_u^x \frac{1}{\overline{N}_r}\frac{d}{dr}\overline{N}_r\, dr = -\int_u^x \frac{d}{dr}\ln \overline{N}_r\, dr$$
$$= \ln \overline{N}_u - \ln \overline{N}_x = \ln \frac{\overline{N}_u}{\overline{N}_x} = \ln \frac{\overline{a}_u}{{}_{x-u|}\overline{a}_u} ,$$

from which we obtain

$$e^{\int_u^x \frac{1}{\overline{a}_r}\, dr} = \frac{\overline{a}_u}{{}_{x-u|}\overline{a}_u} , \tag{5}$$

and insertion in (4) gives

$$B_x = N\left(\frac{\overline{a}_u}{{}_{x-u|}\overline{a}_u}\right)^{\varphi} = B\left(\frac{\overline{N}_u}{\overline{N}_x}\right)^{\varphi} .$$

In practice, the work-rate will be piece-wise constant, so we assume that

$$\varphi_r = \widetilde{\varphi}_j \,. \qquad \begin{array}{l} r_{j-1} \le r < r_j; \quad j = 1, 2, \dots, j_x\,; \\ u = r_0 < r_1 < \cdots < r_{j_x} = x \end{array}$$

Insertion in (3) and application of (5) gives

$$\begin{aligned} B_x &= B e^{\int_u^x \frac{\varphi_r}{\bar{a}_r}\, dr} = B e^{\sum_{j=1}^{j_x} \widetilde{\varphi}_j \int_{r_{j-1}}^{r_j} \frac{1}{\bar{a}_r}\, dr} \\ &= B \prod_{j=1}^{j_x} \left(e^{\int_{r_{j-1}}^{r_j} \frac{1}{\bar{a}_r}\, dr} \right)^{\widetilde{\varphi}_j} = B \prod_{j=1}^{j_x} \left(\frac{\bar{a}_{r_{j-1}}}{{}_{r_j - r_{j-1}|}\bar{a}_{r_{j-1}}} \right)^{\widetilde{\varphi}_j} \\ &= B \prod_{j=1}^{j_x} \left(\frac{{}_{r_{j-1}-u|}\bar{a}_u}{{}_{r_j - u|}\bar{a}_u} \right)^{\widetilde{\varphi}_j} = B \prod_{j=1}^{j_x} \left(\frac{\overline{N}_{r_{j-1}}}{\overline{N}_{r_j}} \right)^{\widetilde{\varphi}_j} . \end{aligned}$$

Let us now consider a member who works full time until he fully retires at age x. Then $\varphi_r = 1$ for $u \le r < x$, and we obtain

$$B_x = B \frac{\bar{a}_u}{{}_{x-u|}\bar{a}_u} \,. \tag{6}$$

In this case, no pension has been taken out before full retirement, and, hence, we can also find B_x by (1). We have $V_u = B\bar{a}_u$ and $V_x = B_x \bar{a}_x$, and insertion in (1) gives

$$B\bar{a}_u = \frac{D_x}{D_u} B_x \bar{a}_x = B_x \, {}_{x-u|}\bar{a}_u \,,$$

from which we obtain (6).

3 The discrete case

3A. In this section we consider the case when the pension is paid annually at the beginning of the year, the first time at age u. The payment at age x is $B_x \, \varphi_x$ ($x = u, u+1, \dots$) with φ_x denoting the work-rate between age x and age $x+1$. We shall consider two cases:

1. The pension that is not taken out, is applied to increase the pension from the following payment.

2. The pension that is not taken out, is applied to increase the pension with immediate effect.

3B. We assume the pension not taken out at age x is applied to increase the pension from the following payment. Then we obtain analogous to (2)

$$(B_{x+1} - B_x)a_x = B_x\varphi_x\,,$$

with $B_u = B$, that is,

$$\frac{B_{x+1}}{B_x} = 1 + \frac{\varphi_x}{a_x}\,,$$

which gives

$$B_x = B_u \prod_{r=u}^{x-1} \frac{B_{r+1}}{B_r} = B \prod_{r=u}^{x-1} \left(1 + \frac{\varphi_r}{a_r}\right).$$

In the special case with $\varphi_r = 1$ for $r = u, u+1, \dots, x-1$ we get

$$B_x = B \prod_{r=u}^{x-1} \left(1 + \frac{1}{a_r}\right). \tag{7}$$

We have

$$\prod_{r=u}^{x-1} \left(1 + \frac{1}{a_r}\right) = \prod_{r=u}^{x-1} \left(1 + \frac{D_r}{N_{r+1}}\right) = \prod_{r=u}^{x-1} \frac{N_r}{N_{r+1}} = \frac{N_u}{N_x} = \frac{\ddot{a}_u}{{}_{x-u|}\ddot{a}_u}\,,$$

which corresponds to (5). Insertion in (7) gives

$$B_x = B\frac{\ddot{a}_u}{{}_{x-u|}\ddot{a}_u} = B\frac{N_u}{N_x}\,,$$

which could also have been found by (1).

3C. We now assume that the pension not taken out at age x is applied to increase the pension with immediate effect. As already the pension taken out at age u will be increased through the pension not taken out at that age, we have $B_u \geq B$. Before the increase at age x, full pension was B_{x-1}. From that amount we apply $B_{x-1}\varphi_x$ as premium for an annuity that should immediately give a payment of

$(B_x - B_{x-1})(1-\varphi_x)$ and at the ages $x+1$, $x+2$, ... payments of $B_x - B_{x-1}$ as long the member is alive. This gives

$$(B_x - B_{x-1})(1 - \varphi_x + a_x) = B_{x-1}\varphi_x$$

with $B_{u-1} = B$, that is,

$$\frac{B_x}{B_{x-1}} = \frac{1+a_x}{1-\varphi_x+a_x} = \frac{\ddot{a}_x}{\ddot{a}_x - \varphi_x},$$

from which we obtain

$$B_x = B_{u-1}\prod_{r=u}^{x}\frac{B_r}{B_{r-1}} = B\prod_{r=u}^{x}\frac{\ddot{a}_r}{\ddot{a}_r - \varphi_r}.$$

We finally consider the case when the member works full time until he fully retires at age x, that is, $\varphi_r = 1$ for $r = u, u+1, \dots, x-1$ and $\varphi_x = 0$. Then

$$B_x = B\prod_{r=u}^{x-1}\frac{\ddot{a}_r}{a_r} = B\prod_{r=u}^{x-1}\frac{N_r}{N_{r+1}} = B\frac{N_u}{N_x} = B\frac{\ddot{a}_u}{{}_{x-u|}\ddot{a}_u},$$

which could also have been found by (1). It is not surprising that we get the same result as in Case 1 as when the pension increase is equal to zero, then it does not matter from what age that increase is made.

Reference

Jordan. C. W. (1967). *Life contingencies* (2. ed.). The Society of Actuaries, Chicago.

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